Stability Analysis of Discrete-Time Stochastic Delay Systems with Impulses

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Abstract: This paper is concerned with stability analysis of discrete-time stochastic delay systems with impulses. By using the sums average value of the time-varying coefficients and the average impulsive interval, two sufficient criteria for exponential stability of discrete-time impulsive stochastic delay systems are derived, which are more convenient to be applied than those Razumikhin-type conditions in previous literature. Both $p$th moment asymptotic stability and $p$th moment exponential stability are considered. Finally, two numerical examples to illustrate the effectiveness.

Keywords: discrete-time stochastic systems; average impulsive interval; asymptotic stability; exponential stability

MSC: 34A38; 34H15

1. Introduction

Over the past decades, the impulsive phenomena have been intensively investigated since their significance and applications in areas such as economics, mechanics, chemical, biological phenomena, population dynamics, see other works and the references therein [1–6]. On the other hand, time delays occur frequently in many evolution processes and it is the inherent feature of many physical processes. Therefore, the study of impulsive delay systems has attracted great attention over the past few years [7–11]. For example, in [7], the authors have studied the $p$th moment exponential stability of a class of impulsive delay stochastic functional differential systems, by using the Lyapunov functions and Razumikhin techniques, some stability results have been given. Li and Song [8] have proposed an impulsive delay inequality and studied the stabilization problem of delayed systems via impulsive control. In particular, Liu [9] have studied the $p$th moment asymptotical stability for impulsive stochastic differential equations by employing Lyapunov function method and Itô’s formula. Recently, some important and interesting results for stability of impulsive systems have been obtained, see [12–17] and the references therein. Hence, it is necessary to investigate the exponential stability for discrete-time stochastic systems with impulses.

Formally speaking, discrete-time systems is more challenging than continuous-time systems [18,19]. Hence the study on discrete-time systems will become more and more important, and attract a lot of researchers’ attention [20–23]. In [20], the global exponential stability results for discrete-time delay systems with impulsive controllers have been considered. By using Razumikhin technique, the robust exponential stability results for discrete-time neural network with uncertainty have been given in [21]. The discrete-time Markovian jump delay systems with impulses have been investigated in [22], where the impulses act as perturbations. The analysis and synthesis problems for stability of discrete-time impulsive systems have been extensively studied in recent years [24–29]. However, both of the above results require that the time delay in system is always greater than time delays in impulses, which leads to very conservative results. Hence, the existing methods and tools on stability of discrete-time stochastic systems with impulsive control are very...
insufficient. Therefore, it is more useful to consider the stability or stabilization problem for more general discrete-time stochastic systems with impulsive control.

Based on the above discussion, the aim of the present paper is to establish exponential stability criterion on discrete-time stochastic delay systems, the criterion of this paper allows stabilizing impulses and destabilizing impulses is simultaneously effective under average impulsive interval condition. This paper mainly employ the sums average value of time-varying coefficients and the average impulsive interval, which are quite different from existing results in literature [30–34].

The rest of the paper is structured as follows. In Section 2, we introduce some basic definitions and notations. In Section 3, some criteria for exponential stability of discrete-time stochastic time-varying systems under impulsive control are obtained. In Section 4, two examples and their simulations are presented to illustrate the effectiveness of the proposed results. Finally, some conclusions are given in Section 5.

2. Preliminaries

Throughout this paper, let \((\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n \geq 0}, \mathbb{P})\) be a complete probability space with a filtration \(\{\mathcal{F}_n\}_{n \geq 0}\) satisfying the usual conditions (\(\mathcal{F}_0\) contains all \(\mathbb{P}\)-null sets). Let \(\mathbb{R} = (-\infty, +\infty), \mathbb{R}^+ = [0, +\infty), \mathbb{R}^d\) the \(d\)-dimensional Euclidean space, and \(\mathbb{R}^{d \times r}\) the space of \(d \times r\) real matrices. For a set \(A \subseteq \mathbb{R}\), we denote \(I_A\) the family of all integer in \(A\). Denote \(N = I[0, \infty) = \{0, 1, 2, \ldots\}\) and \(N_+ = I[1, \infty)\). Let \(|\cdot|\) be the Euclidean norm in \(\mathbb{R}^d\). \(A^T\) denote the transpose of \(A\). \(I\) represents the identity matrix. Let \(\Psi = \{\psi : \mathbb{R}^+ \to \mathbb{R} | \psi(n) \) is continuous and \(\sup_{n \in \mathbb{N}} \dot{\psi}(n) \) and \(\inf_{n \in \mathbb{N}} \ddot{\psi}(n) \) exist\}, where \(\dot{\psi}(n) = \max\{\psi(n), 1\}\) and \(\ddot{\psi}(n) = \min\{\psi(n), 1\}\). Let \(G(N_{-m}, \mathbb{R}^d) = \{\psi : N_{-m} \to \mathbb{R}^n | \psi(s)\) is continuous for all but at most countable points \(s \in N_{-m}\) and at these points \(s \in N_{-m}\), where \(N_{-m} = I[-\tau, 0]\) and \(\tau \in N_+\). For \(p > 0\), we denote by \(G_{\mathcal{F}_n}^p(N_{-m}, \mathbb{R}^d)\) the family of all \(\mathcal{F}_n\)-measurable \(\mathbb{R}\)-valued functions \(\xi = \{\xi(s); s \in N_{-m}\}\) satisfying \(\|\xi\|_{C^p}^p = \sup_{s \in N_{-m}} \mathbb{E}[|\xi(s)|^p] < \infty\), where \(\mathbb{E}\) denotes mathematical expectation operator. Furthermore, \(G_{\mathcal{F}_n}^p(N_{-m}, \mathbb{R}^d)\) is denoted by class of all bounded \(\mathcal{F}_n\) measurable \(G(N_{-m}, \mathbb{R}^d)\)-valued functions.

