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The Functional Equation $\max\{\chi(xy), \chi(xy^{-1})\} = \chi(x)\chi(y)$ on Groups and Related Results

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Abstract: This research paper focuses on the investigation of the solutions $\chi\colon G \to \mathbb{R}$ of the maximum functional equation $\max\{\chi(xy), \chi(xy^{-1})\} = \chi(x)\chi(y)$, for every $x, y \in G$, where G is any group. We determine that if a group G is divisible by two and three, then every non-zero solution is necessarily strictly positive; by the work of Toborg, we can then conclude that the solutions are exactly the $e^{|\alpha|}$ for an additive function $\alpha\colon G \to \mathbb{R}$. Moreover, our investigation yields reliable solutions to a functional equation on any group G, instead of being divisible by two and three. We also prove the existence of normal subgroups Z_χ and N_χ of any group G that satisfy some properties, and any solution can be interpreted as a function on the abelian factor group G/N_χ.

Keywords: additive function; normal subgroup; strictly positive solution; commutators; maximum functional equation

MSC: 20D60; 54E35; 11T71

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1. Introduction

By the maximum functional equation on the group $(G, +)$, we mean the functional equation

$$\max\{\chi(x+y), \chi(x-y)\} = \chi(x) + \chi(y), \qquad x, y \in G, \tag{1}$$

where $\chi\colon G \to \mathbb{R}$ is an unknown function to be determined.

The solutions of the functional Equation (1) for real-valued functions on an abelian group G have been given by Volkmann and Simon [1]; they demonstrated that χ satisfies Equation (1) if and only if there is an additive function $\alpha\colon G \to \mathbb{R}$ such that $\chi(g) = |\alpha(g)|$ for any $g \in G$. It has been shown that the functional Equation (1) characterizes the functions of the form $\chi(g) = |\alpha(g)|$. Subsequently, an interesting application of their result and a new proof were obtained by Kochanek [2]. In [3], Tabor generalized the functional Equation (1) for real and complex values.

In [4], Fechner dealt with four functional inequalities, which were derived from the results of [1] concerning the functional Equation (1) and also properties of the inequalities compared with some classical functional inequalities, such as the inequality of the Jensen quasi-convexity and the inequality of subadditivity.

Moreover, by rewriting the functional Equation (1) in the form

$$\max\{\chi((x \circ y) \circ y), \chi(x)\} = \chi(x \circ y) + \chi(y), \qquad x, y \in G, \tag{2}$$

Gilanyi et al. [5] determined the stability results of the functional Equation (2) on a square-symmetric groupoid. As a consequence, they also analyzed the stability of the functional Equation (1) on an abelian group. In addition, in [6] a Pexider version of the functional Equation (1) has been investigated. Later, their results were generalized by

Badora et al. [7], who demonstrated the stability of the functional equation on a certain class of groupoids.

In [8], Jarczyck and Volkmann investigated the stability of Equation (1) on an abelian group and also presented a characterization of the functional Equation (1), which is formulated in the following theorem as follows:

Theorem 1 (see [8])**.** *Let G be an abelian group. Then, a function* $\chi\colon G \to \mathbb{R}$ *satisfies Equation* (1) *if and only if* $\min\{\,\chi(x+y), \chi(x-y)\,\} = |\chi(x) - \chi(y)|$ *and* $\chi(2x) = 2\chi(x)$ *for all* $x, y \in G$.

The most exhaustive study of the functional Equation (1) on groups has been accomplished by Volkmann [9] and Toborg [10]. Volkmann generalized the functional Equation (1) in the form of

$$\max\{\,\chi(xy), \chi(xy^{-1})\,\} = \chi(x) + \chi(y), \qquad x, y \in G, \tag{3}$$

with the additional assumption that χ satisfies the Kannappan condition [11], that is, $\chi(xyu) = \chi(xuy)$ for all $u, x, y \in G$, and showed that it characterizes the absolute value of additive functions (see the theorem below). Recently, Toborg [10] gave the characterization of such functions in terms of (3) without additional assumptions (χ fulfills Kannappan condition and G is an abelian group). Their main theorem is stated as follows:

Theorem 2 (see [10] for the general case and [9] for the special case of an abelian group G)**.** *Let G be any group; then a function* $\chi\colon G \to \mathbb{R}$ *satisfies Equation* (3) *if and only if there exists an additive function* $\alpha\colon G \to \mathbb{R}$ *such that* $\chi(g) = |\alpha(g)|$ *for any* $g \in G$.

