On Graded 2-Prime Ideals

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Abstract: The purpose of this paper is to introduce the concept of graded 2-prime ideals as a new generalization of graded prime ideals. We show that graded 2-prime ideals and graded semi-prime ideals are different. Furthermore, we show that graded 2-prime ideals and graded weakly prime ideals are also different. Several properties of graded 2-prime ideals are investigated. We study graded rings in which every graded 2-prime ideal is graded prime, we call such a graded ring a graded 2-P-ring. Moreover, we introduce the concept of graded semi-primary ideals, and show that graded 2-prime ideals and graded semi-primary ideals are different concepts. In fact, we show that graded semi-primary, graded 2-prime and graded primary ideals are equivalent over \( \mathbb{Z} \)-graded principal ideal domain.

Keywords: graded prime ideal; graded primary ideal; graded semi-primary ideals

1. Introduction

This article is devoted to the study of a generalization of graded prime ideals. We follow [1] to introduce the concept of graded 2-prime ideals. A proper graded ideal \( P \) of \( R \) is said to be a graded 2-prime ideal of \( R \) if whenever \( x, y \in h(R) \) (the set of homogeneous elements of \( R \)) such that \( xy \in P \), then \( x^2 \in P \) or \( y^2 \in P \). Clearly, every graded prime ideal is graded 2-prime. However, the converse is not necessarily true (Example 1). In fact, we show that graded 2-prime ideals and graded semi-prime ideals are completely different (Remark 1, Example 2). Also, we show that graded 2-prime ideals and graded weakly prime ideals are totally different (Remark 2, Example 3). On the other hand, a graded ideal \( P \) of \( R \) is graded prime if and only if \( P \) is a graded 2-prime and a graded semi-prime ideal of \( R \) (Proposition 1). Also, in this article, we introduce the concept of graded semi-primary ideals. A proper graded ideal \( P \) of \( R \) is said to be a graded semi-primary ideal of \( R \) if whenever \( x, y \in h(R) \) such that \( xy \in P \), then \( x \in \text{Grad}(P) \) or \( y \in \text{Grad}(P) \). Clearly, every graded 2-prime ideal is graded semi-primary. However, we show that the converse is not necessarily true (Example 4).

Among several results, the main results of the article are Corollary 1, Proposition 6, Proposition 7 and Propositions 10–14. We prove that graded semi-primary, graded 2-prime and graded primary ideals are equivalent over \( \mathbb{Z} \)-graded principal ideal domain (Corollary 1). We study graded 2-prime ideals over graded epimorphism (Proposition 6), over direct product of graded rings (Proposition 7) and over graded factor rings (Proposition 10). Example 5 shows that the intersection of two graded 2-prime ideals need not be graded 2-prime. In the rest of our article, we study graded rings in which every graded 2-prime ideal is graded prime; we call such a graded ring a graded 2-P-ring. We prove that if \( R \) is a graded 2-P-ring and graded local, then \( R \) is a graded field (Proposition 11). We show that if \( (R, X)_\alpha \) is a graded local ring and \( R \) is a graded 2-P-ring, then \( PX = K \), for every minimal graded prime ideal \( K \) over an arbitrary graded 2-prime ideal \( P \) (Proposition 12). We prove that the converse is true if \( R \) is \( \mathbb{Z} \)-graded (Proposition 13). We prove that if \( R \) is a graded 2-P-ring, then \( P^2 = P \), for every graded prime ideal \( P \) of \( R \) (Proposition 14).

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2. Preliminaries

Throughout, $G$ will be a group with identity $e$ and $R$ a commutative ring with nonzero unity $1$. We say that $R$ is $G$-graded whenever $R = \bigoplus_{g \in G} R_g$ with $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$ where $R_g$ is an additive subgroup of $R$ for all $g \in G$. The elements of $R_g$ are called homogeneous of degree $g$. If $x \in R$, then $x$ can be written as $\sum_{g \in G} x_g$, where $x_g$ is the component of $x$ in $R_g$. Also, we set $h(R) = \bigcup_{g \in G} R_g$. Moreover, it has been proved in [2] that $R_g$ is a subring of $R$ and $1 \in R_g$. Let $I$ be an ideal of a graded ring $R$. Then $I$ is said to be graded ideal if $I = \bigoplus_{g \in G} (I \cap R_g)$, i.e., for $x \in I$, $x = \sum_{g \in G} x_g$ where $x_g \in I$ for all $g \in G$.

