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Sequences of Groups, Hypergroups and Automata of Linear Ordinary Differential Operators

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Abstract: The main objective of our paper is to focus on the study of sequences (finite or countable) of groups and hypergroups of linear differential operators of decreasing orders. By using a suitable ordering or preordering of groups linear differential operators we construct hypercompositional structures of linear differential operators. Moreover, we construct actions of groups of differential operators on rings of polynomials of one real variable including diagrams of actions–considered as special automata. Finally, we obtain sequences of hypergroups and automata. The examples, we choose to explain our theoretical results with, fall within the theory of artificial neurons and infinite cyclic groups.

Keywords: hyperstructure theory; linear differential operators; ODE; automata theory

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1. Introduction

This paper discusses sequences of groups, hypergroups and automata of linear differential operators. It is based on the algebraic approach to the study of linear ordinary differential equations. Its roots lie in the work of Otakar Borůvka, a Czech mathematician, who tied the algebraic, geometrical and topological approaches, and his successor, František Neuman, who advocated the algebraic approach in his book [1]. Both of them (and their students) used the classical group theory in their considerations. In several papers, published mainly as conference proceedings such as [2–4], the existing theory was extended by the use of hypercompositional structures in place of the usual algebraic structures. The use of hypercompositional generalizations has been tested in the automata theory, where it has brought several interesting results; see, e.g., [5–8]. Naturally, this approach is not the only possible one. For another possible approach, investigations of differential operators by means of orthognal polynomials, see, e.g., [9,10].

Therefore, in this present paper we continue in the direction of [2,4] presenting results parallel to [11]. Our constructions, no matter how theoretical they may seem, are motivated by various practical issues of signal processing [12–16]. We construct sequences of groups and hypergroups of linear differential operators. This is because, in signal processing (but also in other real-life contexts), two or more connecting systems create a standing higher system, characteristics of which can be determined using characteristics of the original systems. Cascade (serial) and parallel connecting of systems of signal transfers are used in this. Moreover, series of groups motivated by the Galois theory of solvability of algebraic equations and the modern theory of extensions of fields, are often discussed in literature. Notice also paper [11] where the theory of artificial neurons, used further on in some examples, has been studied.

Another motivation for the study of sequences of hypergroups and their homomorphisms can be traced to ideas of classical homological algebra which comes from the algebraic description of topological spaces. A homological algebra assigns to any topological space a family of abelian groups and to any continuous mapping of topological spaces a family of group homomorphisms. This allows us to express properties of spaces and their mappings (morphisms) by means of properties of the groups or modules or their homomorphisms. Notice that a substantial part of homology theory is devoted to the study of exact short and long sequences of the above mentiones structures.

2. Sequences of Groups and Hypergroups: Definitions and Theorems

2.1. Notation and Preliminaries

It is crucial that one understands the notation used in this paper. Recall that we study, by means of algebra, linear ordinary differential equations. Therefore, our notation, which follows the original model of Borůvka and Neuman [1], uses a mix of algebraic and functional notation.

First, we denote intervals by J and regard open intervals (bounded or unbounded). Systems of functions with continuous derivatives of order k on J are denoted by $\mathbb{C}^k(J)$; for $k = 0$ we write $\mathbb{C}(J)$ instead of $\mathbb{C}^0(J)$. We treat $\mathbb{C}^k(J)$ as a ring with respect to the usual addition and multiplication of functions. We denote by δ_{ij} the Kronecker delta, $i, j \in \mathbb{N}$, i.e., $\delta_{ii} = \delta_{jj} = 1$ and $\delta_{ij} = 0$, whenever $i \neq j$; by $\overline{\delta_{ij}}$ we mean $1 - \delta_{ij}$. Since we will be using some notions from the theory of hypercompositional structures, recall that by $\mathcal{P}(X)$ one means the power set of X while $(P)^*(X)$ means $\mathcal{P}(X) \setminus \emptyset$.

We regard linear homogeneous differential equations of order $n \geq 2$ with coefficients, which are real and continuous on J, and–for convenience reasons–such that $p_0(x) > 0$ for all $x \in J$, i.e., equations

$$y^{(n)}(x) + p_{n-1}(x)y^{(n-1)}(x) + \cdots + p_0(x)y(x) = 0. \tag{1}$$

By $\mathbb{A}_n$ we, adopting the notation of Neuman [1], mean the set of all such equations.

Example 1. *The above notation can be explained on an example taken from* [17]*, in which Neuman considers the third-order linear homogeneous differential equation*

$$y'''(x) - \frac{q_1'(x)}{q_1(-x)}y''(x) + (q_1(x) - 1)^2 y'(x) - \frac{q_1'(x)}{q_1(x)}y(x) = 0$$

on the open interval $J \in \mathbb{R}$. One obtains this equation from the system

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -y_1 + q_1(x)y_3 \\ y_3' &= -q_1(x)y_2 \end{aligned}$$

Here $q_1 \in C^+(J)$ satisfies the condition $q_1(x) \neq 0$ on J. In the above differential equation we have $n = 3$, $p_0(x) = -\frac{q_1'(x)}{q_1(x)}$, $p_1(x) = (q_1(x) - 1)^2$ and $p_2(x) = -\frac{q_1'(x)}{q_1(-x)}$. It is to be noted that the above three equations form what is known as set of global canonical forms for the third-order equation on the interval J.

Denote $L_n(p_{n-1}, \ldots, p_0) : \mathbb{C}^n(J) \to \mathbb{C}^n(J)$ the above linear differential operator defined by

$$L_n(p_{n-1}, \ldots, p_0)y(x) = y^{(n)}(x) + \sum_{k=0}^{n-1} p_k(x)y^{(k)}(x), \tag{2}$$

where $y(x) \in \mathbb{C}^n(J)$ and $p_0(x) > 0$ for all $x \in J$. Further, denote by $\mathbb{LA}_n(J)$ the set of all such operators, i.e.,

$$\mathbb{LA}_n(J) = \{L(p_{n-1}, \ldots, p_0) \mid p_k(x) \in \mathbb{C}(J), p_0(x) > 0\}. \tag{3}$$

By $\mathbb{LA}_n(J)_m$ we mean subsets of $\mathbb{LA}_n(J)$ such that $p_m \in \mathbb{C}_+(J)$, i.e., there is $p_m(x) > 0$ for all $x \in J$. If we want to explicitly emphasize the variable, we write $y(x)$, $p_k(x)$, etc. However, if there is no specific need to do this, we write y, p_k, etc. Using vector notation $\vec{p}(x) = (p_{n-1}(x), \ldots, p_0(x))$, we can write

$$L_n(\vec{p})y = y^{(n)} + \sum_{k=0}^{n-1} p_k y^{(k)}. \tag{4}$$

Writing $L(\vec{p}) \in \mathbb{LA}_n(J)$ (or $L(\vec{p}) \in \mathbb{LA}_n(J)_m$) is a shortcut for writing $L_n(\vec{p})y \in \mathbb{LA}_n(J)$ (or, $L_n(\vec{p})y \in \mathbb{LA}_n(J)_m$).

On the sets of linear differential operators, i.e., on sets $\mathbb{LA}_n(J)$, or their subsets $\mathbb{LA}_n(J)_m$, we define some binary operations, hyperoperations or binary relations. This is possible because our considerations happen within a ring (of functions).

For an arbitrary pair of operators $L(\vec{p}), L(\vec{q}) \in \mathbb{LA}_n(J)_m$, where $\vec{p} = (p_{n-1}, \ldots, p_0)$, $\vec{q} = (q_{n-1}, \ldots, q_0)$, we define an operation "$\circ_m$" with respect to the m-th component by $L(\vec{p}) \circ_m L(\vec{q}) = L(\vec{u})$, where $\vec{u} = (u_{n-1}, \ldots, u_0)$ and

$$u_k(x) = p_m(x)q_k(x) + (1 - \delta_{km})p_k(x) \tag{5}$$

for all $k = n-1, \ldots, 0$, $k \neq m$ and all $x \in J$. Obviously, such an operation is not commutative.