Additionally, the characterization of the generalized functional Equation (3) and stability results can be found in [12]. In [1], Simon and Volkmann investigated the functional equation

$$\max\{\,\chi(x+y), \chi(x-y)\,\} = \chi(x)\chi(y), \qquad x, y \in G, \tag{4}$$

for abelian groups. The Equations (1) and (4) have complex analogues: Let V be a complex vector space and assume that $A = \{\alpha \mid \alpha \in \mathbb{C}, |\alpha| = 1\}$ is the unit circle in $\mathbb{C}$, and for functions $\chi\colon V \to \mathbb{R}$, consider the equations

$$\sup_{\alpha \in A} \chi(x + \alpha y) = \chi(x) + \chi(y) \tag{5}$$

$$\sup_{\alpha \in A} \chi(x + \alpha y) = \chi(x)\chi(y) \tag{6}$$

for every $x, y \in V$. In [13] it has been proven that the solutions of the functional Equation (5) are given by $\chi(x) = |\eta(x)|$, $x \in V$ where $\eta\colon V \to \mathbb{C}$ is a linear functional, and by Przebieracz [14] it was proven that the non-identically vanishing solutions of Equation (6) are of the form $\chi(x) = e^{|\eta(x)|}$, $x \in V$ where $\eta\colon V \to \mathbb{C}$ is also a linear functional; and in [14], Przebieracz proved the general theorem about the superstability of Equation (6).

Baron with Volkmann [13] considered a characterization of the absolute value of complex determinants, which was similar to the results of Volkmann [15], who analyzed the real case using the functional Equation (1). Readers are encouraged to refer to [14,16] and the references cited therein to obtain useful results regarding the functional Equation (4).

The functional Equation (4) in a generalized form exhibited in equation

$$\max\{\,\chi(xy), \chi(xy^{-1})\,\} = \chi(x)\chi(y), \qquad x, y \in G, \tag{7}$$

is the focus of this research. No additional assumptions (G is an abelian group) are required, and thus, we derive results about solutions of the functional Equation (7) that are valid for any group rather than only in the special case of an abelian group G.

Additionally, to determine the solutions of the functional Equation (7) on a group G divisible by two and three, our objective is to implement the generalized result of the functional Equation (3) provided by Toborg [10] concerning the characterization of the modulus of an additive function. We will also present the existence of a certain normal subgroup N_χ of G when a non-zero solution χ is unitary. Finally, we will present some results whenever the solutions of Equation (7) are strictly positive on an arbitrary group G.

2. Main Results

Throughout this article, 1 is considered to be the identity element of a group G, and G is an arbitrary group and divisible by two and three.

Lemma 1. *Let G be any group and $\chi\colon G \to \mathbb{R}$ is a strictly positive solution of the functional Equation (7). Then there is an additive function $\alpha\colon G \to \mathbb{R}$ such that $\chi(x) = e^{|\alpha(x)|}$ for all $x \in G$.*

Proof. Since χ is a strictly positive solution of the functional Equation (7), we can compute

$$\log \chi(x) + \log \chi(y) = \max\{\, \log \chi(xy), \log \chi(xy^{-1}) \,\}. \tag{8}$$

Let $\eta(x) = \log \chi(x)$; then η satisfies Equation (8). By applying the main theorem of Toborg [10], for any $x \in G$, there exists an additive function $\alpha\colon G \to \mathbb{R}$ such that $\eta(x) = |\alpha(x)|$. Hence, by utilizing (8), for any $x, y \in G$, we can conclude that

$$\max\{\, \chi(xy), \chi(xy^{-1}) \,\} = \chi(x)\chi(y)$$

if and only if there exists an additive function $\alpha\colon G \to \mathbb{R}$ such that $\chi(x) = e^{|\alpha(x)|}$ for any $x \in G$. □

One of the main theorems of this article is stated as follows:

Theorem 3. *Assume that each element of a group G is divisible by two and three; then a function $\chi\colon G \to \mathbb{R}$ fulfills the functional Equation (7) if and only if χ vanishes identically on a group G or there exists an additive function $\alpha\colon G \to \mathbb{R}$ such that $\chi(x) = e^{|\alpha(x)|}$ for any $x \in G$.*

Proof. According to Lemma 1, we only have to prove that a nonzero solution χ is strictly positive.