Consider the following discrete-time stochastic delay system with impulses

\[
\begin{align*}
    x(n+1) &= f(n, x_n) + g(n, x_n)\omega(n), \quad n \neq n_k - 1, \quad n \in \mathbb{N}, \\
    x(n_k) &= I_k(n_k - 1, x(n_k - 1)), \quad k \in \mathbb{N}_+
\end{align*}
\]

with initial value \(x_0 = \xi = \{\xi(s); s \in N_{-m}\} \in G_{\mathcal{F}_n}^b(N_{-m}, \mathbb{R}^d)\), where \(x(n) = (x_1(n), \ldots, x_d(n))^T\), \(x_0 = x_{d}(\theta) = (x(n + \theta) ; \theta \in N_{-m}) \in G_{\mathcal{F}_n}^b(N_{-m}, \mathbb{R}^d)\), \(f : \mathbb{N} \times G_{\mathcal{F}_n}^b(N_{-m}, \mathbb{R}^d) \to \mathbb{R}^d\) and \(g : \mathbb{N} \times G_{\mathcal{F}_n}^b(N_{-m}, \mathbb{R}^d) \to \mathbb{R}^{d \times r}\) are Borel measurable. \(\omega(n) = (\omega_1(n), \ldots, \omega_r(n))^T\) be \(r\)-dimensional mutually independent stochastic sequence defined on the complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\) satisfying \(\mathbb{E}\omega(n) = 0\), \(\mathbb{E}|\omega(n)|^2 = 1\) and \(\mathbb{E}\omega(i)\omega(j) = 0\) for \(i \neq j\). The impulsive functions \(I_k(n_k - 1, x(n_k - 1)) : \mathbb{N}_+ \times G_{\mathcal{F}_n}(N_{-m}; \mathbb{R}^d) \to \mathbb{R}^d\), and the impulsive moments \(n_k\) satisfy \(0 = n_0 < n_1 < \cdots < n_k < \cdots, n_k \to \infty\) (as \(k \to \infty\)).

For the purpose of stability, we assume the functions \(f(n, 0) = g(n, 0) \equiv 0\) and \(I_k(n, 0) \equiv 0, k \in \mathbb{N}_+\), which implies that \(x(n) \equiv 0\) is an equilibrium solution.

**Definition 1.** The trivial solution of system (1) is said to be \(p\)th moment stable, if for any \(\epsilon > 0\), there exists \(\delta = \delta(\epsilon)\) such that

\[
\mathbb{E}|x(n)|^p < \epsilon, \quad n \geq n_0
\]

for any initial value \(\|\xi\|_{C^p}^p < \delta\).

**Definition 2.** The trivial solution of system (1) is said to be \(p\)th moment asymptotically stable, if it is \(p\)th moment stable, and for any \(\epsilon > 0\), there exists \(\delta = \delta(\epsilon)\) and \(T = T(\epsilon)\) such that

\[
\mathbb{E}|x(n)|^p < \epsilon, \quad n \geq T
\]
for any initial value $||\xi||_{G^b}^p < \delta$.

**Definition 3.** The trivial solution of system (1) is said to be $p$th ($p > 0$) moment exponentially stable if there is a pair of positive constants $\lambda$ and $C$ such that

$$E|x(n)|^p \leq CE||\xi||_{G^b}^p e^{-\lambda(n-n_0)}, \quad n \geq n_0$$

for any initial value $||\xi||_{G^b}^p < \delta$. When $p = 2$, it is usually said to be exponentially stable in mean square.

**Definition 4.** The impulsive sequence $\zeta = \{n_k : k \in \mathbb{N}_+\}$ is said to have an average impulsive interval $T_a$ if there exist $N_0 \geq 0$ and $T_a > 0$ such that for any $n \geq s \geq 0$,

$$\frac{n-s}{T_a} - N_0 \leq N(n,s) \leq \frac{n-s}{T_a} + N_0,$$

(2)

where $N(n,s)$ denotes the number of impulsive times of the impulsive sequence $\zeta$ on the interval $(s,n]$. The $N_0$ is said to be the elasticity number.

**Remark 1.** For most real-world impulsive signals, the occurrence of impulses is not uniformly distributed. For this impulsive signals, the lower bound of the impulsive intervals is small, meanwhile the upper bound of the impulsive intervals is quite large. Hence, for non-uniformly distributed impulsive signals, taking $\sup_{k \in \mathbb{N}_+} \{n_{k+1} - n_k\}$ (upper bound) or $\inf_{k \in \mathbb{N}_+} \{n_{k+1} - n_k\}$ (lower bound) to characterize the frequency of the impulses’ occurrence would make the obtained results very conservative.

### 3. Main Results

#### 3.1. Asymptotically Stable

In this subsection, we will consider the $p$th moment globally asymptotically stable for discrete-time impulsive stochastic systems (1) by using average impulsive interval.