Moreover, apart from the above binary operation we can define also a relation "$\leq_m$" comparing the operators by their m-th component, putting $L(\vec{p}) \leq_m L(\vec{q})$ whenever, for all $x \in J$, there is

$$p_m(x) = q_m(x) \text{ and at the same time } p_k(x) \leq q_k(x) \tag{6}$$

for all $k = n-1, \ldots, 0$. Obviously, $(\mathbb{LA}_n(J)_m, \leq_m)$ is a partially ordered set.

At this stage, in order to simplify the notation, we write $\mathbb{LA}_n(J)$ instead of $\mathbb{LA}_n(J)_m$ because the lower index m is kept in the operation and relation. The following lemma is proved in [2].

Lemma 1. *Triads $(\mathbb{LA}_n(J), \circ_m, \leq_m)$ are partially ordered (noncommutative) groups.*

Now we can use Lemma 1 to construct a (noncommutative) hypergroup. In order to do this, we will need the following lemma, known as Ends lemma; for details see, e.g., [18–20]. Notice that a join space is a special case of a hypergroup–in this paper we speak of hypergroups because we want to stress the parallel with groups.

Lemma 2. *Let $(H, \cdot, \leq)$ be a partially ordered semigroup. Then $(H, *)$, where $* : H \times H \to \mathcal{H}$ is defined, for all $a, b \in H$ by*

$$a * b = [a \cdot b)_{\leq} = \{x \in H \mid a \cdot b \leq x\},$$

is a semihypergroup, which is commutative if and only if "$\cdot$" is commutative. Moreover, if $(H, \cdot)$ is a group, then $(H, *)$ is a hypergroup.

Thus, to be more precise, defining

$$\star_m : \mathbb{LA}_n(J) \times \mathbb{LA}_n(J) \to \mathcal{P}(\mathbb{LA}_n(J)), \tag{7}$$

by

$$L(\vec{p}) \star_m L(\vec{q}) = \{L(\vec{u}) \mid L(\vec{p}) \circ_m L(\vec{q}) \leq_m L(\vec{u})\} \tag{8}$$

for all pairs $L(\vec{p}), L(\vec{q}) \in \mathbb{LA}_n(J)_m$, lets us state the following lemma.

Lemma 3. *Triads $(\mathbb{LA}_n(J), \star_m)$ are (noncommutative) hypergroups.*

Notation 1. *Hypergroups* $(\mathbb{LA}_n(J), \star_m)$ *will be denoted by* $\mathbb{HLA}_n(J)_m$ *for an easier distinction.*

Remark 1. *As a parallel to* (2) *and* (3) *we define*

$$\overline{L}(q_n, \ldots, q_0)y(x) = \sum_{k=0}^{n} q_k(x)y^{(k)}(x), q_0 \neq 0, q_k \in \mathbb{C}(J) \tag{9}$$

and

$$\overline{\mathbb{LA}}_n(J) = \{q_n, \ldots, q_0) \mid q_0 \neq 0, q_k(x) \in \mathbb{C}(J)\} \tag{10}$$

and, by defining the binary operation "$\circ_m$" and "$\leq_m$" in the same way as for $\mathbb{LA}_n(J)_m$*, it is easy to verify that also* $(\overline{\mathbb{LA}}_n(J), \circ_m, \leq_m)$ *are noncommutative partially ordered groups. Moreover, given a hyperoperation defined in a way parallel to* (8)*, we obtain hypergroups* $(\overline{\mathbb{LA}}_n(J)_m, \star_m)$*, which will be, in line with Notation 1, denoted* $\overline{\mathbb{HLA}}_n(J)_m$*.*

2.2. Results

In this subsection we will construct certain mappings between groups or hypergroups of linear differential operators of various orders. The result will have a form of sequences of groups or hypergroups.

Define mappings $F_n : \mathbb{LA}_n(J) \to \mathbb{LA}_{n-1}(J)$ by

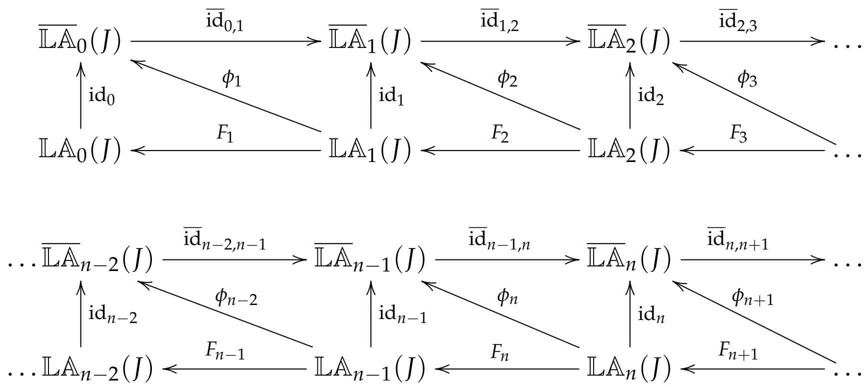
$$F_n(L(p_{n-1}, \ldots, p_0)) = L(p_{n-2}, \ldots, p_0)$$

and $\phi_n : \mathbb{LA}_n(J) \to \overline{\mathbb{LA}}_{n-1}(J)$ by

$$\phi_n : (L(p_{n-1}, \ldots, p_0)) = \overline{L}(p_{n-2}, \ldots, p_0).$$

It can be easily verify that both F_n and ϕ_n are, for an arbitrary $n \geq 2$, group homomorphisms.

Evidently, $\mathbb{LA}_n(J) \subset \overline{\mathbb{LA}}_n(J), \overline{\mathbb{LA}}_{n-1}(J) \subset \overline{\mathbb{LA}}_n(J)$ for all admissible $n \in \mathbb{N}$. Thus we obtain two complete sequences of ordinary linear differential operators with linking homomorphisms F_n and ϕ_n :



(11)

where $\overline{id}_{k,k+1}$, id_k are corresponding inclusion embeddings.

Notice that this diagram, presented at the level of groups, can be lifted to the level of hypergroups. In order to do this, one can use Lemma 3 and Remark 1. However, this is not enough. Yet, as Lemma 4 suggests, it is possible to show that the below presented assignment is functorial, i.e., not only objects are mapped onto objects but also morphisms (isotone group homomorphisms) are mapped onto morphisms (hypergroup homomorphisms). Notice that Lemma 4 was originally proved in [4]. However, given the minimal impact of the proceedings and its very limited availability and accessibility, we include it here with a complete proof.

Lemma 4. *Let $(G_k, \cdot_k, \leq_k), k = 1, 2$ be preordered groups and $f : (G_1, \cdot_1, \leq_1) \to (G_2, \cdot_2, \leq_2)$ a group homomorphism, which is isotone, i.e., the mapping $f : (G_1, \leq_1) \to (G_2, \leq_2)$ is order-preserving. Let $(H_k, *_k), k = 1, 2$ be hypergroups constructed from $(G_k, \cdot_k, \leq_k), k = 1, 2$ by Lemma 2, respectively. Then $f : (H_1, *_1) \to (H_2, *_2)$ is a homomorphism, i.e., $f(a *_1 b) \subseteq f(a) *_2 f(b)$ for any pair of elements $a, b \in H_1$.*

Proof. Let $a, b \in H_1$ be a pair of elements and $c \in f(a *_1 b)$ be an arbitrary element. Then there is $d \in a *_1 b = [a \cdot_1 b)_{\leq_1}$, i.e., $a \cdot_1 b \leq_1 d$ such that $c = f(d)$. Since the mapping f is an isotone homomorphism, we have $f(a) \cdot_2 f(b) = f(a \cdot_1 b) \leq f(d) = c$, thus $c \in [f(a) \cdot_2 f(b))_{\leq_2}$. Hence

$$f(a *_1 b) = f([a \cdot_1 b)_{\leq_1}) \subseteq [f(a) \cdot_2 f(b))_{\leq} = f(a) *_2 f(b).$$

□

Consider a sequence of partially ordered groups of linear differential operators

$$\mathbb{LA}_0(J) \xleftarrow{F_1} \mathbb{LA}_1(J) \xleftarrow{F_2} \mathbb{LA}_2(J) \xleftarrow{F_3} \dots$$
$$\dots \xleftarrow{F_{n-2}} \mathbb{LA}_{n-2}(J) \xleftarrow{F_{n-1}} \mathbb{LA}_{n-1}(J) \xleftarrow{F_n} \mathbb{LA}_n(J) \xleftarrow{F_{n+1}} \mathbb{LA}_{n+1}(J) \leftarrow \dots$$

given above with their linking group homomorphisms $F_k : \mathbb{LA}_k(J) \to \mathbb{LA}_{k-1}(J)$ for $k = 1, 2, \dots$. Since mappings $F_n : \mathbb{LA}_n(J) \to \mathbb{LA}_{n-1}(J)$, or rather

$$F_n : (\mathbb{LA}_n(J), \circ_m, \leq_m) \to (\mathbb{LA}_{n-1}(J), \circ_m, \leq_m),$$

for all $n \geq 2$, are group homomorphisms and obviously mappings isotone with respect to the corresponding orderings, we immediately get the following theorem.