When setting $x = y = 1$ in Equation (7), we have $\chi(1)\chi(1) = \max\{\, \chi(1), \chi(1) \,\}$, so $\chi(1)\chi(1) = \chi(1)$; therefore, we can evaluate that either $\chi(1) = 1$ or $\chi(1) = 0$.

Suppose $y = 1$; then $\chi(x)\chi(1) = \max\{\, \chi(x), \chi(x) \,\}$; therefore, $\chi(x) = \chi(x)\chi(1)$. Since $\chi(1) = 0$ implies $\chi(x) = 0$ for any $x \in G$, we may assume that $\chi(1) = 1$; then, $\chi(x) \neq 0$. Thus, we attempt to show that $\chi(x) > 0$ for all $x \in G$. Assume that $x, y \in G$, such that $y = x$; therefore, Equation (7) yields that

$$\chi(x)\chi(x) = \max\{\, \chi(x^2), \chi(1) \,\}$$

$$\chi(x)\chi(x) = \max\{\, \chi(x^2), 1 \,\}. \tag{9}$$

$$\chi(x)\chi(x^2) = \max\{\, \chi(x^3), \chi(x) \,\}. \tag{10}$$

$$\chi(x)\chi(x^3) = \max\{\, \chi(x^4), \chi(x^2) \,\}. \tag{11}$$

By (9), if $\chi(x^2) < 0$, then $\chi(x)\chi(x) = 1$, so we can evaluate

$$\chi(x) = \pm 1. \tag{12}$$

By (10), if $\chi(x)\chi(x^2) = \chi(x)$, then it is not possible. Thus, $\chi(x^3) > \chi(x)$ and (10) implies that

$$\chi(x^3) = \chi(x)\chi(x^2) > \chi(x). \tag{13}$$

Therefore, (13) is not possible when $\chi(x) > 0$ and $\chi(x^2) < 0$. Hence

$$\chi(x) = -1. \tag{14}$$

Then, by (13), $\chi(x^3) = (-1)\chi(x^2) > 0$; therefore,

$$\chi(x^3) > 0. \tag{15}$$

From (11), (14) and (15), we can obtain $-\chi(x^3) = \max\{\,\chi(x^4), \chi(x^2)\,\}$; then, again, from (15), we have $\max\{\,\chi(x^4), \chi(x^2)\,\} = \chi(x^2)$; therefore, $\chi(x^4) \leq \chi(x^2)$, so (11) implies that $\chi(x^4) \leq \chi(x^2)$. Then,

$$\chi(x^4) \leq \chi(x)\chi(x^3) = -\chi(x^3) < 0,$$

so $\chi(x^4) < 0$. Hence, we conclude that

$$If \;\; \chi(x^2) < 0 \;\; then \;\; \chi(x) = -1, \;\; \chi(x^3) > 0 \;\; and \;\; \chi(x^4) < 0. \tag{16}$$

We also prove that

(i) $\chi(x) < 0$ implies $\chi(x^2) < 0$.
(ii) $\chi(x) > 0$ implies $\chi(x^2) > 0$.

Suppose that $\chi(x) < 0$ and G is divisible by two; then there exists an element $y \in G$ such that $x = y^2$; therefore, $\chi(y^2) = \chi(x) < 0$, so $\chi(y^2) < 0$. From (16), we have $\chi(x^2) = \chi(y^4) < 0$, then $\chi(x^2) < 0$.

Furthermore, let $\chi(x) > 0$. Suppose on the contrary that $\chi(x^2) < 0$; then by (16), $\chi(x) = -1 < 0$, which is a contradiction, so $\chi(x) > 0$ implies that $\chi(x^2) > 0$.

Additionally, to obtain the required result, we need to prove that the solution $\chi(x^3) > 0$ for any $x \in G$. If $\chi(x) < 0$, then from (i), we get $\chi(x^2) < 0$, so by (16), we obtain that $\chi(x^3) > 0$. Let $\chi(x) > 0$; then from (ii), we have $\chi(x^2) > 0$. Utilizing (11), we obtain that $\chi(x)\chi(x^3) = \max\{\,\chi(x^4), \chi(x^2)\,\}$; therefore, in either case $\chi(x^3) > 0$. Hence, combining both cases we get that $\chi(x^3) > 0$ for all $x \in G$.

Assume that G is divisible by two; then there exists an element $y \in G$ such that $x = y^3$; then $\chi(x) = \chi(y^3) > 0$. Therefore, the nonzero solution χ is strictly positive. Then by Lemma 1, the proof is completed. □

The following corollary is a direct consequence of Theorem 3, so we omit the proof.