**Theorem 1.** Assume that there exists a non-negative function $V(n,x)$, and $a(n), \beta(n) \in \Psi$ with $\beta(n) \geq 0$ and exist real constants $p > 0, c_1 > 0, c_2 > 0, \rho > 0$ such that

(i) for any $x \in \mathbb{R}^d$ and $n \in \mathbb{N}_+$, $c_1|x|^p \leq V(n,x) \leq c_2|x|^p$;

(ii) for any $n \neq n_k - 1, k \in \mathbb{N}_+$, $\mathbb{E}V(n+1,x(n+1)) \leq (1 + a(n))\mathbb{E}V(n,x(n)) + \beta(n) \sup_{s \in \mathbb{N}_-^m} \mathbb{E}V(n+s,x(n+s))$;

(iii) for any $x \in \mathbb{R}^d, k \in \mathbb{N}_+$, $V(n_k, I(x)) \leq \rho V(n_k - 1, x)$;

(iv) for $\bar{\lambda} = \inf_{n \in \mathbb{N}_+} \bar{\lambda}(n)$ and $q = \min\{\frac{\rho T_a + N_0}{\bar{\lambda}}, 1\}$,

$$\lim_{n \rightarrow \infty} \sum_{s = n_0}^{n} (a(s) + \frac{\bar{\lambda}^{1-\tau}}{q} \beta(s) + \ln \rho T_a) = -\infty.$$

Then the trivial solution of system (1) is $p$th moment globally asymptotically stable.

**Proof.** For any initial date $\xi \in \mathcal{C}_{\mathbb{Z}_+}^{p}(\mathbb{N}_{-m}; \mathbb{R}^d)$, and write $x(n;n_0,\xi) = x(n)$ and $V(n,x(n)) = V(n)$ simply. For any $\phi > 0$, construct an auxiliary function

$$u(n) = (1 + \bar{\lambda}(n))u(n-1) + \beta(n) \sup_{\theta \in \mathbb{N}_-^m} u(n+\theta) + \phi, \quad n \neq n_k - 1, \quad n \in \mathbb{N},$$

$$u(n_k) = \rho u(n_k - 1), \quad k \in \mathbb{N}_+,$$

(3)

where $u(n)$ the solution of (3) with initial condition $u(s) = u_0 = ||\xi||_{G^b}^p$ for $s \in I[\tau,0]$. We first prove that

$$\sup_{\theta \in \mathbb{N}_-^m} u(n+\theta) \leq \frac{\bar{\lambda}^{1-\tau}}{q} u(n)$$

(4)
holds for any \( n \in \mathbb{N} \). \( \square \)

For \( n \in I[n_0, n_1] \), we have

\[
  u(n) = (1 + \bar{\Phi}(n))u(n - 1) + \beta(n) \sup_{\theta \in \mathbb{N}_-} u(n + \theta) + \phi
  \]

\[
  = \prod_{j=n_0}^{n} (1 + \bar{\Phi}(j)) u(n_0) + \sum_{s=n_0}^{n_1-1} \prod_{j=s+1}^{n} (1 + \bar{\Phi}(j)) [\beta(s) \sup_{\theta \in \mathbb{N}_-} u(s + \theta) + \phi].
\]

(5)

From (5) and condition (iii), we have

\[
u(n_1) = \rho u(n_1 - 1) = \prod_{j=n_0}^{n_1-1} (1 + \bar{\Phi}(j)) u(n_0) + \sum_{s=n_0}^{n_1-1} \prod_{j=s+1}^{n_1} (1 + \bar{\Phi}(j)) [\beta(s) \sup_{\theta \in \mathbb{N}_-} u(s + \theta) + \phi].
\]

(6)

Next, it is similar to the proof of (5), for any \( n \in I[n_1, n_2] \), we can prove

\[
u(n) = \prod_{j=n_0}^{n_1} (1 + \bar{\Phi}(j)) u(n_1) + \sum_{s=n_0}^{n_1-1} \prod_{j=s+1}^{n_1} (1 + \bar{\Phi}(j)) [\beta(s) \sup_{\theta \in \mathbb{N}_-} u(s + \theta) + \phi] + \sum_{s=n_2}^{n_1} \prod_{j=s+1}^{n_1} (1 + \bar{\Phi}(j)) [\beta(s) \sup_{\theta \in \mathbb{N}_-} u(s + \theta) + \phi].
\]

By the induction principle, for any \( n \in I[n_k, n_{k+1}], k \in \mathbb{N}_+ \), we have

\[
u(n) = u(n_0) \prod_{s=n_k < n} (1 + \bar{\Phi}(j))
  + \sum_{s=n_0}^{n} \prod_{j=s+1}^{n_1} (1 + \bar{\Phi}(j)) [\beta(s) \sup_{\theta \in \mathbb{N}_-} u(s + \theta) + \phi].
\]

Then it can be deduced that

\[
u(n) = u(n_0)K(n, n_0) + \sum_{s=n_0}^{n} K(n, s) [\beta(s) \sup_{\theta \in \mathbb{N}_-} u(s + \theta) + \phi], \quad n \in \mathbb{N},
\]

(7)