Theorem 1. *Suppose $J \subseteq \mathbb{R}$ is an open interval, $n \in \mathbb{N}$ is an integer $n \geqq 2, m \in \mathbb{N}$ such that $m \leqq n$. Let $(\mathbb{HLA}_n(J)_m, *_m)$ be the hypergroup obtained from the group $(\mathbb{LA}_n(J)_m, \circ_m)$ by Lemma 2. Suppose that $F_n : (\mathbb{LA}_n(J)_m, \circ_m) \to (\mathbb{LA}_{n-1}(J)_m, \circ_m)$ are the above defined surjective group-homomorphisms, $n \in \mathbb{N}, n \geqq 2$. Then $F_n : (\mathbb{HLA}_n(J)_m, *_m) \to \mathbb{HLA}_{n-1}(J)_m, *_m)$ are surjective homomorphisms of hypergroups.*

Proof. See the reasoning preceding the theorem. □

Remark 2. *It is easy to see that the second sequence from (11) can be mapped onto the sequence of hypergroups*

$$\mathbb{HLA}_0(J)_m \xleftarrow{F_1} \mathbb{HLA}_1(J)_m \xleftarrow{F_2} \mathbb{HLA}_2(J)_m \xleftarrow{F_3} \dots$$
$$\dots \xleftarrow{F_{n-2}} \mathbb{HLA}_{n-1}(J)_m \xleftarrow{F_{n-1}} \mathbb{HLA}_n(J)_m \leftarrow \dots$$

This mapping is bijective and the linking mappings are surjective homomorphisms F_n. Thus this mapping is functorial.

3. Automata and Related Concepts

3.1. Notation and Preliminaries

The concept of an automaton is mathematical interpretation of diverse real-life systems that work on a discrete time-scale. Various types of automata, called also machines, are applied and used in numerous forms such as money changing devices, various calculating machines, computers, telephone switch boards, selectors or lift switchings and other technical objects. All the above mentioned devices have one aspect in common–states are switched from one to another based on outside influences (such as electrical or mechanical impulses), called inputs. Using the binary operation of concatenation of chains of input symbols one obtains automata with input alphabets in the form of semigroups or a groups. In the case of our paper we work with input sets in the form of hypercompositional structures. When focusing on the structure given by transition function and simultaneously neglecting the output functions and output sets, one reaches a generalization of automata–quasi-automata (or semiautomata); see classical works such as, e.g., [3,18,21–24].

To be more precise, a quasi-automaton is a system (A, S, δ) which consists of a non-void set A, an arbitrary semigroup S and a mapping $\delta : A \times S \to A$ such that

$$\delta(\delta(a, r, s)) = \delta(a, r, s) \tag{12}$$

for arbitrary $a \in A$ and $r, s \in S$. Notice that the concept of quasi-automaton has been introduced by S. Ginsberg as quasi-machine and was meant to be a generalization of the Mealy-type automaton. Condition (12) is sometimes called Mixed Associativity Condition (MAC). With most authors it is nameless, though.

For further reading on automata theory and its links to the theory of hypercompositional structures (also known as algebraic hyperstructures), see, e.g., [24–26]. Furthermore, for clarification and evolution of terminology, see [8]. For results obtained by means of quasi-multiautomata, see, e.g., [5–8,27].

Definition 1. *Let A be a nonempty set, $(H, \cdot)$ be a semihypergroup and $\delta : A \times H \to A$ a mapping satisfying the condition*

$$\delta(\delta(s, a), b) \in \delta(s, a \cdot b) \tag{13}$$

for any triad $(s, a, b) \in A \times H \times H$, where $\delta(s, a \cdot b) = \{\delta(s, x); x \in a \cdot b\}$. Then the triad (A, H, δ) is called quasi-multiautomaton with the state set A and the input semihypergroups $(H, \cdot)$. The mapping $\delta : A \times H \to A$ is called the transition function (or the next-state function) of the quasi-multiautomaton (A, H, δ). Condition (13) is called Generalized Mixed Associativity Condition (or GMAC).

In this section, $\mathbb{R}_n[x]$ means, as usually, the ring of polynomials of degree at most n.

3.2. Results

Now, consider linear differential operators $L(m, p_{n-1}, \dots, p_0) : \mathbb{C}^\infty(\mathbb{R}) \to \mathbb{C}^\infty(\mathbb{R})$ defined by

$$L(m, p_{n-1}, \dots, p_0)f = m\frac{\mathrm{d}^n f(x)}{\mathrm{d}x^n} + \sum_{k=0}^{n-1} p_k(x)\frac{\mathrm{d}^k f(x)}{\mathrm{d}x^k}. \tag{14}$$

Denote by $\mathbb{L}_{A1}\mathbb{A}_n(\mathbb{R})$ the additive abelian group of differential operators $L(m, p_{n-1}, \dots, p_0)$, where for $L(m, p_{n-1}, \dots, p_0), L(k, q_{n-1}, \dots, q_0) \in \mathbb{L}_{A1}\mathbb{A}_n(\mathbb{R})$ we define

$$L(m, p_{n-1}, \dots, p_0) + L(k, q_{n-1}, \dots, q_0) = L(m + k, p_{n-1} + q_{n-1}, \dots, p_0 + q_0), \tag{15}$$

where

$$L(m + k, p_{n-1} + q_{n-1}, \dots, p_0 + q_0)f = (m + k)\frac{\mathrm{d}^n f(x)}{\mathrm{d}x^n} + \sum_{k=0}^{n-1}(p_k(x) + q_k(x))\frac{\mathrm{d}^k f(x)}{\mathrm{d}x^k}. \tag{16}$$

Suppose that $p_k \in \mathbb{R}_{n-1}[x]$ and define

$$\delta_n : \mathrm{R}_n[x] \times \mathrm{L}_{A1}\mathbb{A}_n(\mathbb{R}) \to \mathbb{R}_n[x] \tag{17}$$

by

$$\delta_n(f, L(m, p_{n-1}, \dots, p_0)) = m\frac{\mathrm{d}^n f(x)}{\mathrm{d}x^n} + f(x) + m + \sum_{k=0}^{n-1} p_k(x), f \in \mathbb{R}_n[x]. \tag{18}$$

Theorem 2. *Let $\mathbb{L}_{A1}\mathbb{A}_n(\mathbb{R})$, $\mathbb{R}_n[x]$ be structures and $\delta_n : \mathbb{R}_n[x] \times \mathbb{L}_{A1}\mathbb{A}_n(\mathbb{R}) \to \mathbb{R}_n[x]$ the mapping defined above. Then the triad $(\mathbb{R}_n[x], \mathbb{L}_{A1}\mathbb{A}_n(\mathbb{R}), \delta_n)$ is a quasi-automaton, i.e., an action of the group $\mathbb{L}_{A1}\mathbb{A}_n(\mathbb{R})$ on the group $\mathbb{R}_n[x]$.*

Proof. We are going to verify the mixed associativity condition (MAC) which should satisfy the above defined action:

Suppose $f \in \mathbb{R}_n[x]$, $f(x) = \sum_{k=0}^{n} a_k x^k$, $L(m, p_{n-1}, \dots, p_0)$, $L(k, q_{n-1}, \dots, q_0) \in \mathbb{L}_{A1}\mathbb{A}_n(\mathbb{R})$. Then

$$\begin{aligned}
&\delta_n(\delta_n(f, L(m, p_{n-1}, \dots, p_0)), L(k, q_{n-1}, \dots, q_0)) = \\
&= \delta_n\left(m\frac{\mathrm{d}^n f(x)}{\mathrm{d}x^n} + f(x) + m + \sum_{k=0}^{n-1} p_k(x), L(k, q_{n-1}, \dots, q_0) \right) = \\
&= \delta_n\left(m \cdot n! \cdot a_n + m + f(x) + \sum_{k=0}^{n-1} p_k(x), L(k, q_{n-1}, \dots, q_0) \right) = \\
&= k\frac{\mathrm{d}^n f(x)}{\mathrm{d}x^n} + m \cdot n! \cdot a_n + m + f(x) + \sum_{k=0}^{n-1} p_k(x) + \sum_{k=0}^{n-1} q_k(x) + k = \\
&= (m+k)n! \cdot a_n + (m+k) + f(x) + \sum_{k=0}^{n-1} (p_k(x) + q_k(x)) = \\
&= (m+k)(n! \cdot a_n + 1) + f(x) + \sum_{k=0}^{n-1} (p_k(x) + q_k(x)) = \\
&= (m+k)\frac{\mathrm{d}^n f(x)}{\mathrm{d}x^n} + f(x) + (m+k) + \sum_{k=0}^{n-1} (p_k(x) + q_k(x)) = \\
&= \delta_n(f, L(m+k, p_{n-1} + q_{n-1}, \dots, p_0 + q_0)) = \\
&= \delta_n(f, L(m, p_{n-1}, \dots, p_0) + L(k, q_{n-1}, \dots, q_0)),
\end{aligned} \tag{19}$$

thus the mixed associativity condition is satisfied. □

Since $\mathbb{R}_n[x], \mathbb{L}_{A1}\mathbb{A}_n(\mathbb{R})$ are endowed with naturally defined orderings, Lemma 2 can be straightforwardly applied to construct semihypergroups from them.

Indeed, for a pair of polynomials $f, g \in \mathbb{R}_n[x]$ we put $f \le g$, whenever $f(x) \le g(x), z \in \mathbb{R}_n[x]$. In such a case $(\mathbb{R}_n[x], \le)$ is a partially ordered abelian group. Now we define a binary hyperoperation

$$\# : \mathbb{R}_n[x] \times \mathbb{R}_n[x] \to \mathcal{P}^\star(\mathbb{R}_n[x]) \tag{20}$$

by

$$f\#g = \{h; h \in \mathbb{R}_n[x], f(x) + g(x) \le h(x), x \in \mathbb{R}\} = [f+g)_\le. \tag{21}$$

By Lemma 2 we have that $(\mathbb{R}_n[x], \#)$ is a hypergroup.

Moreover, defining

$$\# : \mathbb{L}_{A1}\mathbb{A}_n(\mathbb{R}) \times \mathbb{L}_{A1}\mathbb{A}_n(\mathbb{R}) \to \mathcal{P}^\star(\mathbb{L}_{A1}\mathbb{A}_n(\mathbb{R})) \tag{22}$$

by $L(m, \overrightarrow{p(x)})\#L(k, \overrightarrow{q(x)}) = \left[L(m, \overrightarrow{p(x)}) + L(k, \overrightarrow{q(x)})\right)_\le = \left[L(m+k, \overrightarrow{p(x)} + \overrightarrow{q(x)}\right)_\le = \{L(r, \overrightarrow{u(x)}); m+k \le r, \overrightarrow{p(x)} + \overrightarrow{q(x)} \le \overrightarrow{u(x)}\}$, which means

$$p_j(x) + q_j(x) \le u_j(x),$$

where $j = 0, 1, \dots, n-1$, we obtain, again by Lemma 2 that the hypergroupoid $(\mathbb{L}_{A1}\mathbb{A}_n(\mathbb{R}), \#)$ is a commutative semihypergroup.

Finally, define a mapping

$$\sigma_n : \mathbb{L}_{A1}\mathbb{A}_n(\mathbb{R}) \times \mathbb{R}_n[x] \to \mathbb{R}_n[x] \tag{23}$$

by

$$\sigma_n(L(m, p_{n-1}, \dots, p_0, f)) = L(m, p_0 \circ f + p_{n-1}, \dots, p_0 \circ f + p_1, p_0). \tag{24}$$

Below, in the proof of Theorem 3, we show that the mapping satisfies the GMAC condition.

This allows us to construct a quasi-multiautomaton.

Theorem 3. *Suppose* $(\mathbb{L}_{A_1}\mathbb{A}_n(\mathbb{R}), \#), (\mathbb{R}_n[x], \#)$ *are hypergroups constructed above and* $\sigma_n : \mathbb{L}_{A1}\mathbb{A}_n(\mathbb{R}) \times \mathbb{R}_n[x] \to \mathbb{R}_n[x]$ *is the above defined mapping. Then the structure*

$$((\mathbb{L}_{A1}\mathbb{A}_n(\mathbb{R}), \#), (\mathbb{R}_n[x], \#), \sigma_n)$$

is a quasi-multiautomaton.

Proof. Suppose $L(m, \vec{p}) \in \mathbb{L}_{A1}\mathbb{A}_n(\mathbb{R}), f, g \in \mathbb{R}_n[x]$. Then

$$\begin{aligned}
&\sigma_n(\sigma_n(L(m,\vec{p}), f), g) = \sigma_n(L(m, p \circ f + p_{n-1}, \dots, p \circ f + p_1, p_0), g) = \\
&= L(m, p \circ g + p \circ f + p_{n-1}, \dots, p \circ g + p \circ f + p_1, p_0) = \\
&= L(m, p \circ (g + f) + p_{n-1}, \dots p \circ (g + f) + p_1, p_0) \in \\
&\in \{\sigma_n(L(m, p \circ h + p_{n-1}, \dots p \circ h + p_1, p_0); f, g, h \in \mathbb{R}_n[x], f + g \leq h\} = \\
&= \sigma_n(L, m, p_{n-1}, \dots, p_1, p_0), [f + g)_{\leq}) = \sigma_n(L(m, \vec{p}), f\#g),
\end{aligned} \tag{25}$$

hence the GMAC condition is satisfied. □

Now let us discuss actions on objects of different dimensions. Recall that a homomorphism of automaton (S, G, δ_S) into the automaton (T, H, δ_T) is a mapping $F = \phi \times \psi : S \times G \to T \times H$ such that $\phi : S \to T$ is a mapping and $\psi : G \to H$ is a homomorphism (of semigroups or groups) such that for any pair $[s, g] \in S \times G$ we have

$$\phi(\delta_S(s, g)) = \delta_T(\phi(s), \psi(g)), \text{ i.e., } \phi \circ \delta_S = \delta_T \circ (\phi \times \psi). \tag{26}$$

In order to define homomorphisms of our considered actions and especially in order to construct a sequence of quasi-automata with decreasing dimensions of the corresponding objects, we need a different construction of a quasiautomaton.

If $f \in \mathbb{R}_n[x], f(x) = \sum_{k=0}^{n} a_n x^k$ and $L(m, \vec{p}) \in \mathbb{L}_{A1}\mathbb{A}_n(\mathbb{R})$, we define

$$\tau_n(L(m, p_{n-1}, \dots, p_0), f) = L(m, a_n + p_{n-1}, \dots, a_1 + p_0 + a_0). \tag{27}$$

Now, if $g \in \mathbb{R}_n[x], g(x) = \sum_{k=0}^{n} b_k x^k$, we have

$$\begin{aligned}
&\tau_n(\tau_n(L(m, p_{n-1}, \dots, p_0), f), g) = \\
&= \tau_n(L(m, a_n + p_{n-1}, \dots, a_1 + p_1 + a_0), g) = \\
&= L(m, a_n + b_n + p_{n-1}, \dots, a_1 + b_1 + p_0 + a_0 + b_0) = \\
&= \tau_n\left(L(m, p_{n-1}, \dots, p_0), \sum_{k=0}^{n} (a_k + b_k)x\right) = \\
&= \tau_n(L(m, p_{n-1}, \dots, p_0), f + g).
\end{aligned} \tag{28}$$

Hence $\tau_n : \mathbb{L}_{A1}\mathbb{A}_n(\mathbb{R}) \times \mathbb{R}_n[x] \to \mathbb{L}_{A1}\mathbb{A}_n(\mathbb{R})$ is the transition function (satisfying MAC) of the automaton $\mathcal{A} = (\mathbb{L}_{A1}\mathbb{A}_n(\mathbb{R}), \mathbb{R}_{n-1}[x], \tau_n)$.