Corollary 1. *Assume that χ is a strictly positive solution of the functional Equation (7) on an arbitrary group G; then following hold:*

(1) $\chi(x^n) = [\chi(x)]^n$, for all $n \in \mathbb{N}$, $x \in G$.
(2) $|\chi(x)| \geq 1$ for any $x \in G$.
(3) For an infinitely divisible group G, $\chi(x^q) = q\chi(x)$ for any rational number $q > 0$.

Example 1. *Assume that $G = \mathbb{Z}_{6n-1}$ or $G = \mathbb{Z}_{6n+1}$ for $n \in \mathbb{N}$, then it can be perceived that each element of group G is divisible by two and three. Accordingly, by employing Theorem 3, we can obtain that $\chi(x) > 0$ for every $x \in G$, and therefore, a nonzero function $\chi\colon G \to \mathbb{R}$ satisfies Equation (7) if and only if there exists an additive function $\alpha\colon G \to \mathbb{R}$ such that $\chi(x) = e^{|\alpha(x)|}$ for any $x \in G$.*

Example 2. *Let $G = \mathbb{Z}_2$; then each element of group G is divisible by three. Assume that $\chi(x) = 1$ if $x = 0$ and $\chi(x) = -1$ if $x \neq 0$; then χ satisfies Equation (7) but $\chi(x) \neq e^{|\alpha(x)|}$ for $x \neq 0$.*

On the other hand, let $G = \mathbb{Z}_3$; then each element of G is divisible by two. Let $\chi(x) = 1$ if $x = 0$ and $\chi(x) = -1$ if $x \neq 0$; then χ also satisfies Equation (7) but $\chi(x) = e^{|\alpha(x)|}$ does not hold for all $x \in G$.

Let us consider G to be any group instead of divisible by two and three. Functional Equation (7) inspires us to investigate results about its non-zero solutions and numerous other results valid on any group G. Then we derive the following lemma that will be used a couple of times during proofs:

Lemma 2. *Let χ be a non-zero solution of the functional Equation (7) on any group G; then we can derive the following:*
(1) $\chi(1) = 1$;
(2) $\chi(x^{-1}) = \chi(x)$;
(3) $\chi(y^{-1}xy) = \chi(x)$;
(4) χ is central; i.e., $\chi(xy) = \chi(yx)$.

Proof. (1). Since $\chi(x)\chi(1) = \max\{\,\chi(x), \chi(x)\,\}$ and χ is non-zero, so, we have $\chi(1) = 1$.
(2). By setting $x = 1$, for any $y \in G$, from Equation (7) we can deduce that

$$\begin{aligned} \chi(1)\chi(y) &= \max\{\,\chi(1.y), \chi(1.y^{-1}) \\ \chi(y) &= \max\{\,\chi(y), \chi(y^{-1})\,\}; \end{aligned} \tag{17}$$

writing y^{-1} instead of y in Equation (17) yields

$$\chi(y^{-1}) = \max\{\,\chi(y^{-1}), \chi(y)\,\}. \tag{18}$$

Equations (17) and (18) imply that $\chi(y^{-1}) = \chi(y)$. Hence $\chi(x^{-1}) = \chi(x)$ for every $x \in G$.
(3). Let $x, y \in G$ and write y instead of x and $y^{-1}xy$ instead of y in (7); then (3) is the result of the following computation:

$$\begin{aligned} \chi(y)\chi(y^{-1}xy) &= \max\{\,\chi(yy^{-1}xy), \chi(y[y^{-1}xy]^{-1})\,\} \\ &= \max\{\,\chi(xy), \chi(y(y^{-1}x^{-1}y)\,\} \\ &= \max\{\,\chi(xy), \chi(x^{-1}y)\,\} \\ &= \max\{\,\chi(y^{-1}x^{-1}), \chi((x^{-1}y)^{-1})\,\} \qquad \text{(by Lemma 2 (2))} \\ &= \max\{\,\chi(y^{-1}x^{-1}), \chi(y^{-1}x)\,\} \\ \chi(y)\chi(y^{-1}xy) &= \chi(y^{-1})\chi(x) \\ \chi(y)\chi(y^{-1}xy) &= \chi(y)\chi(x) \\ \chi(y^{-1}xy) &= \chi(x). \end{aligned}$$

(4). By applying Lemma 2 (3) and writing yx instead of x, we have $\chi(y^{-1}(yx)y) = \chi(yx)$, which implies that $\chi(xy) = \chi(yx)$. Hence, χ is central. □

Moreover, we prove the following theorem by utilizing Lemma 2 to characterize the solutions χ of the functional Equation (7), without using additional assumptions about divisibility by two and three.