where \( K(n, s) = \prod_{s \leq n < s} (1 + \bar{\Phi}(j)) \) for any \( n > s \geq n_0 \). We only need to consider the following two possible cases.
Case I. If \( n + \theta \geq 0 \), then it follows from (7) and the fact \( n + \theta \leq n \) that
\[
\begin{align*}
    u(n) &= u(n_0)K(n, n_0) + \sum_{s=n_0}^{n} K(n, s)[\beta(s) \sup_{\theta \in \mathbb{N}_m} u(s + \theta) + \phi] \\
    &= u(n_0)K(n, n + \theta)K(n + \theta, n_0) + \sum_{s=n_0}^{n+\theta} K(n, s)[\beta(s) \sup_{\theta \in \mathbb{N}_m} u(s + \theta) + \phi] \\
    &\quad + \sum_{s=n+\theta+1}^{n} K(n, s)[\beta(s) \sup_{\theta \in \mathbb{N}_m} u(s + \theta) + \phi] \\
    &\geq u(n_0)K(n, n + \theta)K(n + \theta, n_0) + \sum_{s=n_0}^{n+\theta} K(n, s)[\beta(s) \sup_{\theta \in \mathbb{N}_m} u(s + \theta) + \phi] \\
    &= u(n + \theta) \prod_{n+\theta \leq n_k < n} \rho \prod_{j=n+\theta+1}^{n} (1 + \tilde{\kappa}(j)). \quad (8)
\end{align*}
\]
Noting that
\[
\prod_{j=n+\theta+1}^{n} (1 + \tilde{\kappa}(j)) \geq \prod_{j=n+\theta+1}^{n} (1 + \tilde{\kappa}(j)) \geq \delta^{\tau - 1}. \quad (9)
\]
Moreover, it easy to deduce that
\[
\prod_{n+\theta \leq n_k < n} \rho = \rho^{N(n, n + \theta)}.
\]
Hence, from (2), then we have \( \rho^{N(n, n + \theta)} \geq 1 \) for \( \rho \geq 1 \), and for \( 0 < \rho < 1 \), it follows from Definition 4 that
\[
N(n, n + \theta) \leq -\frac{\theta}{I_a} + N_0 \leq \frac{\tau}{I_a} + N_0,
\]
which further implies that \( \rho^{N(n, n + \theta)} \geq \rho^{\frac{\theta}{I_a} + N_0} \geq \rho^{\frac{\tau}{I_a} + N_0} \). Therefore, we get
\[
\prod_{n+\theta \leq n_k < n} \rho \geq \eta. \quad (10)
\]
Substituting (9) and (10) into (8) yields that
\[
u(n) \geq u(n + \theta)\eta^{\delta^{\tau - 1}} \quad (11)
\]
hold for \( n + \theta \geq 0 \).

Case II. If \( n + \theta < 0 \), then it can be deduced form (7) that
\[
\begin{align*}
    u(n) &= u(n_0)K(n, n_0) + \sum_{s=n_0}^{n} K(n, s)[\beta(s) \sup_{\theta \in \mathbb{N}_m} u(s + \theta) + \phi] \\
    &= u(n + \theta)K(n, n_0) + \sum_{s=n_0}^{n} K(n, s)[\beta(s) \sup_{\theta \in \mathbb{N}_m} u(s + \theta) + \phi] \\
    &\geq u(n + \theta)K(n, n_0) \\
    &= u(n + \theta) \prod_{n_0 \leq n_k < n} \rho \prod_{j=n_0 + 1}^{n} (1 + \tilde{\kappa}(j)). \quad (12)
\end{align*}
\]
Noting that \( u(n + \theta) = u(n_0) = \|\xi\|_{C^0} \).
Since \( n + \theta < 0 \), one can see \( n < -\theta \leq \tau \). Thus, in view of (9) and (10), we obtain
\[
\prod_{j=n_0+1}^n (1 + \hat{\alpha}(j)) \geq \hat{\alpha}^{\tau - 1}
\]  
(13)

and
\[
\prod_{n_0 \leq n_k < n} \rho \geq q.
\]  
(14)

Substituting (13) and (14) into (12) results in
\[
u(n) \geq u(n + \theta)q\hat{\alpha}^{\tau - 1},
\]  
(15)

for \( n + \theta < 0 \).

Hence, for both of the above cases, we always have
\[
u(n + \theta) \leq \hat{\alpha}^{1 - \tau}q u(n),
\]
for any \( n \in \mathbb{N} \) and \( \theta \in \mathbb{N}_{-m} \), which means that (4) holds.

Next we claim that
\[\mathbb{E}V(n) \leq \|\xi\|_{\mathcal{G}^p}^p \prod_{n_0 \leq n_k < n} \rho \prod_{j=n_0}^n (1 + \hat{\alpha}(j) + \hat{\alpha}^{1 - \tau}q \beta(j))\]

hold for \( n \in \mathbb{N} \).

We first prove that
\[
\mathbb{E}V(n) \leq \|\xi\|_{\mathcal{G}^p}^p \prod_{n_0 \leq n_k < n} \rho \prod_{j=n_0}^n (1 + \hat{\alpha}(j) + \hat{\alpha}^{1 - \tau}q \beta(j)), \quad n \in \mathbb{N}_{[n_0, n_1)}.
\]  
(17)

Actually, by substituting (4) into the auxiliary function (3), we derive that
\[
u(n) = (1 + \hat{\alpha}(n) + \frac{\hat{\alpha}^{1 - \tau}q \beta(n)}{q})u(n - 1) + \phi, \quad n \neq n_k - 1, \quad n \in \mathbb{N},
\]
\[x(n_k) = \rho u(n_k - 1), \quad k \in \mathbb{N}_+.
\]  
(18)

System (18) can be solved as
\[
u(n) = \|\xi\|_{\mathcal{G}^p}^p \prod_{n_0 \leq n_k < n} \rho \prod_{j=n_0}^n (1 + \hat{\alpha}(j) + \hat{\alpha}^{1 - \tau}q \beta(j))
\]
\[+ \phi \sum_{s=n_0}^{n_k} \prod_{s \leq n_k < n} \rho \prod_{j=s+1}^n (1 + \hat{\alpha}(j) + \hat{\alpha}^{1 - \tau}q \beta(j))
\]
\[= \|\xi\|_{\mathcal{G}^p}^p \prod_{n_0 \leq n_k < n} \rho \prod_{j=n_0}^n (1 + \gamma_j) + \phi \sum_{s=n_0}^{n_k} \prod_{s \leq n_k < n} \rho \prod_{j=s+1}^n (1 + \gamma_j), \quad n \in \mathbb{N},
\]  
(19)

where \( \gamma_j = \hat{\alpha}(j) + \frac{\hat{\alpha}^{1 - \tau}q \beta(j)}{q} \).