Consider now two automata–$\mathcal{A}_{n-1} = (\mathbb{L}_{A1}\mathbb{A}_{n-1}(\mathbb{R}), \mathbb{R}_{n-1}[x], \tau_{n-1})$ and the above one. Define mappings

$$\phi_n : \mathbb{L}_{A1}\mathbb{A}_n(\mathbb{R}) \to \mathbb{L}_{A1}\mathbb{A}_{n-1}(\mathbb{R}), \quad \psi_n : \mathbb{R}_n[x] \to \mathbb{R}_{n-1}[x] \tag{29}$$

in the following way: For $L(m, p_{n-1}, \dots, p_0) \in \mathbb{L}_{A1}\mathbb{A}_n(\mathbb{R})$ put

$$\phi_n(L(m, p_{n-1}, \dots, p_0)) = L(m, p_{n-2}, \dots, p_0) \in \mathbb{L}_{A1}\mathbb{A}_{n-1}(\mathbb{R}) \tag{30}$$

and for $f \in \mathbb{R}_n[x]$, $f(x) = \sum_{k=0}^{n} a_k x^k$ define

$$\psi_n(f) = \psi_n\left(\sum_{k=0}^{n} a_k x^k\right) = \sum_{k=0}^{n-1} a_k x^k \in \mathbb{R}_{n-1}[x]. \tag{31}$$

Evidently, there is $\psi_n(f+g) = \psi_n(f) + \psi(g)$ for any pair of polynomials $f, g \in \mathbb{R}_n[x]$.

Theorem 4. *Let $\phi_n : \mathbb{L}_{A1}\mathbb{A}_n(\mathbb{R}) \to \mathbb{L}_{A1}\mathbb{A}_{n-1}(\mathbb{R})$, $\psi_n : \mathbb{R}_n[x] \to \mathbb{R}_{n-1}[x]$, $\tau_n : \mathbb{L}_{A1}\mathbb{A}_n(\mathbb{R}) \times \mathbb{R}_n[x] \to \mathbb{L}_{A1}\mathbb{A}_n(\mathbb{R})$, $n \in \mathbb{N}, n \geqq 2$, be mappings defined above. Define $F_n : \mathcal{A}_n \to \mathcal{A}_{n-1}$ as mapping*

$$F_n = \phi_n \times \psi_n : \mathbb{L}_{A1}\mathbb{A}_n(\mathbb{R}) \times \mathbb{R}_n[x] \to \mathbb{L}_{A1}\mathbb{A}_{n-1}(\mathbb{R}) \times \mathbb{R}_{n-1}[x].$$

Then the following diagram

$$\begin{array}{ccc} \mathbb{L}_{A1}\mathbb{A}_n(\mathbb{R}) \times \mathbb{R}_n[x] & \xrightarrow{\tau_n} & \mathbb{L}_{A1}\mathbb{A}_n(\mathbb{R}) \\ \downarrow{\scriptstyle \phi_n \times \psi_n} & & \downarrow{\scriptstyle \phi_n} \\ \mathbb{L}_{A1}\mathbb{A}_{n-1}(\mathbb{R}) \times \mathbb{R}_{n-1}[x] & \xrightarrow{\tau_{n-1}} & \mathbb{L}_{A1}\mathbb{A}_{n-1}(\mathbb{R}) \end{array} \tag{32}$$

is commutative, thus the mapping $F_n = \phi_n \times \psi_n$ is a homomorphism of the automaton $\mathcal{A}_n = (\mathbb{L}_{A1}\mathbb{A}_n(\mathbb{R}), \mathbb{R}_n[x], \tau_n)$ into the automaton $\mathcal{A}_{n-1} = (\mathbb{L}_{A1}\mathbb{A}_{n-1}(\mathbb{R}), \mathbb{R}_{n-1}[x], \tau_{n-1})$.

Proof. Let $[L(m, \vec{p}), f] \in \mathbb{L}_{A1}\mathbb{A}_n(\mathbb{R}) \times \mathbb{R}_n[z]$, $f(x) = \sum_{k=0}^{n} a_k x^k$. Then

$$\begin{aligned} (\phi_n \circ \tau_n)(L(m, \vec{p}), f) &= \phi_n\left(\tau_n\left(L(m, p_{n-1}, \dots, p_0), \sum_{k=0}^{n} a_k x^k\right)\right) = \\ &= \phi_n((m, a_n + p_{n-1}, \dots, a_1 + p_0 + a_0)) = L(m, a_{n-1} + p_{n-2}, \dots, a_1 + p_0 + a_0) = \\ &= \tau_{n-1}\left(L(m, p_{n-2}, \dots, p_0), \sum_{k=0}^{n-1} a_k x^k\right) = \\ &= \tau_{n-1}\left((\phi_n \times \psi_n)\left(L(m, p_{n-1}, \dots, p_0), \sum_{k=0}^{n} a_k x^k\right)\right) = \\ &= (\tau_{n-1} \circ (\phi_n \times \phi_n)))(L(m, \vec{p}), f), \end{aligned} \tag{33}$$

Thus the diagram (32) is commutative. □

Using the above defined homomorphism of automata we obtain the sequence of automata with linking homomorphisms $F_k : \mathcal{A}_k \to \mathcal{A}_{n-1}$, $k \in \mathbb{N}$, $k \geq 2$:

$$\begin{aligned} \dots \xleftarrow{F_{n-1}} (\mathbb{L}_{A1}\mathbb{A}_{n-1}(\mathbb{R}), \mathbb{R}_{n-1}[x], \tau_{n-1}) \xleftarrow{F_n} (\mathbb{L}_{A1}\mathbb{A}_n(\mathbb{R}), \mathbb{R}_n[x], \tau_n) \\ (\mathbb{L}_{A1}\mathbb{A}_1(\mathbb{R}), \tau_1) \xleftarrow{F_2} (\mathbb{L}_{A1}\mathbb{A}_2(\mathbb{R}), \tau_2) \xleftarrow{F_3} \dots \xleftarrow{F_{n-2}} (\mathbb{L}_{A1}\mathbb{A}_{n-2}(\mathbb{R}), \mathbb{R}_{n-2}[x], \tau_{n-2}) \xleftarrow{F_{n-1}} \dots \end{aligned} \tag{34}$$

Here, for $L(m, P-1, P-0) \in \mathbb{L}_{A1}\mathbb{A}_2(\mathbb{R})$ we have

$$L(m, p_1, p_0)y(x) = m\frac{\mathrm{d}^2 y(x)}{\mathrm{d}x^2} + p_1(x)\frac{\mathrm{d}y(x)}{\mathrm{d}x} + p_0(x)y(x) \tag{35}$$

for any $y(x) \in \mathbb{C}^2(\mathbb{R})$ and any $L(m, p_0) \in \mathbb{L}_{A1}\mathbb{A}_1(\mathbb{R})$ it holds $L(m, p_0)y(x) = m\frac{\mathrm{d}y(x)}{\mathrm{d}x} + p_0(x)y(x)$.

The obtained sequence of automata can be transformed into a countable sequence of quasi-multiautomata. We already know that the transition function

$$\sigma_n : \mathbb{L}_{A1}\mathbb{A}_n(\mathbb{R}) \times \mathbb{R}_n[x] \to \mathbb{L}_{A1}\mathbb{A}_n(\mathbb{R})$$

satisfies *GMAC*. Further, suppose $f, g \in \mathbb{R}_n[x]$, $f(x) = \sum_{k=0}^{m} a_k x^k$, $g(x) = \sum_{k=0}^{m} b_k x^k$. Then

$$\begin{aligned}
\psi_n(f\#g) = \psi_n(\{h; h \in \mathbb{R}_n[x], f+g \leqq h\}) = \\
= \{\psi_n(h); h \in \mathbb{R}_n[x], f+g \leqq h\} = \\
= \{u; u \in \mathbb{R}_{n-1}[x], \psi_n(f) + \psi_n(g) \leqq u\} = \\
= \psi_n(f)\#\psi_n(g);
\end{aligned}$$

here $\operatorname{grad}\psi_n(h) < \operatorname{grad} h$ for any polynomial $h \in f\#g$. Thus the mapping $\psi_n : (\mathbb{R}_n[x], \#) \to (\mathbb{R}_{n-1}[x], \#)$ is a good homomorphism of corresponding hypergroups.