Theorem 4. *Let G be any group; then a function $\chi\colon G \to \mathbb{R}$ fulfills the functional Equation (7) if and only if χ vanishes identically on a group G or there exists a normal subgroup N_χ of G such that*

$$N_\chi = \{\, x \in G \mid \chi(x) = 1\,\}$$

and

$$\text{either} \quad xy^{-1} \in N_\chi \quad \text{or} \quad xy \in N_\chi \quad \text{for every} \quad x, y \in G \text{ and } x, y \notin N_\chi; \tag{19}$$

or $\chi(x) = e^{|\alpha(x)|}$ *for any* $x \in G$ *and for some additive function* $\alpha \colon G \to \mathbb{R}$.

Proof. The "if" part can be easily demonstrated. To prove the converse inclusion, we assume that χ fulfills the functional Equation (7), and we will prove that it has one of the forms presented in the statement of the theorem. Setting $x = y = 1$ in Equation (7) gives that $\chi(1)\chi(1) = \max\{\chi(1), \chi(1)\}$; then $\chi(1)\chi(1) = \chi(1)$; therefore, either $\chi(1) = 1$ or $\chi(1) = 0$.

Setting $y = 1$ in Equation (7) implies that $\chi(x)\chi(1) = \max\{\chi(x), \chi(x)\}$; therefore, $\chi(x) = \chi(x)\chi(1)$. If $\chi(1) = 0$, then χ vanishes identically on a group G. Setting $y = x$ in Equation (7), we can compute that $\chi(x)^2 \geq 1$, so $\chi(x) \leq -1$ and $\chi(x) \geq 1$. Further, suppose that $\chi(1) = 1$ and there exists an element $\bar{x} \in G$ such that $\chi(\bar{x}) \leq -1$. We need to prove that $\chi(\bar{x}) = -1$, but on the contrary suppose that $\chi(\bar{x}) < -1$; then from Equation (7) we obtain that

$$\begin{aligned} \max\{\chi(\bar{x}\bar{x}), \chi(\bar{x}\bar{x}^{-1})\} &= \chi(\bar{x})\chi(\bar{x}) \\ \max\{\chi(\bar{x}^2), \chi(1)\} &= \chi(\bar{x})^2 \\ \max\{\chi(\bar{x}^2), 1\} &= \chi(\bar{x})^2 > 1, \end{aligned}$$

which provides that $\chi(\bar{x}^2) = \chi(\bar{x})^2$. Moreover

$$\begin{aligned} \chi(\bar{x}) &> \chi(\bar{x})^3 = \chi(\bar{x})^2\chi(\bar{x}) \\ &= \chi(\bar{x}^2)\chi(\bar{x}) \\ &= \max\{\chi(\bar{x}^2\bar{x}), \chi(\bar{x}^2\bar{x}^{-1})\} \\ &= \max\{\chi(\bar{x}^3), \chi(\bar{x})\} \geq \chi(\bar{x}) \\ \chi(\bar{x}) &> \chi(\bar{x}), \end{aligned}$$

which is a contradiction; therefore, $\chi(\bar{x}) = -1$. Suppose again that $\chi(1) = 1$ and there exists an element $x_\circ \in G$ such that $\chi(x_\circ) \geq 1$. We will show that $\chi(x_\circ) = 1$, but on the contrary, suppose that $\chi(x_\circ) > 1$ for some $x_\circ \in G$; then Equation (7) gives that

$$\begin{aligned} \max\{\chi(\bar{x}x_\circ), \chi(\bar{x}x_\circ^{-1})\} &= \chi(\bar{x})\chi(x_\circ) \\ \max\{\chi(\bar{x}x_\circ), \chi(\bar{x}x_\circ^{-1})\} &= -\chi(x_\circ) \\ \max\{\chi(\bar{x}x_\circ), \chi(\bar{x}x_\circ^{-1})\} &< -1, \end{aligned}$$

