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# Several Limit Theorems on Fuzzy Quantum Space

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**Abstract:** The probability theory using fuzzy random variables has applications in several scientific disciplines. These are mainly technical in scope, such as in the automotive industry and in consumer electronics, for example, in washing machines, televisions, and microwaves. The theory is gradually entering the domain of finance where people work with incomplete data. We often find that events in the financial markets cannot be described precisely, and this is where we can use fuzzy random variables. By proving the validity of the theorem on extreme values of fuzzy quantum space in our article, we see possible applications for estimating financial risks with incomplete data.

**Keywords:** fuzzy quantum space; convergences on fuzzy quantum space; law of large numbers; central limit theorem; Fisher–Tippett–Gnedenko theorem; Balkema; de Haan–Pickands theorem



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## 1. Introduction

Selected limit theorems, which we shall deal with in the article, are well known from Kolmogorov's classical probability theory. Kolmogorov's work [1] introduced the theoretical axiomatic model in which events connected with the experiment form the Boolean $\sigma$-algebra of subsets $\mathcal{S}$ of the set $\Omega$. Thus, the probability is that for the $\sigma$-additive, nonnegative final function $P$ on $\mathcal{S}$, with values in the interval $[0,1]$, if $\{A_n\}$ is a sequence of mutually exclusive events from $\mathcal{S}$, then $P(\cup_n A_n) = \sum_n P(A_n)$ and $P(\Omega) = 1$. Limit theorems have a wide range of use in this theory. Their validity has already been proven for other structures (spaces), e.g., MV-algebras defined in [2]. We want to extend their use; therefore, in this article we prove that they also apply to sets in which we are working with incomplete data. Specifically, they also apply to Fuzzy quantum space, and that is the most significant finding in this article.

After some time, it became apparent that Kolmogorov's classical model of the probability theory was not sufficient for describing quantum mechanics situations. Birkhoff and von Neumann [3] referred to the fact that the set of experimentally verifiable statements about the quantum mechanical system does not have the same algebraic structure as Boolean algebra. Heisenberg [4] and Schrödinger [5] put forth the earliest attempts at the mathematical formulation of quantum mechanics. Schrödinger presented the formalism of wave mechanics, while Heisenberg proposed the formalism of matrix mechanics.

Zadeh [6] wrote about the theory of fuzzy sets in the 1960s. The current quantum theory, basic mathematical model is that of von Neumann, grounded in the geometry of Hilbert space (Varadarajan, [7]). If we define all closed subspaces of a given Hilbert space (where, according to Varadarajan, the notion "a state of system" means a measure of probability on $\mathcal{M}$) as system $\mathcal{M}$, and such a definition is compared with that of the $P$-measure on fuzzy sets (according to Piasecki [8]), it follows that both objects have a similar algebraic structure. Piasecki submitted a model called soft $\sigma$-algebra in the fuzzy set theory in 1985. His model demonstrated several characteristics identical to quantum logics. That comparison was first noted by Riečan [9] and then by Pykacz [10], and it led us to the idea to build a quantum theory based on fuzzy sets. If $A$ is a non-empty set called a universum and $\mathcal{M}$ is a system of fuzzy subsets of universum $A$, i.e., the system of functions on $A$ with values in the interval $[0,1]$, then according to Riečan [9] we say that $(A,\mathcal{M})$ is an $F$-quantum space, also referred to by Dvurečenskij and Chovanec [11] as a fuzzy quantum space, or by Dvurečenskij [12] as a fuzzy measurable space.

Many writers have attempted to prove some known assertions from the classical probability theory in the theory of fuzzy quantum spaces. For example, Dvurečenskij [12], Navara [13], and Navara and Pták [14,15] studied the existence of a fuzzy state on fuzzy quantum space while Dvurečenskij and Riečan [16,17] examined joint fuzzy observables and joint distributions of fuzzy observables. The representation theorem was proved by Dvurečenskij, Kôpka, and Riečan [18]; it also includes the case in fuzzy quantum space. Riečan [19,20] looked at the theory of an indefinite integral on fuzzy quantum space. Mesiar [21–23], Piasecki [24,25], and Piasecki and Svitalski [26] investigated the extension of the validity of the Bayes formula for fuzzy sets. Markechová [27–29] researched the entropy on fuzzy quantum space, and Tirpáková and Markechová [30] investigated the fuzzy analogies of some ergodic theorems and Birkhoff's individual ergodic theorem and maximal ergodic theorem for fuzzy dynamical systems [31].

The existence of the sum of fuzzy observables is a key fact for the analysis of many assertions in the fuzzy sets theory. The existence of the sum of compatible fuzzy observables was proved by Harman and Riečan [32].

Among the vital concepts of probability theory are the different kinds of convergence of random variables. They are especially significant for parts dealing with the validity of various forms of the law, the central limit theorem, and big numbers. As a result, the problem of generalizing different types of convergence for fuzzy quantum space $(A,\mathcal{M})$ became topical. A few authors studied particular types of convergences on quantum logic. Here, we mention the writings which were the basic material for the study of various types of convergences of fuzzy observables on fuzzy quantum space $(A,\mathcal{M})$: Dvurečenskij and Pulmanová [33], Jajte [34], Ochs [35,36], Cushen [37], Gudder [38], and Révesz [39]. Some types of convergences of fuzzy observables on fuzzy quantum space were dealt with by Dvurečenskij [40], Riečan [41,42], Chovanec and Kôpka [43], Kôpka and Chovanec [44], and others.