Now we are going to construct a sequence of automata with increasing dimensions, i.e., in a certain sense sequence "dual" to the previous sequence. First of all, we need a certain "reduction" member to the definition of a transition function

$$\lambda_n^* : \mathbb{L}_{A1}\mathbb{A}_n(\mathbb{R}) \times \mathbb{R}_n[x] \to \mathbb{L}_{A1}\mathbb{A}_n(\mathbb{R}), \tag{36}$$

namely $\mathrm{red}_{n-1} : \mathbb{R}_r[x] \to \mathbb{R}_{n-1}[x]$ whenever $r > n-1$. In detail, if $f(x) = \sum_{k-0}^{r} a_k x^k \in \mathbb{R}_r[x]$, then

$$\begin{aligned}
\mathrm{red}_{n-1}(f) = \mathrm{red}_{n-1}\left(\sum_{k=0}^{r} a_k x^k\right) = \mathrm{red}_{n-1}\left(\sum_{k=n}^{r} a_k x^k + \sum_{k=0}^{n-1} a_k x^{k-1}\right) = \\
= \sum_{k=0}^{n-1} a_k x^{k-1} \in \mathbb{R}_{n-1}[x].
\end{aligned} \tag{37}$$

Further, $L(m, p_{n-2}, \ldots, p_0)_n \in \mathbb{L}_{A1}\mathbb{A}_n(\mathbb{R})$ is acting by

$$L(m, p_{n-2}, \ldots, p_0)_n y(x) = m\frac{\mathrm{d}^n y(x)}{\mathrm{d}x^n} + \sum_{k=0}^{n-2} p_k(x)\frac{\mathrm{d}^k y(x)}{\mathrm{d}x^k} \tag{38}$$

whereas $L(m, p_{n-2}, \ldots, p_0)_{n-1} \in \mathbb{L}_{A1}\mathbb{A}_{n-1}(\mathbb{R})$, i.e.,

$$L(m, p_{n-2}, \ldots, p_0)_{n-1} y(x) = m\frac{\mathrm{d}^{n-1} y(x)}{\mathrm{d}x^{n-1}} + \sum_{k=0}^{n-2} p_k(x)\frac{\mathrm{d}^k y(x)}{\mathrm{d}x^k}. \tag{39}$$

Then for any pair $(L(m, p_{n-1}, \ldots, p_0), f) \in \mathbb{L}_{A1}\mathbb{A}_n(\mathbb{R}) \times \mathrm{R}_n[x]$, where $f(x) = \sum_{k=0}^{n} a_k x^k$, we obtain

$$\lambda_n = \lambda_n^{\star} \circ (\mathrm{id} \times \mathrm{red}_{n-1}) \tag{40}$$

and

$$(\lambda_n^\star \circ (\mathrm{id} \times \mathrm{red}_{n-1}))(L(m, p_{n-1}, \ldots, p_0, f) = \lambda_n^\star \left(L(m, p_{n-1}, \ldots, p_0), \sum_{k=0}^{n-1} a_k x^k \right) =$$
$$= L(m, p_{n-1} + a_{n-1}, \ldots, p_0 + a_0) \in \mathbb{L}_{A1}\mathbb{A}_n(\mathbb{R}), \tag{41}$$

thus the mapping $\lambda_n : \mathbb{L}_{A1}\mathbb{A}_n(\mathbb{R}) \times \mathbb{R}_n[x] \to \mathbb{L}_{A1}\mathbb{A}_n(\mathbb{R})$ is well defined. We should verify validity of MAC and commutativity of the square diagram determining a homomorphism between automata.

Suppose $L(m, p_{n-1}, \ldots, p_0) \in \mathbb{L}_{A1}\mathbb{A}_n(\mathbb{R})$, $f, g \in \mathbb{R}_n[x]$, $f(x) = \sum_{k=0}^{n} a_k x^k$, $g(x) = \sum_{k=0}^{n} b_k x^k$. Then

$$\begin{aligned}
&\lambda_n(\lambda_n(L(m, p_{n-1}, \ldots, p_0), f), g) = \\
&= \lambda_n(L(m, p_{n-1} + +a_{n-1}, \ldots, p_0 + a_0, g) = \\
&= L(m, p_{n-1} + a_{n-1} + b_{n-1}, \ldots, p_0 + a_0 + b_0) = \\
&= \lambda_n \left(L(m, p_{n-1}, \ldots, p_0), \sum_{k=0}^{n} (a_k + b_k) x^k \right) = \\
&= \lambda_n(L(m, p_{n-1}, \ldots, p_0), f + g),
\end{aligned} \tag{42}$$

thus MAC is satisfied.

Further, we are going to verify commutativity of this diagram

$$\begin{array}{ccc}
\mathbb{L}_{A1}\mathbb{A}_{n-1}(\mathbb{R}) \times \mathbb{R}_{n-1} & \xrightarrow{\lambda_{n-1}} & \mathbb{L}_{A1}\mathbb{A}_{n-1}(\mathbb{R}) \\
\downarrow{\scriptstyle \xi_{n-1} \times \eta_{n-1}} & & \downarrow{\scriptstyle \eta_{n-1}} \\
\mathbb{L}_{A1}\mathbb{A}_n(\mathbb{R}) \times \mathbb{R}_n[x] & \xrightarrow{\lambda_n} & \mathbb{L}_{A1}\mathbb{A}_n(\mathbb{R})
\end{array} \tag{43}$$

where $\xi_{n-1}(L(m, p_{n-2}, \ldots, p_0)_{n-1}) = L(m, p_{n-1}, \ldots, p_0)_n$, i.e.,

$$m \frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}} + p_{n-1}(x) \frac{\mathrm{d}^{n-2}}{\mathrm{d}x^{n-1}} + \cdots + p_0(x)\mathrm{id} \to m \frac{\mathrm{d}^n}{\mathrm{d}x^n} + p_{n-2}(x) \frac{\mathrm{d}^{n-2}}{\mathrm{d}x^{n-2}} + \cdots + p_0(x)\mathrm{id} \tag{44}$$

and $\eta_{n-1}\left(\sum_{k=0}^{n-1} a_k x^k\right) = a_{n-1}x^n + \sum_{k=0}^{n-1} a_k x^k$.

Considering $L(m, p_{n-2}, \ldots, p_0)$ and the polynomial $f(x) = \sum_{k=0}^{n-1} a_k x^k$ similarly as above, we have

$$\begin{aligned}
&(\eta_{n-1})(L(m, p_{n-2}, \ldots, p_0), f) = \eta_{n-1}(L(m, p_{n-2} + a_{n-2}, \ldots, p_0 + a_0)_{n-1}) = \\
&= \eta_{n-1}(\lambda_{n-1}(L(m, p_{n-2}, \ldots, p_0), f) = (\eta_{n-1} \circ \lambda_{n-1})(L(m, p_{n-2}, \ldots, p_0), f).
\end{aligned} \tag{45}$$

Thus the above diagram is commutative. Now, denoting by T_{n-1} the pair of mappings (ξ_{n-1}, η_{n-1}), we obtain that $T_{n-1} : (\mathbb{L}_{A1}\mathbb{A}_{n-1}(\mathbb{R}), \mathbb{R}_{n-1}[x], \lambda_{n-1}) \to (\mathbb{L}_{A1}\mathbb{A}_n(\mathbb{R}), \mathbb{R}_n[x], \lambda_n)$ is a homomorphism of the given automata. Finally, using T_k as connecting homomorphism, we obtain the sequence

$$\begin{aligned}
&(\mathbb{L}_{A1}\mathbb{A}_1(\mathbb{R}), \mathbb{R}_1[x], \lambda_1) \xrightarrow{T_1} (\mathbb{L}_{A1}\mathbb{A}_2(\mathbb{R}), \mathbb{R}_2[x], \lambda_2) \xrightarrow{T_2} (\mathbb{L}_{A1}\mathbb{A}_3(\mathbb{R}), \mathbb{R}_3[x], \lambda_3) \xrightarrow{T_3} \ldots \\
&\ldots \xrightarrow{T_{n-2}} (\mathbb{L}_{A1}\mathbb{A}_{n-1}(\mathbb{R}), \mathbb{R}_{n-1}[x], \lambda_{n-1}) \xrightarrow{T_{n-1}} (\mathbb{L}_{A1}\mathbb{A}_n(\mathbb{R}), \mathbb{R}_n[x], \lambda_n) \xrightarrow{T_n} \ldots
\end{aligned} \tag{46}$$