which is a contradiction; therefore, $\chi(x_\circ) = 1$. Hence, $\chi(x) \in \{-1, 1\}$ for every $x \in G$. Moreover, let $N_\chi = \{x \in G \mid \chi(x) = 1\}$. It is clear that $1 \in N_\chi$ because $\chi(1) = 1$. Let $z \in N_\chi$; then by Lemma 2 we have $\chi(z^{-1}) = \chi(z) = 1$; therefore, $z^{-1} \in N_\chi$. Suppose that $y, z \in N_\chi$; then, $\chi(y) = \chi(z) = 1$, so from Equation (7) we obtain that

$$\begin{aligned} \chi(yz) &= \chi(yz)\chi(z) \\ &= \max\{\chi(yz^2), \chi(y)\} \\ &\geq \chi(y) \\ \chi(yz) &\geq \chi(y)\chi(z). \end{aligned} \tag{20}$$

$$\begin{aligned} \chi(y)\chi(z) &= \max\{\chi(yz), \chi(yz^{-1})\} \\ &\geq \chi(yz) \\ \chi(y)\chi(z) &\geq \chi(yz). \end{aligned} \tag{21}$$

From inequalities (20) and (21), we get $\chi(yz) = \chi(y)\chi(z) = 1$, which implies that $yz \in N_\chi$. Hence, N_χ is a subgroup of G. Let $z \in N_\chi$; then by Lemma 2(3), it can be seen that $\chi(y^{-1}zy) = \chi(z)$ for every $y \in G$ and $z \in N_\chi$; hence, N_χ is a normal subgroup of G.

Moreover, let $x, y \in G$ such that $x, y \notin N_\chi$; therefore, $\chi(x) \neq 1$ and $\chi(y) \neq 1$; then we have $\chi(x) = \chi(y) = -1$. Thus, Equation (7) implies that $\max\{\chi(xy), \chi(xy^{-1})\} = \chi(x)\chi(y) = 1$. In either case, $\chi(xy) = 1$ or $\chi(xy^{-1}) = 1$, which implies that either $xy \in N_\chi$ or $xy^{-1} \in N_\chi$.

Furthermore, assume that $\chi(x) > 0$ for every $x \in G$; then, by applying the main theorem of [10], there exists an additive function $\alpha\colon G \to \mathbb{R}$ such that $\chi(x) = e^{|\alpha(x)|}$ for every $x \in G$. □

Corollary 2. *Let χ be a non-zero solution of the functional Equation (7) on any group G; then the commutator subgroup G' is a normal subgroup of N_χ.*

Proof. Let χ be a non-zero solution of Equation (7). Then from Theorem 4, in the first case, assume that $\chi(x) > 0$ for every $x \in G$; then there exists an additive function $\alpha\colon G \to \mathbb{R}$ such that $\chi(x) = e^{|\alpha(x)|}$ for every $x \in G$. Therefore, using the fact that $\alpha([x, y]) = 0$, it can be seen that $\chi([x, y]) = 1$ for every $x, y \in G$. In the second case, there exists a normal subgroup N_χ of G such that $\chi(x) = 1$ for all $x \in N_\chi$ that also satisfies the condition (19); then the simple computation gives that

$$
\begin{aligned}
(xy^{-1})^{-1} \in N_\chi &\vee (xy)^{-1} \in N_\chi \\
yx^{-1} \in N_\chi &\vee y^{-1}x^{-1} \in N_\chi \\
x^{-1}y \in N_\chi &\vee x^{-1}y^{-1} \in N_\chi \\
xy^{-1}x^{-1}y \in N_\chi &\vee xyx^{-1}y^{-1} \in N_\chi \\
(xy^{-1}x^{-1}y)^{-1} \in N_\chi &\vee [x, y] \in N_\chi \\
y^{-1}xyx^{-1} &\in N_\chi \\
xyx^{-1}y^{-1} &\in N_\chi,
\end{aligned}
$$

which implies that $\chi([x, y]) = 1$; therefore, in either case the commutator subgroup G' is a normal subgroup of N_χ. □

Definition 1. *Let G be an arbitrary group. We say a function $\chi\colon G \to \mathbb{R}$ satisfies the Kannappan condition [11] if*

$$\chi(xyu) = \chi(xuy) \qquad \text{for all } u, x, y \in G.$$

Corollary 3. *Any solution of the functional Equation (7) on an arbitrary group G satisfies the Kannappan condition.*

Proof. The proof depends on three different cases, but in each case, the proof is quite obvious:

Case 1: If $\chi = 0$, then the result is trivial.

Case 2: Let χ be strictly positive. The proof is quite simple because the additive function $\alpha(x) = \log \chi(x)$ for all $x \in G$.