An ideal of a graded ring need not be graded. Let $R$ be a $G$-graded ring and $I$ is a graded ideal of $R$. Then $R/I$ is $G$-graded by $(R/I)_g = (R_g + I)/I$ for all $g \in G$ (see [3]). Also, if $R$ and $S$ are two $G$-graded rings, then $R \times S$ is a $G$-graded ring by $(R \times S)_g = R_g \times S_g$ for all $g \in G$, (see [3]).

The graded radical of $P$ is denoted by $\text{Grad}(P)$, and is defined to be the set of all $r \in R$ such that for each $g \in G$; there exists a positive integer $n_g$ satisfies $r^n_g \in P$, see [4]. It is clear that if $r \in h(R)$, then $r \in \text{Grad}(P)$ if and only if $r^n \in P$ for some positive integer $n$.

A proper graded ideal $P$ of $R$ is said to be graded prime if whenever $x, y \in h(R)$ such that $xy \in P$, then either $x \in P$ or $y \in P$ ([4]). Graded prime ideals play an essential role in graded commutative ring theory. Thus, this concept has been generalized and studied in several directions. The significance of some of these generalizations is same as the graded prime ideals. In a feeling of animate being, they determine how far an ideal is from being graded prime. For instance, a proper graded ideal $P$ of $R$ is said to be graded primary if for $x, y \in h(R)$ such that $xy \in P$, then either $x \in P$ or $y \in \text{Grad}(P)$ ([5]). A proper graded ideal $P$ of $R$ is said to be a graded weakly prime ideal of $R$ if whenever $x, y \in h(R)$ such that $0 \neq xy \in P$, then $x \in P$ or $y \in P$ ([6]). A proper graded ideal $P$ of $R$ is said to be graded almost prime if for $x, y \in h(R)$ such that $xy \in P - P^n$, then either $x \in P$ or $y \in P$ (Ref. [7]). Thus, a graded weakly prime ideal is graded almost prime. Also, graded almost prime ideals were stunningly generalized in [7] to graded $n$-almost prime as follows; for $x, y \in h(R)$ such that $xy \in P - P^n$, then either $x \in P$ or $y \in P$. A proper graded ideal $P$ of $R$ is called graded semi-prime if whenever $x \in h(R)$ such that $x^2 \in P$, then $x \in P$ ([8]). A proper graded ideal $P$ of $R$ is said to be graded 2-absorbing if whenever $x, y, z \in h(R)$ such that $xyz \in P$, then either $xy \in P$ or $xz \in P$ or $yz \in P$ ([9]). Graded 2-absorbing ideals have been admirably studied in [10].

3. Graded 2-Prime Ideals

Here, we shall introduce the concept of graded 2-prime ideals along with its properties and characteristics.

Definition 1. Let $R$ be a graded ring and $P$ be a proper graded ideal of $R$. Then $P$ is said to be a graded 2-prime ideal of $R$ if whenever $x, y \in h(R)$ such that $xy \in P$, then $x^2 \in P$ or $y^2 \in P$.

Clearly, every graded prime ideal is graded 2-prime; while the converse is not necessarily true.

Example 1. Consider $R = \mathbb{Z}[i]$ and $G = \mathbb{Z}_2$. Then $R$ is $G$-graded by $R_0 = \mathbb{Z}$ and $R_1 = i\mathbb{Z}$. Now, $P = 9R$ is a graded ideal of $R$ as $9 \in h(R)$. We show that $P$ is a graded 2-prime ideal of $R$.

Let $x, y \in h(R)$ such that $xy \in P$.