4. Practical Applications of the Sequences

In this section, we will include several examples of the above reasoning. We will apply the theoretical results in the area of artificial neurons, i.e., in a way, continue with the paper [11] which focuses on artificial neurons. For notation, recall [11]. Further on we consider a generalization of the usual concept of artificial neurons. We assume that the inputs ux_i and weight w_i are functions of an argument t, which belongs into a linearly ordered (tempus) set T with the least element 0. The index set is, in our case, the set $C(J)$ of all continuous functions defined on an open interval $J \subset \mathbb{R}$. Now, denote by W the set of all non-negative functions $w : T \to \mathbb{R}$. Obviously W is a subsemiring of the ring of all real functions of one real variable $x : \mathbb{R} \to \mathbb{R}$. Further, denote by $Ne(\vec{w}_r) = Ne(w_{r1}, \ldots, w_{rn})$ for $r \in C(J)$, $n \in \mathbb{N}$ the mapping

$$y_r(t) = \sum_{k=1}^{n} w_{r,k}(t)x_{r,k}(t) + b_r$$

which will be called the artificial neuron with the bias $b_r \in \mathbb{R}$. By $\mathbb{AN}(T)$ we denote the collection of all such artificial neurons.

4.1. Cascades of Neurons Determined by Right Translations

Similarly as in the group of linear differential operators we will define a binary operation in the system $\mathbb{AN}(T)$ of artificial neurons $Ne(\cdot)$ and construct a non-commutative group.

Suppose $Ne(\vec{w}_r)$, $Ne(\vec{w}_s) \in \mathbb{AN}(T)$ such that $r, s \in C(J)$ and $\vec{w}_r = (w_{r,1}, \ldots, w_{r,n})$, $\vec{w}_s = (w_{s,1}, \ldots, w_{s,n})$, where $n \in \mathbb{N}$. Let $m \in \mathbb{N}$, $1 \leq m \leq n$ be a such an integer that $w_{r,m} > 0$. We define

$$Ne(\vec{w}_r) \cdot_m Ne(\vec{w}_s) = Ne(\vec{w}_u),$$

where

$$\vec{w}_u = (w_{u,1}, \ldots, w_{u,n}) = (w_{u,1}(t), \ldots, w_{u,n}(t)),$$
$$\vec{w}_{u,k}(t) = w_{r,m}(t)w_{s,k}(t) + (1 - \delta_{m,k})w_{r,k}(t), t \in T$$

and, of course, the neuron $Ne(\vec{w}_u)$ is defined as the mapping $y_u(t) = \sum_{k=1}^{n} w_k(t)x_k(t) + b_u$, $t \in T$, $b_u = b_r b_s$.

The algebraic structure $(\mathbb{AN}(T), \cdot_m)$ is a non-commutative group. We proceed to the construction of the cascade of neurons. Let $(\mathbb{Z}, +)$ be the additive group of all integers. Let $Ne(\vec{w}_s(t)) \in \mathbb{AN}(T)$ be an arbitrary but fixed chosen artificial neuron with the output function

$$y_s(t) = \sum_{k=1}^{n} w_{s,k}(t)x_{s,k}(t) + b_s.$$

Denote by $\rho_s : \mathbb{AN}(T) \to \mathbb{AN}(T)$ the right translation within the group of time varying neurons determined by $Ne(\vec{w}_s(t))$, i.e.,

$$\rho_s(Ne(\vec{w}_p(t)) = Ne(\vec{w}_p(t) \cdot_m Ne(\vec{w}_s(t))$$

for any neuron $Ne(\vec{w}_p(t)) \in \mathbb{AN}(T)$. In what follows, denote by ρ_s^r the r-th iteration of ρ_s for $r \in \mathbb{Z}$. Define the projection $\pi_s : \mathbb{AN}(T) \times \mathbb{Z} \to \mathbb{AN}(T)$ by

$$\pi_s(Ne(\vec{w}_p(t)), r) = \rho_s^r(Ne(\vec{w}_p(t)).$$

One easily observes that we get a usual (discrete) transformation group, i.e., the action of $(\mathbb{Z}, +)$ (as the phase group) on the group $\mathbb{AN}(T)$. Thus the following two requirements are satisfied:

1. $\pi_s(Ne(\vec{w}_p(t)), 0) = Ne(\vec{w}_p(t))$ for any neuron $Ne(\vec{w}_p(t)) \in \mathbb{AN}(T)$.
2. $\pi_s(Ne(\vec{w}_p(t)), r + u) = \pi_s(\pi_s(Ne(\vec{w}_p(t)), r), u)$ for any integers $r, u \in \mathbb{Z}$ and any artificial neuron $Ne(\vec{w}_p(t))$. Notice that the just obtained structure is called a cascade within the framework of the dynamical system theory.

4.2. An Additive Group of Differential Neurons

As usually denote by $\mathbb{R}_n[t]$ the ring of polynomials of variable t over $\mathbb{R}$ of the grade at most $n \in \mathbb{N}_0$. Suppose $\vec{w} = (w_1(t), \ldots, w_n(t))$ be the fixed vector of continuous functions

$w_k : \mathbb{R} \to \mathbb{R}$, b_p be the bias for any polynomial $p \in \mathbb{R}_n[t]$. For any such polynomial $p \in \mathbb{R}_n[t]$ we define a differential neuron $DNe(\vec{w})$ given by the action

$$y(t) = \sum_{k=1}^{n} w_k(t) \frac{d^{k-1}p(t)}{dt^{k-1}} + b_0 \frac{d^n p(t)}{dt^n}. \tag{47}$$

Considering the additive group of $\mathbb{R}_n[t]$ we obtain an additive commutative group $\mathbb{DN}(T)$of differential neurons which is assigned to the group of $\mathbb{R}_n[t]$. Thus for $DNe_p(\vec{w})$, $DNe_q(\vec{w}) \in \mathbb{DN}(T)$ with actions

$$y(t) = \sum_{k=1}^{n} w_k(t) \frac{d^{k-1}p(t)}{dt^{k-1}} + b_0 \frac{d^n p(t)}{dt^n}$$

and

$$z(t) = \sum_{k=1}^{n} w_k(t) \frac{d^{k-1}q(t)}{dt^{k-1}} + b_0 \frac{d^n q(t)}{dt^n}$$

we have $DNe_{p+q}(\vec{w}) = DNe_p(\vec{w}) + DNe_q(\vec{w}) \in \mathbb{DN}(T)$ with the action

$$u(t) = y(t) + z(t) = \sum_{k=1}^{n} w_k(t) \frac{d^{k-1}(p(t)+q(t))}{dt^{k-1}} + b_0 \frac{d^n (p(t)+q(t))}{dt^n}$$

Considering the chain of inclusions

$$\mathbb{R}_n[t] \subset \mathbb{R}_{n+1}[t] \subset \mathbb{R}_{n+2}[t] \ldots$$

we obtain the corresponding sequence of commutative groups of differential neurons.