Next, we claim that for all \( n \in \mathbb{N}_{[n_0, n_1)} \) it follows
\[
\mathbb{E}V(n) \leq \nu(n).
\]  
(20)

We assume, on the contrary, there exists \( \check{n} \in \mathbb{N}_{[n_0, n_1 - 1]} \) such that
\[
\mathbb{E}V(n + 1) > \nu(n + 1)
\]  
(21)
and

\[ \mathbb{E}V(n) \leq u(n), \quad n \in I[n_0 - \tau, n]. \]

Hence, by (4), (18) and (21), we have for any \( n \in I[n + 1, n_1] \),

\[ \mathbb{E}V(n + 1) \geq (1 + \bar{\alpha}(n + 1))u(n) + \beta(n + 1) \sup_{\theta \in \mathbb{N} - \bar{n}} u(\bar{n} + \theta). \tag{22} \]

In view of condition (ii), we have

\[ \mathbb{E}V(\bar{n} + 1) \leq (1 + \bar{\alpha}(\bar{n} + 1))u(\bar{n}) + \beta(\bar{n} + 1) \sup_{\theta \in \mathbb{N} - \bar{n}} u(\bar{n} + \theta), \]

which is a contradiction to (22), and so (20) holds. Let \( \phi \to 0 \) on both sides of (19), then we obtain (17) immediately.

Combining (17) and condition (iii), we obtain that

\[ \mathbb{E}V(n_1) \leq \rho \mathbb{E}V(n_1 - 1) \leq \|\xi\|_{G, P}^P \prod_{j = n_0}^{n_1 - 1} (1 + \gamma_j). \]

It is similar to the proof of (20), one can prove that for any \( n \in I[n_1, n_2] \)

\[ \mathbb{E}V(n) \leq u(n). \]

From condition (iii), we get

\[ \mathbb{E}V(n_2) \leq \rho \mathbb{E}V(n_2 - 1) \leq \|\xi\|_{G, P}^P \prod_{j = n_0}^{n_2 - 1} (1 + \gamma_j). \]

By a simple derivation, we can prove in general that

\[ \mathbb{E}V(n) \leq \|\xi\|_{G, P}^P \prod_{n_0 \leq n < u} \rho \prod_{j = n_0}^{n} (1 + \gamma_j), \quad n \in \mathbb{N}. \tag{23} \]

Finally, we show that system (1) is \( p \)th moment asymptotical stability. By using Definition 4, we get that

\[ \prod_{n_k \in I[n_0, n]} \rho = \rho^{N(n_0, n_0)} \geq \begin{cases} \rho^{\frac{n}{P}} N_0 = \rho^{\frac{\ln \rho}{P} (n-n_0)}, & \rho \geq 1, \\ \rho^{\frac{n}{P} - N_0} = \rho^{\frac{\ln \rho}{P} (n-n_0)}, & 0 < \rho < 1. \end{cases} \tag{24} \]

Hence, (16), together with the condition (i) and (24), we have

\[ \mathbb{E}|x(n)|^P \leq \|\xi\|_{G, P}^P e^{\frac{\ln \rho}{P} (n-n_0)} \prod_{j = n_0}^{n} (1 + \gamma_j) \leq \|\xi\|_{G, P}^P e^{\frac{\ln \rho}{P} (n-n_0)} \prod_{j = n_0}^{n} e^{\gamma_j} \]

\[ \leq \|\xi\|_{G, P}^P e^{\frac{\ln \rho}{P} (n-n_0)} \sum_{j = n_0}^{n} \gamma_j \]

\[ \leq \|\xi\|_{G, P}^P e^{\frac{\ln \rho}{P} (n-n_0)} \sum_{j = n_0}^{n} (\hat{\alpha}(j) + \frac{1 + \gamma_j}{P} \beta(j) + \ln \rho) \]

\[ \leq \|\xi\|_{G, P}^P e^{\frac{\ln \rho}{P} (n-n_0)} \sum_{j = n_0}^{n} (\hat{\alpha}(j) + \frac{1 + \beta(j) + \ln \rho}{P}), \quad n \in \mathbb{N}. \tag{25} \]
where \( c = \max \{ \rho^{-N_0}, \rho^{N_0} \} \). Note that \( \prod_{j=\tau_0}^{\tau} (1 + \gamma_j) \leq \sum_{j=\tau_0}^{\tau} e^{\gamma_j} \leq e^{\sum_{j=\tau_0}^{\tau} \gamma_j} \).

Combining this with the condition (iv) yields

\[
\lim_{n \to \infty} e^{\sum_{j=\tau_0}^{n} \left( \alpha(j) + \frac{1 - \tau}{\tau} \beta(j) + \frac{\ln \rho}{T_a} \right)} = 0,
\]

which implies that the trivial solution of system (1) is \( p \)-th moment asymptotically stable. The proof is complete.

**Remark 2.** The parameters \( \rho \) in condition (iii) describe the influence of impulses on the stability of the underlying discrete-time systems. If \( \rho > 1 \), the impulses are destabilizing, which means that the impulses do not occur too frequently. If \( \rho < 1 \), the impulses are stabilizing, which means that the impulses act appropriately frequently.

**Remark 3.** Both destabilizing and stabilizing impulses are discussed with the aid of average impulsive interval technique. Compared with [18,23], we obtained results have a greater range of applications.

### 3.2. Almost Sure Exponential Stability

At the end of this section, under an irrestrictive condition, we shall establish a theorem about the almost exponentially stable of system (1).

**Theorem 2.** Assume that there exist constants \( n^* \geq 0, T^* \geq 0 \) and \( \sigma > 0 \) such that conditions (i)-(iii) and the following condition hold:

\[
(iiv') \quad \sum_{s=n^*+kT^*}^{n^*+(k+1)T^*} \left( \alpha(s) + \beta(s) + \frac{\ln \rho}{T_a} \right) \leq -\sigma T^*, \quad k \in \mathbb{N}_+.
\]

Then the trivial solution of system (1) is \( p \)-th moment exponentially stable for \( \tau \in I[1, 1 + \log_\tau \left( \frac{\kappa T^*}{\rho} \right)] \).