Case 3: Let χ be unitary. N_χ is a normal subgroup containing the commutator subgroup G'. Therefore, $\chi(xyz) = 1$ if and only if $xyz \in N_\chi$ if and only if $xyz[y^{-1}, x^{-1}] = xzy \in N_\chi$ if and only if $\chi(xzy) = 1$. As χ only takes the values 1 and -1, this is enough to prove the Kannappan condition. □

Remark 1. *(1). Any function on an abelian group G satisfies the Kannappan condition.*

(2). A function is central and fulfills the Kannappan condition if and only if it is invariant under arbitrary permutations of factors of its arguments, if and only if it is the lift of a function on the abelian quotient group G/G'.

Remark 2. *(1). It can be seen that Theorem 3 can also be concluded quite easily from Theorem 4, because if G is divisible by two and three, there is no nontrivial subgroup N_χ with the required condition* (19).

(2). $\{1\}$ *is a normal subgroup of the cyclic group* $U(3) = \langle 2\rangle = \{1, 2\}$ *and* $U(5) = \langle 2\rangle = \{1, 2, 3, 4\}$ *satisfies the condition* (19).

(3). $\{1, 6\}$ *is a normal subgroup of the cyclic group* $U(7) = \langle 3\rangle = \{1, 2, 3, 4, 5, 6\}$ *and* $\{1, 3\}$ *is a normal subgroup of* $U(8) = \{1, 3, 5, 7\}$ *satisfying the condition* (19).

(4). $\{0\}$ *is a normal subgroup of the cyclic group* $\mathbb{Z}_2$ *and* $\mathbb{Z}_3$ *that satisfies the condition* (19). *Moreover,* $2\mathbb{Z}$ *and* $3\mathbb{Z}$ *hold the condition* (19) *as normal subgroups of group* $\mathbb{Z}$.

Proposition 1. *Let G be any group. If* $\chi\colon G \to \mathbb{R}$ *is a non-zero solution of the functional Equation* (7)*, then* $\chi(xy)\chi(xy^{-1}) = \max\{\chi(x^2), \chi(y^2)\}$ *for all* $x, y \in G$.

Proof. According to Corollary 3, the function χ satisfies the Kannappan condition; therefore, we have $\chi(x^2) = \chi(xxyy^{-1}) = \chi(xyxy^{-1})$. Consequently, by Kannappan condition, the proof of proposition is a result of the following computation:

$$
\begin{aligned}
\chi(xy)\chi(xy^{-1}) &= \max\{\chi(xyxy^{-1}), \chi(xy[xy^{-1}]^{-1})\} \\
&= \max\{\chi(xyxy^{-1}), \chi(xyyx^{-1})\} \\
&= \max\{\chi(x^2), \chi(xy^2x^{-1})\} \\
&= \max\{\chi(x^2), \chi(y^2)\} && \text{(by Lemma 2 (3))} \\
\chi(xy)\chi(xy^{-1}) &= \max\{\chi(x^2), \chi(y^2)\}.
\end{aligned}
$$

□

We can get the following proposition, which is a direct consequence of Proposition 1, so we omit the proof.

Proposition 2. *Suppose that a function* $\chi\colon G \to \mathbb{R}$ *satisfies the functional Equation* (7) *such that* χ *is not identically zero on group G; then for any* $x, y \in G$*, we derive the following:*

(1) If $\chi(xy) = \chi(x)\chi(y)$*, then* $\chi(xy^{-1}) = \chi(y)^{-1}\chi(x)^{-1}\max\{\chi(x^2), \chi(y^2)\}$.

(2) If $\chi(xy^{-1}) = \chi(x)\chi(y)$*, then* $\chi(xy) = \chi(y)^{-1}\chi(x)^{-1}\max\{\chi(x^2), \chi(y^2)\}$.

(3) If $\chi(xy) = \chi(xy^{-1})$*, then either* $\chi(x^2) = 1$ *or* $\chi(y^2) = 1$.

The following theorem yields the information about the existence of a normal subgroup of G and will be important to compute more useful results.

Theorem 5. *Let G be any group. Suppose that the non-zero solution* $\chi\colon G \to \mathbb{R}$ *satisfies the functional Equation* (7)*; then the following results hold:*

(1) Let $z \in G$ *and* $\chi(zx) = \chi(zx^{-1})$ *for some arbitrary element* $x \in G$ *satisfying the condition that* $\chi(x^2) \neq 1$*. Then* $\chi(z^2) = 1$.