4.3. A Cyclic Subgroup of the Group $\mathbb{AN}(T)_m$ Generated by Neuron $Ne(\vec{w}_r) \in \mathbb{AN}(T)_m$

First of all recall that if $Ne(\vec{w}_r)$, $Ne(\vec{w}_s) \in \mathbb{AN}(T)_m$, $r, s \in C(J)$, where $\vec{w}_r(t) = (w_{r,1}(t), \ldots, w_{r,n}(t))$, $\vec{w}_s(t) = (w_{s,1}(t), \ldots, w_{s,n}(t))$, are vector function of weights such that $w_{r,m}(t) \neq 0 \neq w_{s,m}(t)$, $t \in T$ with outputs $y_r(t) = \sum_{k=1}^{n} w_{r,k}(t)x_k(t) + b_r$, $y_s(t) = \sum_{k=1}^{n} w_{s,k}(t)x_k(t) + b_s$ (with inputs $x_k(t)$), then the product $Ne(\vec{w}_r) \cdot_m Ne(\vec{w}_s) = Ne(\vec{w}_u)$ has the vector of weights

$$\vec{w}_u(t) = (w_{u,1}(t), \ldots, w_{u,n}(t))$$

$$\text{with } w_{u,k}(t) = w_{r,m}(t)w_{s,k}(t) + (1 - \delta_{m,k})w_{r,k}(t), t \in T.$$

The binary operation "$\cdot_m$" is defined under the assumption that all values of functions which are m-th components of corresponding vectors of weights are different from zero.

Let us denote by $\mathbb{ZAN}_r(T)$ the cyclic subgroup of the group $\mathbb{AN}(T)_m$ generated by the neuron $Ne(\vec{w}_r) \in \mathbb{AN}(T)_m$. Then denoting the neutral element by $N1(\vec{e})_m$ we have $\mathbb{ZAN}_r(T) =$

$$= \{\ldots, [Ne(\vec{w}_r)]^{-2}, [Ne(\vec{w}_r)]^{-1}, N1(\vec{e})_m, Ne(\vec{w}_r), [Ne(\vec{w}_r)]^2, \ldots, [Ne(\vec{w}_r)]^p, \ldots\}.$$

Now we describe in detail objects

$$[Ne(\vec{w}_r)]^2, [Ne(\vec{w}_r)]^p, p \in \mathbb{N}, p \geq 2, N1(\vec{e})_m \text{ and } [Ne(\vec{w}_r)]^{-1}, \tag{48}$$

i.e., the inverse element to the neuron $Ne(\vec{w}_r)$.

Let us denote $[Ne(\vec{w}_r)]^2 = Ne(\vec{w}_s)$, with $\vec{w}_s(t) = (w_{s,1}(t), \ldots, w_{s,n}(t))$. then

$$w_{s,k}(t) = w_{r,m}(t)w_{r,k}(t) + (1 - \delta_{m,k})w_{r,k}(t) = (w_{r,m}(t) + 1 - \delta_{m,k})w_{r,k}(t).$$

Then the vector of weights of the neuron $[Ne(\vec{w}_r)]^2$ is

$$\vec{w}_s(t) = ((w_{r,m}(t)+1)w_{r,1}(t), \ldots, w_{r,m}^2(t), \ldots, (w_{r,m}(t)+1)w_{r,n}(t)),$$

the output function is of the form

$$y_s(t) = \sum_{\substack{k=1 \\ k\neq m}}^{n} (w_{r,m}(t)+1)w_{r,k}(t)x_k(t)) + w_{r,m}^2 x_n(t) + b_r^2.$$

It is easy to calculate the vector of weights of the neuron $[Ne(\vec{w}_r)]^3$:

$$((w_{r,m}^2(t)+1)w_{r,1}(t), \ldots, w_{r,m}^3(t), \ldots, (w_{r,m}^2(t)+1)w_{r,n}(t)).$$

Finally, putting $[Ne(\vec{w}_r)]^p = Ne(\vec{w}_v)$ for $p \in \mathbb{N}$, $p \geq 2$, the vector of weights of this neuron is

$$\vec{w}_v(t) = ((w_{r,m}^{p-1}(t)+1)w_{r,1}(t), \ldots, w_{r,m}^p(t), \ldots, (w_{r,m}^{p-1}(t)+1)w_{r,n}(t)).$$

Now, consider the neutral element (the unit) $N1(\vec{e})_m$ of the cyclic group $\mathbb{ZAN}_r(T)$. Here the vector $\vec{e}$ of weights is $\vec{e} = (e_1, \ldots, e_m, \ldots, e_n)$, where $e_m = 1$ and $e_k = 0$ for each $k \neq m$. Moreover the bias $b = 1$.

We calculate products $Ne(\vec{w}_s) \cdot N1(\vec{e})_m$, $N1(\vec{e})_m) \cdot Ne(\vec{w}_s)$. Denote $Ne(\vec{w}_u)$, $Ne(\vec{w}_v)$ results of corresponding products, respectively–we have $\vec{w}_u(t) = (w_{u,1}(t), \ldots, w_{u,n}(t))$, where

$$w_{u,k}(t) = w_{s,m}(t)e_k(t) + (1-\delta_{m,k})w_{s,k}(t) = w_{s,k}(t)$$

if $k \neq m$ and $w_{u,k}(t) = w_{s,m}(t)(e_m(t) + 0 \cdot w_{s,m}(t)) = w_{s,m}(t)$ for $k = m$. Since the bias is $b = 1$, we obtain $y_u(t) = x_m(t) + 1$. Thus $Ne(\vec{w}_u) = Ne(\vec{w}_s)$. Similarly, denoting $\vec{w}_v(t) = (w_{v,1}(t), \ldots, w_{v,n}(t))$, we obtain $w_{v,k}(t) = e_m(t)w_{s,k}(t) + (1-\delta_{m,k})e_k(t) = w_{s,k}(t)$ for $k \neq m$ and $w_{v,k}(t) = w_{s,m}(t)$ if $k = m$, thus $\vec{w}_v(t) = (w_{s,1}(t), \ldots, w_{s,n}(t))$, consequently $Ne(\vec{w}_u) = Ne(\vec{w}_s)$ again.

Consider the inverse element $[Ne(\vec{w}_r)]^{-1}$ to the element $Ne(\vec{w}_r \in \mathbb{ZAN}(T)_m$. Denote $Ne(\vec{w}_s) = [Ne(\vec{w}_r)]^{-1}$, $\vec{w}_s(t) = (w_{s,1}(t), \ldots, w_{s,n}(t))$, $t \in T$. We have $Ne(\vec{w}_r) \cdot Ne(\vec{w}_s) = Ne(\vec{w}_r) \cdot_m [Ne(\vec{w}_r)]^{-1} = N1(\vec{e})_m$. Then

$$\begin{aligned}
0 = e_1 &= w_{r,m}(t)w_{s,1}(t) + w_{r,1}(t), \\
0 = e_2 &= w_{r,m}(t)w_{s,2}(t) + w_{r,2}(t), \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
1 = e_m &= w_{r,m}(t)w_{s,m}(t), \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
0 = e_n &= w_{r,m}(t)w_{s,n}(t) + w_{r,n}(t).
\end{aligned}$$

From the above equalities we obtain

$$w_{s,1}(t) = -\frac{w_{r,1}(t)}{w_{r,m}(t)}, \ldots,, w_{s,m}(t) = \frac{1}{w_{r,m}(t)}, \ldots, w_{s,n}(t) = -\frac{w_{r,n}(t)}{w_{r,m}(t)}.$$

Hence, for $[Ne(\vec{w}_r)]^{-1} = Ne(\vec{w}_s)$, we get

$$\vec{w}_s(t) = \left(-\frac{w_{r,1}(t)}{w_{r,m}(t)}, \ldots, \frac{1}{w_{r,m}(t)}, \ldots, -\frac{w_{r,n}(t)}{w_{r,m}(t)}\right) =$$

$$= \frac{1}{w_{r,m}(t)}(-w_{r,1}(t), \ldots, 1, \ldots, -w_{r,n}(t)),$$

where the number 1 is on the m-th position. Of course, the bias of the neuron $[Ne(\vec{w}_r)]^{-1}$ is b_r^{-1}, where b_r is the bias of the neuron $Ne(\vec{w}_r)$.

5. Conclusions

The scientific school of O. Borůvka and F. Neuman used, in their study of ordinary differencial equations and their transformations [1,28–30], the algebraic approach with the group theory as a main tool. In our study, we extended this existing theory with the employment of hypercomposiional structures—semihypergroups and hypergroups. We constructed hypergroups of ordinary linear differential operators and certain sequences of such structures. This served as a background to investigate systems of artificial neurons and neural networks.

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