We formulated convergences and consequently proved many familiar limit theorems [34,36,45] for fuzzy quantum space on the basis of the analogy of the probability theory notions. As the central limit theorem refers to the limit distribution of the averages of independent, equally-distributed random variables, extreme value theory (EVT) addresses the limit distribution of the maximums of the independent, equally-distributed random variables [45,46]. EVT's principal objective is to know or predict the statistical probabilities of events that have never or rarely been observed. Kotz and Nadarajah [47] indicated that the extreme value distributions could be traced back to Bernoulli's 1709 work [48]. The theory of max-stable distribution functions, the counterpart of Feller stable distributions [49], formed the basis of the probability background. First, the statistical analysis of extreme values was performed in order to study flood levels. These days, the areas of application include finance, meteorological events, insurance, industry, or the environmental sciences [50]. Allow $X_1, X_2, \ldots, X_n$ to be a sequence of $n$, independently and identically distributed random variables with distribution function $F$. The corresponding ordered sequence in non-decreasing order is indicated by $X_{1:n}, X_{2:n}, \ldots,$ $X_{n:n}$, where $X_{i:n}, i = 1, \cdots, n$ represents the $i$-th order statistic. $X_{1:n}$ and $X_{n:n}$ stand for the sample minimum and the sample maximum, respectively. Then, examine the sequence of maxima $M_1 = X_1$, $M_n = X_{n:n} = \max(X_1, X_2, \ldots, X_n)$, for $n \geq 2$, obtained from the above sequence. All the sequence minimum results can be obtained from those of the sequence maximum since $m_n = \min(X_1, X_2, \ldots, X_n) = -\max(-X_1, -X_2, \ldots, -X_n)$. $M_n$'s exact distribution can be obtained from the distribution function $F$. In fact, for all $x \in \mathbb{R}$: $F_{M_n}(x) = P(M_n \leq x) = P(X_1 \leq x, X_2 \leq x, \cdots, X_n \leq x) = \prod_{i=1}^{n} P(X_i \leq x) = F^n(x)$. For a single process, the behavior of the maxima can be described by the three extreme value distributions: Gumbel, Fréchet and reversed Weibull distribution as suggested by the Fisher–Tippett–Gnedenko theorem. One can combine these three distributions into a single family of continuous cumulative distribution functions, known as the generalized extreme value (GEV) distributions [50]. A GEV can be identified by the real parameter $\gamma$ and the extreme value index, and as a stable distribution it is a characteristic exponent $\alpha \in [0,2]$. Subsequently, several researchers have provided useful applications of extreme value distributions. They may be found in several works [51–54].

## 2. Fuzzy Quantum Space

First, we recall the definitions of basic notions and some facts that will be used in the following text. In the quantum space approach to the fuzzy quantum theory, the triple $(\Omega, S, P)$ is replaced by the couple $(\mathbb{A}, \mathcal{M})$ where $\mathbb{A}$ is a nonempty set, $\mathcal{M} \subset [0,1]^{\mathbb{A}}$ is fuzzy $\sigma$-algebra of fuzzy subsets of $\mathbb{A}$, such that the following conditions are satisfied:

(i) if $1_{\mathbb{A}}(x) = 1$ for any $x \in \mathbb{A}$, then $1_{\mathbb{A}} \in \mathcal{M}$
(ii) if $f \in \mathcal{M}$, then $f'1_{\mathbb{A}} - f \in \mathcal{M}$
(iii) $\vee_{n=1}^{\infty} f_n \sup_n f_n \in \mathcal{M}$, for any $\{f_n\}_{n=1}^{\infty} \subset \mathcal{M}$
(iv) if $\frac{1}{2_{\mathbb{A}}}(x) = \frac{1}{2}$ for any $x \in \mathbb{A}$, then $\frac{1}{2_{\mathbb{A}}} \notin \mathcal{M}$

Elements of the set $\mathcal{M}$ are called fuzzy subsets of the universe $\mathbb{A}$. In particular, if $f$ is the characteristic function, we call it a crisp set. The symbols $\vee_{n=1}^{\infty} f_n \sup_n f_n$ and $\wedge_{n=1}^{\infty} f_n \inf_n f_n$ indicate a fuzzy union and a fuzzy intersection of the sequence of fuzzy sets $\{f_n\}_{n=1}^{\infty} \subset \mathcal{M}$, respectively. The event $f'1_{\mathbb{A}} - f \in \mathcal{M}$ is the so-called fuzzy complement. By Piasecki [8], the system $\mathcal{M}$ is called a soft $\sigma$-algebra.

To define and prove the law of large numbers and the central limit theorem, we need the following basic notions:

**Definition 1.** *A fuzzy state on fuzzy quantum space $(\mathbb{A}, \mathcal{M})$ is a mapping $m : \mathcal{M} \to [0,1]$, such that*

*(i)* $m(f \vee (1_{\mathbb{A}} - f)) = 1$ *for every* $f \in \mathcal{M}$
*(ii)* *if $\{f_k\}_{k=1}^{\infty}$ is a sequence of pairwise orthogonal fuzzy subsets from $\mathcal{M}$, i.e., $f_i \perp f_j$, $\left(f_i \leq 1_{\mathbb{A}} - f_j\right)$, whenever $i \neq j$, then $m\left(\vee_{k=1}^{\infty} f_k\right) = \sum_{k=1}^{\infty} m(f_k)$*

According to Piasecki [8], a fuzzy state is called the $\sigma$-measure. The triplet $(\mathbb{A}, \mathcal{M}, m)$ where $m$ is a $\sigma$-measure is called a fuzzy probability space. This structure was studied in [55,56].

For illustration, we give the following example of a nontrivial fuzzy quantum space [55].

**Example 1.** *Consider $(\mathbb{A}, \mathcal{M})$ where $\mathbb{A} = [0,1]$, $f : \mathbb{A} \to \mathbb{A}$, $f(x) = x$, $\mathcal{M} = \{f, f', f \vee f', f \wedge f', 0_{\mathbb{A}}, 1_{\mathbb{A}}\}$ for every $x \in \mathbb{A}$. It is evident that $f \vee f' \neq 1_{\mathbb{A}}$. We define the mapping $m : \mathcal{M} \to [0,1]$ by the equalities $m(1_{\mathbb{A}}) = m(f \vee f') = 1$, $m(0_{\mathbb{A}}) = m(f \wedge f') = 0$, and $m(f) = m(f') = \frac{1}{2}$. Then the triplet $(\mathbb{A}, \mathcal{M}, m)$ is a fuzzy probability space.*

**Definition 2.** *A fuzzy observable on fuzzy quantum space $(\mathbb{A}, \mathcal{M})$ maps to $x : \mathcal{B}(\mathbb{R}^1) \to \mathcal{M}$, satisfying the following properties:*

*(i)* $x(E^c) = 1_{\mathbb{A}}(x) - x(E)$ *for every* $E \in \mathcal{B}(\mathbb{R}^1)$
*(ii)* if $\{E_n\}_{n=1}^{\infty} \subset \mathcal{B}(\mathbb{R}^1)$, *then* $x\left(\cup_{n=1}^{\infty} E_n\right) = \vee_{n=1}^{\infty} x(E_n)$

*where $\mathcal{B}(\mathbb{R}^1)$ denotes the Borel $\sigma$-algebra of the real line $\mathbb{R}^1$ and $E^c$ denotes the complement of a set $E$ in $\mathbb{R}^1$.*