(2) Let $Z_\chi = \{z \in G \mid \chi(z^2) = 1\}$*. Then* Z_χ *is a normal subgroup of G.*

Proof. (1). According to the given condition $\chi(zx) = \chi(zx^{-1})$ for some $x \in G$, by applying Proposition 2 (3), we can conclude that either $\chi(x^2) = 1$ or $\chi(z^2) = 1$. Therefore, condition $\chi(x^2) \neq 1$ yields that $\chi(z^2) = 1$.

(2). Given that 1 is a neutral element and $\chi(1^2) = \chi(1) = 1$, $1 \in Z_\chi$. Let $z \in Z_\chi$; then $\chi(z^2) = 1$. By Lemma 2, $\chi(x^{-1}) = \chi(x)$; therefore, $\chi(z^{-2}) = \chi(z^2) = 1$, and hence $z^{-1} \in Z_\chi$. Let $y, z \in Z_\chi$; therefore, by the definition of Z_χ, $\chi(y^2) = 1$ and $\chi(z^2) = 1$.

$$
\begin{aligned}
\chi(y^2z^2) &= \chi(y^2z^2)\chi(z^2) \\
&= \max\{\, \chi(y^2z^4), \chi(y^2) \,\} \\
&\geq \chi(y^2) \\
\chi(y^2z^2) &\geq \chi(y^2)\chi(z^2).
\end{aligned} \tag{22}
$$

$$
\begin{aligned}
\chi(y^2)\chi(z^2) &= \max\{\, \chi(y^2z^2), \chi(y^2z^{-2}) \,\} \\
&\geq \chi(y^2z^2) \\
\chi(y^2)\chi(z^2) &\geq \chi(y^2z^2).
\end{aligned} \tag{23}
$$

From the inequalities (22) and (23), we can conclude that $\chi(y^2z^2) = \chi(y^2)\chi(z^2) = 1$, which implies that $yz \in Z_\chi$. Hence, $Z_\chi \leq G$. In addition, by Lemma 2, χ is central; therefore, $\chi(xz) = \chi(zx)$ for all $x \in G$ and $z \in Z_\chi$. Hence, Z_χ is a normal subgroup of G. □

Proposition 3. *Assume that χ is a non-zero solution of the functional Equation (7) on an arbitrary group G; then $G' \lhd N_\chi \lhd Z_\chi \lhd G$.*

Proof. The proof follows directly from Theorems 4 and 5 and Corollary 2. □

Proposition 4. *If χ is a strictly positive solution of Equation (7) on an arbitrary group G, then $Z_\chi = N_\chi$.*

In [10], Toborg has to be taken to prove that the solutions of the functional Equation (3) are of the form $|\alpha|$ for the additive function $\alpha\colon G \to \mathbb{R}$, and that in the notation of the theorem, the relation $\sim_R$ can be characterized as $x \sim_R y$ if and only if $\alpha(x)$ and $\alpha(y)$ have the same sign (with the convention that 0 has both signs). The following propositions are obvious consequences of Theorem 3 and what Toborg used to prove her main result. The results of the following propositions are quite obvious and can be easily verified, so we overlook the proof.

Proposition 5. *Let χ be a strictly positive solution of the functional Equation (7) on an arbitrary group G; then relation $\sim_R$ on group G satisfies the following:*

(1) $\sim_R$ is reflexive;
(2) $\sim_R$ is symmetric;
(3) Either $x \sim_R y$ or $x \sim_R y^{-1}$;
(4) $x \sim_R y$ if and only if $x^{-1} \sim_R y^{-1}$.

Proposition 6. *Let χ be a strictly positive solution of the functional Equation (7) on an arbitrary group G. If $\chi(z) = 1$ for some $z \in G$, then the following results hold:*

(1) $z \sim_R x$ for any $x \in G$;
(2) If $x \sim_R y$ then $x \sim_R yz$.

Proof. Since G is a strictly positive solution of Equation (7), there exists an additive function $\alpha\colon G \to \mathbb{R}$ such that $\chi(x) = e^{|\alpha(x)|}$ for any $x \in G$; therefore, the results become quite obvious because $\alpha(z) = 0$. □

Proposition 7. *Let χ be a strictly positive solution of Equation (7) on an arbitrary group G; then any solution $\chi\colon N_\chi \to \mathbb{R}$ can be interpreted as a function on the abelian factor group G/N_χ.*

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