**Proof.** For convenience, we denote \( \lambda(n) = \bar{\lambda}(n) + \frac{\delta}{\tau} \beta(n) + \frac{\ln \rho}{T_a}, \quad \bar{\lambda} = \sup_{n \in \mathbb{N}} \lambda(n), \) and \( \bar{\beta} = \sup_{n \in \mathbb{N}} \beta(n). \)

From the proof of Theorem 1, we conclude that conditions (i)-(iii) guarantee (25) holds, which means that

\[
E|x(n)|^p \leq c \| \xi \|^p e^{c_1 \lambda(n)}, \quad n \in \mathbb{N}.
\]  

(26)

When \( n \in I[n_0, n^*] \), it follows from \( \lambda(n) \leq \bar{\lambda} + \frac{\delta}{\tau} \bar{\beta} + \frac{\ln \rho}{T_a} \) that

\[
\sum_{s=n_0}^{n} \lambda(s) \leq \sum_{s=n_0}^{n} \left( \bar{\lambda} + \frac{\delta}{\tau} \bar{\beta} + \frac{\ln \rho}{T_a} \right) \leq \left( \bar{\lambda} + \frac{\delta}{\tau} \bar{\beta} + \frac{\ln \rho}{T_a} \right)n^*.
\]

which together with (26), implies that

\[
E|x(n)|^p \leq c \| \xi \|^p e^{\left( \bar{\lambda} + \frac{\delta}{\tau} \bar{\beta} + \frac{\ln \rho}{T_a} \right)n^*}, \quad n \in I[n_0, n^*].
\]  

(27)

When \( n \in I(n^*, +\infty) \), there exist a constant \( k_0 \in \mathbb{N} \) such that

\[
n^* + k_0 N^* \leq n < n^* + (k_0 + 1)N^*.
\]
Hence, we get that
\[
\sum_{s=n_0}^{n^*} \lambda(s) = \sum_{s=n_0}^{n^*} \lambda(s) + \sum_{k=n_0}^{k_0-1} \left( \sum_{s=n^*+kN^*}^{n^*+(k+1)N^*} \lambda(s) \right) + \sum_{s=n^*+(k_0+1)N^*}^{n} \lambda(s)
\]
\[= I_0 + \sum_{k=n_0}^{k_0-1} I_k + I_{k_0}. \quad (28)\]

It is easy to calculate that
\[
I_0 = \sum_{s=n_0}^{n^*} \lambda(s) \leq (\bar{\alpha} + \frac{\hat{\alpha}^{1-\tau}}{q} \hat{\beta} + \frac{|\ln \rho|}{T_a}) n^*
\]
and
\[
I_{k_0} \leq \sum_{s=n^*+(k_0+1)N^*}^{n^*+(k+1)N^*} (\bar{\alpha} + \frac{\hat{\alpha}^{1-\tau}}{q} \hat{\beta} + \frac{|\ln \rho|}{T_a}) \leq (\bar{\alpha} + \frac{\hat{\alpha}^{1-\tau}}{q} \hat{\beta} + \frac{|\ln \rho|}{T_a}) N^*. \quad (30)\]

It is worth noting that
\[
\lambda(n) = \alpha(n) + \beta(n) + \frac{|\ln \rho|}{T_a} + \left( \frac{\hat{\alpha}^{1-\tau}}{q} - 1 \right) \beta(n), \quad (31)\]
which further leads to
\[
I_k = \frac{n^*+(k+1)N^*}{n^*+kN^*} (\bar{\alpha} + \beta(s) + \frac{|\ln \rho|}{T_a}) \leq (-\sigma + \frac{\hat{\alpha}^{1-\tau}}{q} \hat{\beta} - \hat{\beta}) N^*, \quad (32)\]
for \(k = n_0, 1, \ldots, k_0 - 1\).

Form (28)--(32), it can seen that
\[
\sum_{s=n_0}^{n} \lambda(s) \leq (\bar{\alpha} + \frac{\hat{\alpha}^{1-\tau}}{q} \hat{\beta} + \frac{|\ln \rho|}{T_a}) (n^* + N^*) + (-\sigma + \frac{\hat{\alpha}^{1-\tau}}{q} \hat{\beta} - \hat{\beta}) N^* k_0. \quad (33)\]

Recalling that \(n \in I|n^* + k_0 N^*, n^* + (k_0 + 1) N^*\) and \(\tau \in I|1 + \log \frac{\gamma + \hat{\beta}}{2} \), one gets \(n - n^* - N^* < N^* k_0\) and \(-\sigma + \frac{\hat{\alpha}^{1-\tau}}{q} \hat{\beta} - \hat{\beta} < 0\), which together with (33), means that
\[
\sum_{s=n_0}^{n} \lambda(s) \leq (\bar{\alpha} + \frac{\hat{\alpha}^{1-\tau}}{q} \hat{\beta} + \frac{|\ln \rho|}{T_a}) (n^* + N^*) + (-\sigma + \frac{\hat{\alpha}^{1-\tau}}{q} \hat{\beta} - \hat{\beta}) (n - n^* - N^*)
\]
\[= (\sigma + \bar{\alpha} + \hat{\beta} + \frac{\hat{\alpha}^{1-\tau}}{q} \hat{\beta})(n^* + N^*) + (-\sigma + \frac{\hat{\alpha}^{1-\tau}}{q} \hat{\beta} - \hat{\beta}) n. \quad (34)\]
By substituting (34) into (26), we have
\[ E|x(n)|^p \leq \bar{c} \|\xi\|^p_{\mathcal{C}^d} e^{(\sigma + \hat{\alpha} + \hat{\beta} \frac{1 + \rho}{1 - \rho})(n^* + N^*) \cdot e^{(-\sigma + \hat{\alpha} + \hat{\beta} \frac{1 + \rho}{1 - \rho})n}}, \quad n \in \mathbb{I}[n^*, \infty). \] (35)