**Definition 3.** *Let $f \in \mathcal{M}$. The mapping $x_f : \mathcal{B}(\mathbb{R}^1) \to \mathcal{M}$ is defined by*

$$x_f(E) = \begin{cases} f \wedge f', & if\ 0,1 \notin E \\ f', & if\ 0 \in E, 1 \notin E \\ f, & if\ 0 \notin E, 1 \in E \\ f \vee f', & if\ 0,1 \in E \end{cases}$$

*for every $E \in \mathcal{B}(\mathbb{R}^1)$ there is a fuzzy observable of fuzzy quantum space $(\mathbb{A}, \mathcal{M})$ called the indicator of fuzzy set $f \in \mathcal{M}$.*

In particular, the null fuzzy observable of fuzzy quantum space $(\mathbb{A}, \mathcal{M})$ maps to $o : \mathcal{B}(\mathbb{R}^1) \to \mathcal{M}$ defined by

$$o(E) = \begin{cases} 0_{\mathbb{A}}, & \text{if } 0 \notin E \\ 1_{\mathbb{A}}, & \text{if } 0 \in E \end{cases}$$

where $E \in \mathcal{B}(\mathbb{R}^1)$.

If $\tau : \mathbb{R}^1 \to \mathbb{R}^1$ is a Borel measurable function and $x$ is a fuzzy observable, then $\tau \circ x : E \to x(\tau^{-1}(E)), E \in \mathcal{B}(\mathbb{R}^1)$ is a fuzzy observable, too. In this way, we define the functional calculus of fuzzy observables. For example, if $\tau(t) = t^2, t \in \mathbb{R}^1$, we write $\tau \circ x = x^2$ and the like. In particular, if $a \in \mathbb{R}^1$, then $ax : E \to x(\{t \in \mathbb{R}^1 : at \in E\})$ for any $E \in \mathcal{B}(\mathbb{R}^1)$.

Let $x$ be a fuzzy observable of fuzzy quantum space $(\mathbb{A}, \mathcal{M})$ and let $B_x(t) = x((-\infty, t))$, $t \in \mathbb{R}^1$. Dvurečenskij and Tirpáková [57] proved that the system $\{B_x(t) : t \in \mathbb{R}^1\}$ of fuzzy sets of fuzzy quantum space $(\mathbb{A}, \mathcal{M})$ is a one-to-one correspondence to fuzzy observable $x$. Due to this result, the sum of any pair $x$ and $y$ of fuzzy observables of $(\mathbb{A}, \mathcal{M})$ can be introduced as follows:

**Definition 4.** *Let $x$ and $y$ be two fuzzy observables of fuzzy quantum space $(\mathbb{A}, \mathcal{M})$. If the system $\{B_{x+y}(t) : t \in \mathbb{R}^1\}, B_{x+y}(t) = \vee_{r \in \mathbb{Q}}(B_x(r) \wedge B_y(t-r)), t \in \mathbb{R}^1$, where $\mathbb{Q}$ is the set of all rational numbers, then we determine fuzzy observable $z$ of $(\mathbb{A}, \mathcal{M})$. We call it the sum of $x$ and $y$ and write $z = x + y$.*

In the text [57], it was proved that the sum of two observables always exists, and it coincides with the pointwisely-defined sum of observables for $\sigma$-algebra of crisp subsets. Moreover, $x + y = y + x$, $(x + y) + z = x + (y + z)$ for fuzzy observables $x$, $y$, and $z$. The subtraction of fuzzy observables $x$ and $y$ is defined as follows: $x - y = x + (-y)$, where $(-y)(E) = y(\{t : -t \in E\}), E \in \mathcal{B}(\mathbb{R}^1)$. The mean value of a fuzzy observable on fuzzy quantum space $(\mathbb{A}, \mathcal{M})$ was defined by Riečan [58] as follows: Let $x$ be a fuzzy observable, and let $m$ be a fuzzy state. If the integral $m(x) \int\limits_{\mathbb{R}^1} t dm_x(t)$ exists, then $m(x)$ is called the mean value of $x$ in $m$, where $m_x : E \to m(x(E)), E \in \mathcal{B}(\mathbb{R}^1)$ is a probability measure on $\mathcal{B}(\mathbb{R}^1)$. In addition, if $u$ is a Borel measurable function, then $m(u \circ x) \int\limits_{\mathbb{R}^1} u(t) dm_x(t)$ in the sense that if one side exists, then the second side exists too, and they are equal. Specially, if $u(t) = (t - m(x))^2$, then $D(x)m\left((x - m(x))^2\right)$ is called the dispersion of fuzzy observable $x$ in fuzzy state $m$.

## 3. Convergences on a Fuzzy Space

Various types of convergences of random variables belong among important concepts of the probability theory. Therefore, the notion of a fuzzy observable is an analogy to the notion of a random variable. When defining different types of convergence and for the proof of limit theorems on fuzzy quantum space $(\mathbb{A}, \mathcal{M})$, we used the method of $F$-$\sigma$-ideals, which enabled us to reformulate and prove many of the known limit theorems of the classical probability theory for the fuzzy quantum space $(\mathbb{A}, \mathcal{M})$. The basic idea of the $F$-$\sigma$-ideals method is described in [54], and we can shortly describe it as follows: Let $m$ be a fuzzy state on fuzzy quantum space $(\mathbb{A}, \mathcal{M})$. Denote $I_m = \{f \in \mathcal{M} : m(f) = 0\}$. Dvurečenskij and Riečan [16] proved that $I_m$ is a $\sigma$-algebra of fuzzy quantum space $(\mathbb{A}, \mathcal{M})$. The relation "$\sim_m$" defined on fuzzy quantum space $(\mathbb{A}, \mathcal{M})$ via $f \sim_m g$ if and only if $m(f \wedge g') = 0 = m(g \wedge f')$ is the congruence, and, moreover, $\mathcal{M}/\sim_m = \left\{\overline{f} = \{g \in \mathcal{M} : g \sim_m f\} : f \in \mathcal{M}\right\}$ is the Boolean $\sigma$-algebra (in the sense of Sikorski [59]), where complementation "$'$" in $\mathcal{M}/\sim_m$ is defined with properties $\left(\overline{f}\right)' = \overline{f'}$, $f \in \mathcal{M}$ and $\bigvee_i \overline{f_i} \overline{\bigvee_i f_i}$, $\bigwedge_i \overline{f_i} \overline{\bigwedge_i f_i}$, $f_i \subset \mathcal{M}$). Then, according to these properties, the mapping $h : \mathcal{M} \to \mathcal{M}/\sim_m$ defined by $h(f) = \overline{f}, f \in \mathcal{M}$ is a $\sigma$-homomorphism from $\mathcal{M}$ onto $\mathcal{M}/\sim_m$. The mapping $\mu$ from a Boolean $\sigma$-algebra $\mathcal{M}/\sim_m$ into the interval $[0,1]$, defined by $\mu\left(\overline{f}\right) = m(f)$ for every $\overline{f} \in \mathcal{M}/\sim_m$, is a probability measure on the Boolean $\sigma$-algebra $\mathcal{M}/\sim_m$. According to the Loomis–Sikorski theorem in [60], there is a measurable space $(\Omega, \mathcal{S})$ and $\sigma$-homomorphism $\varphi$ from $\mathcal{S}$ onto $\mathcal{M}/\sim_m$, and due to Varadarajan [7] there are functions $u, u_1, u_2, \ldots : \Omega \to \mathbb{R}^1$, such that

$$\varphi\left(u_i^{-1}(E)\right) = h \circ x_i(E), i = 1, 2, \ldots, \ E \in \mathcal{B}\left(\mathbb{R}^1\right) \tag{1}$$

$$\varphi\left(u^{-1}(E)\right) = h \circ x(E) \tag{2}$$

where $h \circ x$ is an observable of a Boolean $\sigma$-algebra $\mathcal{M}/\sim_m$. Moreover, mapping $\mu_\varphi : \mathcal{S} \to [0,1]$, defined as $\mu_\varphi(\Lambda) = \mu(\varphi(\Lambda))$, $\Lambda \in \mathcal{S}$, is a probability measure on $\mathcal{S}$.

Gudder and Mullikin [51] introduced many types of convergences for observables in quantum logics. Inspired by their definition, Dvurečenskij and Tirpaková introduced [57] the following definition:

**Definition 5.** *We say that sequence $\{x_n\}_{n=1}^{\infty}$ of fuzzy observables on fuzzy quantum space $(\mathbb{A}, \mathcal{M})$ converges to fuzzy observable $x$*

*(i)* *in fuzzy state $m$, if for every $\varepsilon > 0$, we have*

$$\lim_{n\to\infty} m((x_n - x)([-\varepsilon, \varepsilon])) = 1$$

*(ii)* *almost everywhere in the fuzzy state $m$*, if for every $\varepsilon > 0$, *we have*

$$m\left(\bigvee_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} ((x_n - x)([-\varepsilon, \varepsilon]))\right) = 1$$

*(iii)* *everywhere, if*

$$\bigwedge_{p=1}^{\infty} \bigvee_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} \left((x_n - x)\left(\left[-\frac{1}{p}, \frac{1}{p}\right]\right)\right) = 1_{\mathbb{A}}$$

*(iv)* *in a mean $p$, where $1 \le p < \infty$, if*

$$\lim_{n\to\infty} \left(m|\overline{x}_n - \overline{x}|^p\right) = \overline{0}$$

*(v)* *everywhere on $f$, if*

$$f \le \bigwedge_{p=1}^{\infty} \bigvee_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} \left((x_n - x)\left(\left[-\frac{1}{p}, \frac{1}{p}\right]\right)\right)$$

*(vi)* *uniformly on $\overline{f} \in \mathcal{M}$, if for every $\varepsilon > 0$, there is an integer $n_0$, such that*

$$\forall n \ge n_0 : (x_n - x)([-\varepsilon, \varepsilon]) \ge \overline{f}$$

*(vii)* *uniformly, if for every $\varepsilon > 0$, there is an integer $n_0$, such that*

$$\forall n \ge n_0 : (x_n - x)([-\varepsilon, \varepsilon]) = \overline{1}$$

*(viii) almost uniformly in fuzzy state m, if for every $\varepsilon > 0$ there is an element $f \in \mathcal{M}$, such that $m(f') \leq \varepsilon$ and a sequence $\{x_n\}_{n=1}^{\infty}$ converges uniformly to x on f.*

To prove the law of large numbers and the central limit theorem on fuzzy quantum space $(\mathbb{A}, \mathcal{M})$, we also need the next theorem, which was proved in Dvurečenskij and Tirpaková [57].

**Theorem 1.** *Let $m$ be a fuzzy state of fuzzy quantum space $(\mathbb{A}, \mathcal{M})$, $x, x_1, x_2, \ldots$ be fuzzy observables of $(\mathbb{A}, \mathcal{M})$, and $u, u_1, u_2, \ldots$ be functions with properties (1) and (2). Then,*

*(A) The sequence of fuzzy observables $\{x_n\}_{n=1}^{\infty}$ converges to fuzzy observable $x$*
- *(i) in fuzzy state m if and only if the sequence of functions $\{u_n\}_{n=1}^{\infty}$ converges to u in measure $\mu_\varphi$,*
- *(ii) almost uniformly in fuzzy state m if and only if the sequence of functions $\{u_n\}_{n=1}^{\infty}$ converges almost uniformly to u in measure $\mu_\varphi$,*
- *(iii) almost everywhere in fuzzy state m if and only if the sequence of functions $\{u_n\}_{n=1}^{\infty}$ converges almost everywhere to u in measure $\mu_\varphi$,*
- *(iv) in mean p $(1 \leq p \leq \infty)$ if and only if $\{u_n\}_{n=1}^{\infty}$ converges to u in mean p in measure $\mu_\varphi$.*

*(B) If the sequence of fuzzy observables $\{x_n\}_{n=1}^{\infty}$ converges to fuzzy observable $x$*
- *(v) everywhere, then there is $\Lambda \in \mathcal{S}$ such that $\varphi(\Lambda) = \overline{1}_{\mathbb{A}}$ and the sequence $\{u_n\}_{n=1}^{\infty}$ converges to u everywhere on $\Lambda$,*
- *(vi) uniformly, then there is $\Lambda \in \mathcal{S}$ such that $\varphi(\Lambda) = \overline{1}_{\mathbb{A}}$ and the sequence $\{u_n\}_{n=1}^{\infty}$ converges to u uniformly on $\Lambda$,*
- *(vii) uniformly on $f \in \mathcal{M}$, then there is $\Lambda \in \mathcal{S}$ such that $\varphi(\Lambda) \geq \overline{f}$ and the sequence $\{u_n\}_{n=1}^{\infty}$ converges to u uniformly on $\Lambda$.*