Denote
\[ C = \bar{c} \max \left\{ e^{(\sigma + \hat{\alpha} + \hat{\beta} \frac{1 + \rho}{1 - \rho})n^*}, e^{(\sigma + \hat{\alpha} + \hat{\beta} \frac{1 + \rho}{1 - \rho})(n^* + N^*)} \right\}. \]

Then, it follows from (27) and (35) that
\[ E|x(n)|^p \leq C \|\xi\|^p_{\mathcal{C}^d} e^{(-\sigma + \hat{\alpha} + \hat{\beta} \frac{1 + \rho}{1 - \rho})n}, \quad n \in \mathbb{N}. \] (36)

The proof is complete. \(\square\)

**Remark 4.** In [22], Dai and Xu have investigated the exponentially stable for discrete-time delayed systems by using Razumikhin-type method. However, Theorem 2 proposed in this paper removes this restriction and is applicable to discrete-time delayed systems with impulsive control including both \(\rho \geq 1\) and \(\rho < 1\), which makes it suitable for a broader scope of applications than the results in [22].

Consider the special case with \(\alpha(n) \equiv \alpha\) and \(\beta(n) \equiv \beta\), here \(\alpha\) and \(\beta\) are constants. Similar to proof of Theorem 2, we can obtain the following results.

**Corollary 1.** Assume that there exists a non-negative function \(V(n, x)\), and exist real constants \(p > 0, c_1 > 0, c_2 > 0, \alpha \in \mathbb{R}, \beta \geq 0, \rho > 0\) and \(\sigma > 0\), such that
(i) for any \(x \in \mathbb{R}^d\) and \(n \in \mathbb{N}, c_1 |x|^p \leq V(n, x) \leq c_2 |x|^p;\)
(ii) for any \(n \neq n_k - 1, k \in \mathbb{I}_{\mathbb{N}^+}, \mathbb{E}V(n + 1, x(n + 1)) \leq \alpha \mathbb{E}V(n, x(n)) + \beta \sup_{s \in \mathbb{I}_{\mathbb{N}^+}} \mathbb{E}V(n + s, x(s + s));\)
(iii) for any \(x \in \mathbb{R}^d, k \in \mathbb{I}_{\mathbb{N}^+}, \mathbb{E}V(n_k, I(x)) \leq \rho \mathbb{E}V(n_k - 1, x);\)
(iv) \(\alpha + \beta + \frac{\ln p}{1 - \rho} \leq -\sigma.\)

Then the trivial solution of system (1) is \(p\)th moment exponentially stable.

**4. Examples**

In this section, two examples to illustrate the effectiveness of the obtained results.

**Example 1.** Consider the following discrete-time impulsive stochastic delay system
\[
\begin{cases}
    x(n) = f(n - 1, x(n - 1)) + g(n - \tau(n), x(n - \tau(n)) \omega(n), & n \neq n_k, \quad n \in \mathbb{N}, \\
    x(n_k) = I_k(n_k - 1, x(n_k - 1)), & k \in \mathbb{I}_{\mathbb{N}^+},
\end{cases}
\] (37)

where \(\omega(n)\) obeys Gaussian distribution \(\mathcal{N}(0, 1)\), \(f(n, x(n)) = e^{-\frac{1}{2} \sin^2(n - 1)} x(n - 1), g(n - \tau(n), x(n - \tau(n)) = \frac{\cos n}{\sqrt{2(1 + n^2)}} x(n - \tau(n))\) and \(\tau(n) = \tau |n| + 2, \tau \in \mathbb{I}_{\mathbb{N}^+}\). The impulsive perturbation \(I_k(n_k - 1, x(n_k - 1)) = e^2 x(n_k - 1)\) and choosing an impulsive sequence show in (2) for \(N_0 = 0\) and \(T_a = 3\), then the impulsive moments is \(\bar{\xi} = \{n_k \mid n_k = n_{k - 1} + 3, n_0 = 0, k \in \mathbb{I}_{\mathbb{N}^+}\}.\)

Let \(V(n) = |x(n)|^2\), then
\[ \mathbb{E}V(n) = e^{\sin^2(n - 1) - 1} \mathbb{E}V(n - 1) + \frac{\cos^2 n}{2(1 + n^2)} \mathbb{E}V(n - \tau(n)), \] (38)

which implies that
\[ \mathbb{E}V(n) \leq e^{\sin^2(n - 1) - 1} \mathbb{E}V(n - 1) + \frac{\cos^2 n}{2(1 + n^2)} \sup_{\theta \in \mathbb{I}[-\tau - 2, -1]} \mathbb{E}V(n + \theta), \quad n \in \mathbb{N}. \]
So that condition (ii) of Theorem 2 holds by choosing $\alpha(n) = e^{\sin^2(n-1)-1}$ and $\beta(n) = \frac{\cos^2 n}{2(1+n^2)}$ with $\hat{\alpha} = e^{-1}, \hat{\beta} = \frac{1}{2}$.