*Conversely, if the sequence of functions $\{u_n\}_{n=1}^{\infty}$ defined by (1), (2) converges to u*

*(viii) everywhere, then $\{x_n\}_{n=1}^{\infty}$ converges to fuzzy observable x everywhere on $f \in \mathcal{M}$, where $\overline{f} = \overline{1}_{\mathbb{A}}$,*

*(ix) uniformly, then $\{x_n\}_{n=1}^{\infty}$ converges to fuzzy observable x uniformly on $f \in \mathcal{M}$, where $\overline{f} = \overline{1}_{\mathbb{A}}$*

*(x) uniformly on $\Lambda$, $\Lambda \in \mathcal{S}$, then fuzzy observables $\{x_n\}_{n=1}^{\infty}$ converges to fuzzy observable x uniformly on $f \in \mathcal{M}$, where $\overline{f} = \varphi(\Lambda)$.*

In the following, we will continue to introduce the notion of the independence of fuzzy observables $\{x_n\}_{n=1}^{\infty}$ in fuzzy state $m$. Now, we define the joint fuzzy observable of fuzzy observables.

**Definition 6.** *Let $x_1, x_2, \ldots, x_n$, $n \geq 2$, be a finite system of fuzzy observables on fuzzy quantum space $(\mathbb{A}, \mathcal{M})$. A joint fuzzy observable of fuzzy observables $x_1, x_2, \ldots, x_n$ is a $\sigma$-homomorphism $T_n : \mathcal{B}(\mathbb{R}^n) \to \mathcal{M}$, such that*

*(i) $T_n(A^c) = T_n(A)'$ for every $A \in \mathcal{B}(\mathbb{R}^n)$*

*(ii) $T_n\left(\cup_{i=1}^{n} A_i\right) = \wedge_{i=1}^{n} T_n(A_i)$, $A_i \in \mathcal{B}(\mathbb{R}^n)$, $i = 1, 2, \ldots, n$*

*(iii) $T_n\left(\pi_i^{-1}(E)\right) = x_i(E)$ for every $i \in \{1, 2, \ldots, n\}$, $E \in \mathcal{B}(\mathbb{R}^1)$*

*where $\pi_i : \mathbb{R}^n \to \mathbb{R}^1$ is the projection into the i-th coordinate.*

In accordance with Riečan [61] and Riečan and Neubrunn [62], a sufficient condition for the existence of the joint fuzzy observable of fuzzy observables $x_1, x_2, \ldots, x_n$, $n \geq 2$, meets the condition $x_i(\varnothing) = x_j(\varnothing)$ for every $i, j \in \{1, 2, \ldots, n\}$.

**Definition 7.** *Fuzzy observables $x_1, x_2, \ldots, x_n$ on fuzzy quantum space $(\mathbb{A}, \mathcal{M})$ are independent in fuzzy state $m$ if for every $n \geq 2$ there exists joint fuzzy observable $T_n$ and*

$$m(T_n(E_1 \times E_2 \times \cdots \times E_n)) = \prod_{i=1}^{n} m(x_i(E_i))$$

*for any $E_i \in \mathcal{B}(\mathbb{R}^1), i = 1, 2, \ldots, n$.*

According to the assumption of independence of the sequence of fuzzy observables $\{x_n\}_{n=1}^{\infty}$, for every $n \geq 2$ there exists joint fuzzy observable $T_n$. To each fuzzy observable $x_i : \mathcal{B}(\mathbb{R}^1) \to \mathcal{M}, i = 1, 2, \ldots, n$ exists the observable $\overline{x_i} = h \circ x_i : \mathcal{B}(\mathbb{R}^1) \to \mathcal{M}/\sim_m$ and a real function $u_i : \Omega \to \mathbb{R}^1$, such that $\overline{x_i}(E) = \varphi\left(u_i^{-1}(E)\right)$. We define function $\Phi : \Omega \to \mathbb{R}^1$, such that $\Phi_n(\omega) = (u_1(\omega), u_2(\omega), \ldots, u_n(\omega)), \omega \in \Omega$. If $\overline{T_n} = h \circ T_n : \mathcal{B}(\mathbb{R}^n) \to \mathcal{M}/\sim_m$, then $\overline{T_n} = \varphi \circ \Phi_n^{-1}$. The main idea of the proof can be illustrated by Figure 1.

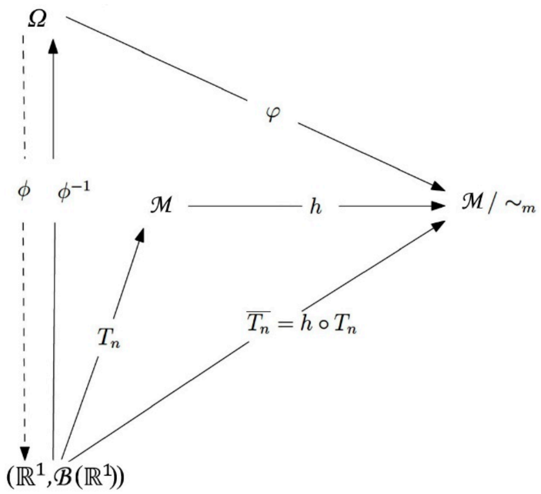


**Figure 1.** The main idea of the proof of the central limit theorem.

## 4. Limit Theorems for Fuzzy Quantum Space

**Theorem 2 (Central limit theorem).** *Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of independent fuzzy observables, identically distributed in fuzzy state m, with mean value a and variance $\sigma^2 \in (0, \infty)$. Then, for any $s \in \mathbb{R}^1$, the following equality holds:*

$$\lim_{n \to \infty} m\left(\frac{1}{\sigma\sqrt{n}} \sum_{i=1}^{n} (x_i - na)(-\infty, s)\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{s} e^{\frac{-t^2}{2}} dt$$

**Proof**. We define the real function $k_n : \mathbb{R}^n \to \mathbb{R}^1$ as follows

$$k_n(r_1, r_2, \ldots, r_n) = \frac{1}{\sigma\sqrt{n}} \sum_{i=1}^{n} (r_i - na), r_i \in \mathbb{R},\ i = 1, 2, \ldots, n.$$

Calculate:

$$m\left(\tfrac{1}{\sigma\sqrt{n}} \textstyle\sum_{i=1}^{n} (x_i - na)(-\infty, s)\right) = m\left(T_n\left(k_n^{-1}((-\infty, s))\right)\right) = \mu\left(\overline{T_n}\left(k_n^{-1}((-\infty, s))\right)\right) =$$
$$= \mu\left(\varphi \circ \Phi_n^{-1}\left(k_n^{-1}((-\infty, s))\right)\right) = \mu_\varphi\left(\Phi_n^{-1}\left(k_n^{-1}((-\infty, s))\right)\right) = \mu_\varphi(\{\omega : k_n(\Phi_n(\omega)) < s\}),$$

where

$$k_n(\Phi_n(\omega)) = \frac{1}{\sigma\sqrt{n}} \sum_{i=1}^{n} (u_i - na).$$

Regarding Theorem 1.7.5 [63], the validity of the following argument is obvious:

$$\lim_{n\to\infty} m\left(\frac{1}{\sigma\sqrt{n}} \sum_{i=1}^{n} (x_i - na)(-\infty, s)\right) = \lim_{n\to\infty} \mu_\varphi(\{\omega : k_n(\Phi_n(\omega)) < s\}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{s} e^{\frac{-t^2}{2}} dt. \square$$

**Theorem 3 (Weak law of large numbers).** *Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of independent fuzzy observables, identically distributed in fuzzy state m, with the mean value a. Then,*

$$\frac{x_1 + x_2 + \ldots + x_n}{n} - a$$

*converges to null fuzzy observable o in fuzzy state m.*

**Proof**. We define real function $k_n : \mathbb{R}^{\mathrm{n}} \to \mathbb{R}^1$ as follows

$$k_n(r_1, r_2, \ldots, r_n) = \frac{1}{n} \sum_{i=1}^{n} r_i - a, r_i \in \mathbb{R}, i = 1, 2, \ldots, n.$$

If $T_n$ is the joint fuzzy observable of fuzzy observables $x_1, x_2, \ldots, x_n$, then we define fuzzy observable $y_n = T_n \circ k_n^{-1}$. According to Definition 5 (i) a sequence of fuzzy observables $\{y_n\}_{n=1}^{\infty}$ converges to null fuzzy observable $o$ in fuzzy state $m$, if for every $\varepsilon > 0$ it holds that

$$\sum_{i=1}^{\infty} m(y_n(-\varepsilon, \varepsilon)) = 1.$$

Calculate:

$$\begin{aligned} &m(y_n((-\varepsilon, \varepsilon))) = m\big(T_n\big(k_n^{-1}((-\varepsilon, \varepsilon))\big)\big) = \mu\big(\overline{T_n}\big(k_n^{-1}((-\varepsilon, \varepsilon))\big)\big) = \\ &= \mu\Big(\varphi \circ \Phi_n^{-1}\big(k_n^{-1}((-\varepsilon, \varepsilon))\big)\Big) = \mu_\varphi\Big(\Phi_n^{-1}\big(k_n^{-1}((-\varepsilon, \varepsilon))\big)\Big). \end{aligned}$$

If $k_n(\Phi_n(\omega)) = \frac{1}{n} \sum_{i=1}^{n} u_i(\omega) - a$, then according to Theorem 1 A) (i) the sequence of fuzzy observables $\{y_n\}_{n=1}^{\infty}$ converges to null fuzzy observable $o$ in fuzzy state $m$ if and only if the sequence of the functions $\{k_n(\Phi_n)\}_{n=1}^{\infty}$ converges to null in measure $\mu_\varphi$. According to Theorem 1.10.3 [61], the validity of the arguments is obvious. □

**Theorem 4 (Strong law of large numbers).** *Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of fuzzy observables independent in fuzzy state m, such that*

$$\sum_{i=1}^{\infty} \frac{1}{i^2} D(x_i) < \infty.$$

*Then,*

$$\frac{1}{n} \sum_{i=1}^{n} (x_i - m(x_i))$$

*converges to null fuzzy observable o almost everywhere in fuzzy state m.*

**Proof**. We define real function $q_n : \mathbb{R}^{\mathrm{n}} \to \mathbb{R}^1$ as follows:

$$q_n(r_1, r_2, \ldots, r_n) = \frac{1}{n}(r_1 - E(r_1) + r_2 - E(r_2) + \ldots + r_n - E(r_n)),$$

where $E(r_i)$ is the mean value $r_i$ defined as $E(r_i) = \int_{\mathbb{R}^1} t d\mu_\varphi(t), r_i \in \mathbb{R}, i = 1, 2, \ldots n$. Then, $q_n(\Phi_n(\omega)) = \frac{1}{n}(\sum_{i=1}^{n}(u_i(\omega) - E(u_i(\omega))))$. If $T_n$ is a joint fuzzy observable of fuzzy observables $x_1, x_2, \ldots, x_n$, then we define fuzzy observable $z_n = T_n \circ q_n^{-1} = \frac{1}{n}(\sum_{i=1}^{n}(x_i - m(x_i)))$. According to Definition 5 (ii), a sequence of fuzzy observables $\{z_n\}_{n=1}^{\infty}$ converges to null fuzzy observable $o$ almost everywhere in fuzzy state $m$ if for any $\varepsilon > 0$ holds:

$$m(\vee_{k=1}^{\infty} \wedge_{n=k}^{\infty} (z_n([-\varepsilon, \varepsilon]))) = 1.$$

Then, according to Theorem 1 A) (iii), a sequence of fuzzy observables $\{z_n\}_{n=1}^{\infty}$ converges to null fuzzy observable $o$ almost everywhere in fuzzy state $m$ if and only if the sequence of functions $\{q_n(\Phi_n)\}_{n=1}^{\infty}$ converges to null almost everywhere in measure $\mu_\varphi$. According to Theorem 1.10.5 [61], the validity of the arguments is obvious. □

## 5. Extreme Value Theorems for Fuzzy Quantum Space

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of independent, identically-distributed fuzzy observables of fuzzy quantum space $(\mathbb{A}, \mathcal{M})$. For any $n \geq 1$ we define the real function $k_n : \mathbb{R}^n \to \mathbb{R}^1$ as follows:

$$k_n(r_1, r_2, \ldots, r_n) = \max\{r_1, r_2, \ldots, r_n\}.$$

Let $T_n$ be the joint fuzzy observable of fuzzy observables $x_1, x_2, \ldots, x_n$. We define the maximum fuzzy observables of $x_1, x_2, \ldots, x_n$ as

$$M_1 = x_1,\ M_n = \max\{x_1, x_2, \ldots, x_n\} = T_n \circ k_n^{-1},\ n \geq 2,$$

where $M_n$ is the fuzzy observable.