Meanwhile, when $n = n_k$, we derive that

$$E\mathbb{V}(n_k) = E[e^{\frac{3}{2} x(n_k - 1)}]^2 = eE\mathbb{V}(n_k - 1),$$

which implies that condition (iii) of Theorem 2 holds with $\rho = e$. Obviously, we conclude that $\hat{\eta} = \min\{1, e\} = 1$. By choosing $n^* = 0$ and $T^* = 4\pi$, we derive that

$$\sum_{s=4kT^*\kappa}^{4(k+1)T^*} \{\ln(\alpha(s) + \beta(s)) + \frac{\ln \rho}{T^*\kappa}\} \leq -0.6\pi, \quad k \in \mathbb{N}$$

(39)

holds, which implies that condition (iv') of Theorem 2 is satisfied with $\sigma = -0.15$.

According to Theorem 2, the trivial solution of (37) is exponentially stable in mean square, the state sample trajectory with $\tau \in I[1, 1+0.7e]$, initial value $\xi = [-2, 3.05]^T$ are simulated in Figure 1. Figure 1 shows that a system can retain its original stability property with some impulse perturbations that are destabilizing.

**Remark 5.** Since the Razumikhin-type method is more conservative than the Lyapunov function method, the Razumikhin-type theorem in [26] is also not convenient to be applied to this system since it is not easy to find an appropriate constant to satisfy the Razumikhin-type condition.

**Remark 6.** Since the system without impulses is exponentially stable and the impulses are destabilizing, the existing results in [35,36] cannot be applied to system (37). From Theorem 1 in this paper, we see that the coefficients $\alpha(n), \beta(n)$ have wider range.

**Example 2.** Consider the following discrete-time stochastic delay system

$$x(n) = f(n-1, x(n-1)) + g(n - \tau(n), x(n - \tau(n)) \omega(n), \quad n \in \mathbb{N},$$

(40)
where \( \omega(n) \) obeys Gaussian distribution \( \mathcal{N}(0,1) \). \( f(n, x(n)) = e^{\frac{1}{2}(\sin n - 1)} x(n - 1) \), \( g(n - \tau(n), x(n - \tau(n))) = \frac{\cos n}{\sqrt{1+n^2}} x(n - \tau(n)) \) and \( \tau(n) = \cos^2 n + \tau, \tau \in \mathbb{N}_+ \).

**Proof.** Define \( V(n) = |x(n)|^2 \), it can be computed that

\[
EV(n) = e^{\sin(n-1)-1}EV(n - 1) + \frac{\cos^2 n}{1+n^2}EV(n - \tau(n)),
\]

which implies

\[
EV(n) \geq e^{\sin(n-1)-1}EV(n - 1), \quad n \in \mathbb{N}.
\]

Similar to the proof of Theorem 1, we can prove that

\[
E|x(n)|^2 \geq E|x(0)|^2 \sum_{s=0}^{\infty} (\sin(s-1)-1), \quad n \in \mathbb{N}.
\]

Thus, when \( n \to +\infty \), we have \( E|x(n)|^2 \to +\infty \), which implies that the trivial solution of (40) is exponentially unstable. The simulation results of system (40) with initial value \( \xi = [-1, 2.15]^T \) are displayed in Figure 2.

![Figure 2](image)

**Figure 2.** State trajectories of system (40) without impulsive.

On the other hand, choose an appropriate impulsive sequence to stabilize the system (40). Setting \( \zeta = \{n_k| n_k = n_{k-1} + 2, n_0 = 0, k \in \mathbb{N}_+ \} \), we can derive from systems (40) that

\[
x(n_k) = e^{-1}x(n_k - 1).
\]

In this case, the average impulsive interval satisfies \( T_a = 2 \) with the elasticity number \( N_0 = 0 \).

By (41), it can be deduced that

\[
EV(n) \leq e^{\sin(n-1)-1}EV(n - 1) + \frac{\cos^2 n}{1+n^2} \sup_{\theta \in [-\tau,-1]} EV(n + \theta).
\]
Thus, it is easy to check that condition (ii) in Theorem 2 holds with \( \alpha(n) = e^{(\sin(n-1)-1)} \) and \( \beta(n) = \cos^2 \frac{\pi n}{10} \). Furthermore, we have \( \hat{\alpha} = e^{-2}, \hat{\beta} = 1 \).

One can derive from systems (40) that

\[
\mathbb{E} \alpha(n) = \mathbb{E} |e^{-1}x(n_k - 1)|^2 = e^{-2} \mathbb{E} \alpha(n_k - 1),
\]

which implies that condition (iii) of Theorem 2 holds with \( \rho = e^{-2} \) and \( q = \min\{e^{-2}, 1\} = e^{-2} \). Selecting \( n^* = 0 \) and \( T^* = 2\pi \), such that

\[
\sum_{s=2k+1}^{2(k+1)} \{ \ln(\alpha(s) + \beta(s)) + \frac{\ln \rho}{I_d} \} \leq -1.06 \pi
\]

holds for any \( k \in \mathbb{N} \), which implies that condition (iv’) of Theorem 2 is satisfied with \( \sigma = 0.53 \) and \( \tau \in [1, 2.31] \).

By Theorem 2, we know that system (40) with impulsive control is exponentially stabilized in the mean square. The state sample trajectories are simulated in Figure 3. Figure 3 illustrates that under the stabilizing impulsive effects, all state trajectories of system (40) are forced to approach the trivial solution with an exponential decay rate.

**Remark 7.** In Example 2, the impulses are used to stabilize an unstable system. In this case, the impulses must be frequent enough, and their amplitude must be suitably related to the growth rate of the continuous flow.

![Figure 3. State trajectories of system (40) with impulsive.](image)

**5. Conclusions**

This paper has studied the stability analysis of discrete-time stochastic delay systems with impulses. The stability analysis is achieved with the help of the sums average value of the time-varying coefficients and the average impulsive interval. Some examples were also presented to illustrate the efficiency of the obtained results.

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