**Theorem 5.** *Let $(\xi_n)_{n=1}^{\infty}$ be a sequence of independent random variables with the same distribution function $F : \mathbb{R} \to [0, 1]$ in fuzzy state m. Put $\eta_n = \max(\xi_1, \cdots, \xi_n), n = 1, 2, \cdots$. Let there exist $a_n > 0$, $b_n \in \mathbb{R}$ such that*

$$\lim_{n\to\infty} m\left(\left\{\omega; \frac{\eta_n(\omega) - b_n}{a_n} < x\right\}\right) = \lim_{n\to\infty} F^n(a_n x + b_n) = H(x),$$

*where $H : \mathbb{R} \to [0, 1]$ is a continuous distribution function, increasing on an interval. Then, H has one of three distributions with parameters $\mu, \sigma, \alpha > 0$ (Figure 2):*

1. *Gumbel* $H_{\mu,\sigma}(x) = \exp\left(-e^{-(\frac{x-\mu}{\sigma})}\right),\ x \in \mathbb{R}$
2. *Fréchet* $H_{\mu,\sigma,\alpha}(x) = \begin{cases} 0, & \text{for } x \leq \mu \\ \exp\left(-\left(\frac{x-\mu}{\sigma}\right)^{-\alpha}\right), & \text{for } x > \mu, \alpha > 0 \end{cases}$
3. *Weibull* $H_{\mu,\sigma,\alpha}(x) = \begin{cases} \exp\left(-\left(-\frac{x-\mu}{\sigma}\right)^{\alpha}\right), & \text{for } x \leq \mu, \alpha > 0 \\ 1, & \text{for } x > \mu \end{cases}$

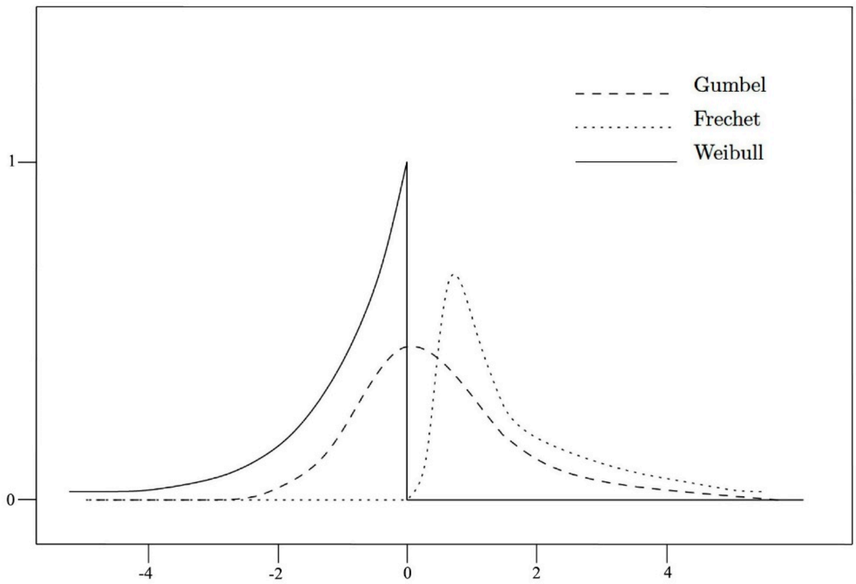


**Figure 2.** Densities of the standard extreme value distribution .

Proof of Theorem 5 can be found in [63].

**Theorem 6 (Fisher–Tippett–Gnedenko theorem).** *Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of fuzzy independent observables identically distributed in fuzzy state m. Let there exist norming constants $a_n > 0$, $b_n \in \mathbb{R}$ and some non-degenerate distribution function H such that*

$$\lim_{n\to\infty} m(\frac{1}{a_n}(M_n - b_n)(-\infty, t)) = H(t) \text{ for any } t \in \mathbb{R}.$$

*Then, H belongs to the type of one of the following three types of standard extreme value distributions: Gumbel, Fréchet, or Weibull.*

**Proof**. We define the real function $q_n : \mathbb{R}^{\mathrm{n}} \to \mathbb{R}^1$ as follows:

$$q_n(r_1, r_2, \ldots, r_n) = \frac{1}{a_n}(\max\{r_1, r_2, \ldots, r_n\} - b_n), \;\; r_i \in \mathbb{R}^1, \; i = 1, 2, \ldots, n.$$

Then $q_n(\Phi_n(\omega)) = \frac{1}{a_n}(\max\{u_1(\omega), u_2(\omega), \ldots, u_n(\omega)\} - b_n)$ and

$$m\Big(\tfrac{1}{a_n}(M_n - b_n)(-\infty, t)\Big) = m\big(T_n\big(q_n^{-1}((-\infty, t))\big)\big) = \mu\big(\overline{T_n}\big(q_n^{-1}((-\infty, t))\big)\big) =$$
$$= \mu\Big(\varphi \circ \Phi_n^{-1}\big(q_n^{-1}((-\infty, t))\big)\Big) = \mu_\varphi\Big(\Phi_n^{-1}\big(q_n^{-1}((-\infty, t))\big)\Big) = \mu_\varphi(\{\omega : q_n(\Phi_n(\omega)) < t\}).$$

We have

$$H(t) = \lim_{n\to\infty} m\left(\frac{1}{a_n}(M_n - b_n)(-\infty, t)\right) = \lim_{n\to\infty} \mu_\varphi(\{\omega : q_n(\Phi_n(\omega)) < t\}).$$

Then, according to Theorem 5, the validity of the arguments is obvious. □

Now, we define the distribution function and excess distribution function on fuzzy quantum space $(\mathbb{A}, \mathcal{M})$.

**Definition 8.** *Let* $m : \mathcal{M} \to [0,1]$ *be a fuzzy state and* $x : \mathcal{B}(\mathbb{R}^1) \to \mathcal{M}$ *be a fuzzy observable on fuzzy quantum space* $(\mathbb{A}, \mathcal{M})$*. For any* $t \in \mathbb{R}^1$ *we define function* $F_x : \mathbb{R}^1 \to [0,1]$ *as*

$$F_x(t) = m(x((-\infty, t))).$$

*Function* $F_x$ *is called the distribution function of an observable* $x$ *on fuzzy quantum space* $(\mathbb{A}, \mathcal{M})$*.*

**Proposition 1.** *If the function* $F_x$ *is the distribution function of an observable* $x$ *on fuzzy quantum space* $(\mathbb{A}, \mathcal{M})$*, then it satisfies the following conditions:*

*(i)* $F_x$ *is non-decreasing*
*(ii)* $F_x$ *is left continuous*
*(iii)* $\lim\limits_{n\to\infty} F_x = 1$
*(iv)* $\lim\limits_{n\to-\infty} F_x = 0$

**Proof.**

(i) Let $s < t,\ s, t \in \mathbb{R}^1$, then $x((-\infty, s)) \le x((-\infty, t))$, it follows that

$$F_x(s) = m(x((-\infty, s))) \le m(x((-\infty, t))) = F_x(t).$$

We proved that the function $F_x$ is non-decreasing.

(ii) Let $t_n \nearrow t,\ t_n, t \in \mathbb{R}^1,\ n = 1, 2, \dots$, then $x((-\infty, t_n)) \nearrow x((-\infty, t))$, it follows that

$$F_x(t_n) = m(x((-\infty, t_n))) \nearrow m(x((-\infty, t))) = F_x(t).$$

We proved that the function $F_x$ is left continuous.

(iii) Let $t_n \nearrow \infty,\ t_n \in \mathbb{R}^1,\ n = 1, 2, \dots$, then $x((-\infty, t_n)) \nearrow x((-\infty, \infty)) = 1_{\mathbb{A}}$, it follows that

$$F_x(t_n) = m(x((-\infty, t_n))) \nearrow m(x((-\infty, \infty))) = m(1_{\mathbb{A}}) = 1.$$

We proved that $\lim\limits_{n\to\infty} F_x = 1$.

(iv) Let $t_n \searrow -\infty,\ t_n \in \mathbb{R}^1,\ n = 1, 2, \dots$, then $x((-\infty, t_n)) \searrow x((-\infty, -\infty)) = 0_{\mathbb{A}}$, it follows that

$$F_x(t_n) = m(x((-\infty, t_n))) \searrow m(x((-\infty, -\infty))) = m(0_{\mathbb{A}}) = 0.$$

We proved that $\lim\limits_{n\to-\infty} F_x = 0$. □

Now we define the excess distribution function $\widetilde{F}_w$ on fuzzy quantum space $(\mathbb{A}, \mathcal{M})$.

**Definition 9.** *For* $w > 0$ *we define excess distribution function* $\widetilde{F}_w$ *on fuzzy quantum space* $(\mathbb{A}, \mathcal{M})$ *as*

$$\widetilde{F}_w(t) = \frac{\widetilde{F}(t+w) - \widetilde{F}(w)}{1 - \widetilde{F}(w)}$$

*for every* $0 < t < \omega\left(\widetilde{F}\right) = \sup\left\{t; \widetilde{F}(t) < 1\right\}$*. Value* $\omega\left(\widetilde{F}\right)$ *is called the right endpoint of distribution function* $\widetilde{F}$*.*

**Theorem 7 (Balkema, de Haan–Pickands).** *For a sufficiently large* $w$*, the excess distribution* $\widetilde{F}_w$ *converges to the generalized pareto distribution. Parameter* $\beta = \beta(w)$ *is dependent on threshold* $w$*, and for every* $\alpha > 0$

$$\lim_{w\to\omega(F_x)} \sup_{0\le t\le\omega(F_x)-w} \left|\widetilde{F}_w(t) - G_{\alpha,\beta(w)}(t)\right| = 0.$$

**Proof**. Let $x_i : \mathcal{B}(\mathbb{R}^1) \to \text{M}, i = 1, 2, \ldots, n$ be fuzzy observables. Then, there exist observables $\overline{x_i} = h \circ x_i : \mathcal{B}(\mathbb{R}^1) \to \mathcal{M}/I_m$ and real functions $u_i : \Omega \to \mathbb{R}^1$, such that $\overline{x_i}(E) = \varphi\left(u_i^{-1}(E)\right)$. Then,

$$\widetilde{F}(t) = m(x((-\infty, t))) = \mu(\overline{x}_i((-\infty, t))) = \mu\left(\varphi \circ u_n^{-1}((-\infty, t))\right) = \\ = \mu_\varphi\left(u_n^{-1}((-\infty, t))\right) = \mu_\varphi(\{\omega : u_n(\omega) < t\}) = F(t),\ t \in \mathbb{R}$$

is the distribution function of real random variable $u$. It is obvious that

$$\widetilde{F}_w(t) = \frac{\widetilde{F}(t+w) - \widetilde{F}(w)}{1 - \widetilde{F}(w)} = \frac{F(t+w) - F(w)}{1 - F(w)} = F_w(t).$$

□

## 6. Conclusions

The seminal theoretical results in probability theory are limit theorems. When using random samples to estimate distributional parameters, we would like to know that as the sample size gets larger, the estimates are probably close to the parameters that they are estimating. In statistical inference, the central limit theorem is the dominant and most useful theorem. It allows us to make the assumption that, for a population, a normal distribution will occur regardless of what the initial distribution looks like for a sufficiently large sample size. When the distribution shape is not known or the population is not normally distributed, the theorem is used to make assumptions. The law of large numbers is an invaluable tool that is expected to state definite things about the real-world results of unexpected events. The law of large numbers is the postulate of statistics and probability theory that states that the greater the number of samples are used from an event, the closer the monitoring results will be to the average population. Thus, the law of large numbers describes the stability of big random variables. Both the strong and weak laws refer to the convergence of the sample mean to the population mean as the sample size gets bigger.

This paper generalizes the central limit theorem, the law of large numbers, and extreme value theorems of classical probability theory to fuzzy quantum spaces. Extreme value theory models rare events outside the range of allowable observations with high impact. This method has become a widely-used tool for risk assessment in recent years. It is used in the areas of insurance, banking, operational risk, market risk, and credit risk [52]. By applying these limit theorems to the Atanassov set, it gives us the space to work with incomplete data, which we can use in the area of finance. The basic advantage of fuzzy logic is the ability to mathematically express information expressed verbally. Thanks to this, fuzzy logic proves to be a very good tool for working with behavioral data. Behavioral finance takes into account the human factor when making financial decisions. For this reason, behavioral finance often uses linguistic data, and therefore it is appropriate to use methods based on fuzzy logic to describe them. Behavioral finance is a financial field examining the effect of social, cognitive, and emotional factors on the economic decisions of individuals and institutions as well as the consequences of these decisions on market prices [64].

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