Case 1: Suppose that $x, y \in R_0$. Then $x, y \in \mathbb{Z}$ such that $xy = 9(z + iw)$ for some $z, w \in \mathbb{Z}$; that is $xy = 9z$ which implies that 9 divides $xy$, and then 3 divides $xy$. Since 3 is prime, we have 3 divides $x$ or 3 divides $y$, and then 9 divides $x^2$ or 9 divides $y^2$, which implies that $x^2 \in P$ or $y^2 \in P$. 
Case 2: Suppose that \(x, y \in R_1\). Then \(x = ia\) and \(y = ib\) for some \(a, b \in \mathbb{Z}\). Since \(xy \in P\), we have \(-ab = 9(z + iw)\) for some \(z, w \in \mathbb{Z}\), i.e., \(-ab = 9z\), which implies that 9 divides \(ab\), and then as in Case 1, either \(a^2 \in P\) or \(b^2 \in P\), which implies that \(x^2 = -a^2 \in P\) or \(y^2 = -b^2 \in P\).

Case 3: Suppose that \(x \in R_0\) and \(y \in R_1\). Then \(x \in \mathbb{Z}\) and \(y = ib\) for some \(b \in \mathbb{Z}\). Since \(xy \in P\), we have \(ixb = 9(z + iw)\) for some \(z, w \in \mathbb{Z}\), i.e., \(xb = 9w\), which implies that 9 divides \(xb\), and then as in Case 1, either \(x^2 \in P\) or \(b^2 \in P\). If \(b^2 \in P\), then \(y^2 = -b^2 \in P\).

Hence, \(P\) is a graded 2-prime ideal of \(R\). On the other hand, \(P\) is not graded prime since \(3, 6 \in h(R)\) such that \(3 \cdot 6 \in P\) but \(3 \notin P\) and \(6 \notin P\).

Remark 1. The concepts of graded 2-prime ideals and graded semi-prime ideals are completely different; it has been proved in Example 1 that \(P = 9R\) is a graded 2-prime ideal of \(R\). However, \(P\) is not graded semi-prime since \(3 \in h(R)\) such that \(3^2 \in P\) but \(3 \notin P\). The next example introduces a graded semi-prime ideal which is not graded 2-prime.

Example 2. Consider \(R = \mathbb{Z}[i]\) and \(G = \mathbb{Z}_2\). Then \(R\) is \(G\)-graded by \(R_0 = \mathbb{Z}\) and \(R_1 = i\mathbb{Z}\). Now, \(P = 6R\) is a graded ideal of \(R\) as \(6 \in h(R)\). We show that \(P\) is a graded semi-prime ideal of \(R\).

Let \(x \in h(R)\) such that \(x^2 \in P\).

Case 1: Suppose that \(x \in R_0\). Then \(x \in \mathbb{Z}\) such that \(x^2 = 6(z + iw)\) for some \(z, w \in \mathbb{Z}\), that is \(2x^2 = 6z\) which implies that 6 divides \(x^2\). Since \(2, 3\) are primes, we have 2 divides \(x\) and 3 divides \(y\), and since \(2, 3\) are relatively prime; we have \(6 = 2 \cdot 3\) divides \(x\), which implies that \(x \in P\).

Case 2: Suppose that \(x \in R_1\). Then \(x = ia\) for some \(a \in \mathbb{Z}\). Since \(x^2 \in P\), we have \(-a^2 = 6(z + iw)\) for some \(z, w \in \mathbb{Z}\), i.e., \(-a^2 = 6z\), which implies that 6 divides \(a^2\), and then as in Case 1, \(a \in P\) and then \(x = ia \in P\).

Hence, \(P\) is a graded semi-prime ideal of \(R\). On the other hand, \(P\) is not graded 2-prime since \(2, 3 \in h(R)\) such that \(2 \cdot 3 \in P\) but \(2^2 \notin P\) and \(3^2 \notin P\).

Proposition 1. Let \(R\) be a graded ring and \(P\) be a graded ideal of \(R\). Then \(P\) is a graded prime ideal of \(R\) if and only if \(P\) is a graded 2-prime and a graded semi-prime ideal of \(R\).

Proof. Suppose that \(P\) is a graded 2-prime and a graded semi-prime ideal of \(R\). Let \(x, y \in h(R)\) such that \(xy \in P\). As \(P\) is graded 2-prime, then either \(x^2 \in P\) or \(y^2 \in P\), and then since \(P\) is graded semi-prime; we have either \(x \in P\) or \(y \in P\). Hence, \(P\) is a graded prime ideal of \(R\). The converse is clear. \(\square\)

Remark 2. The concepts of graded 2-prime ideals and graded weakly prime ideals are completely different; it has been proved in Example 1 that \(P = 9R\) is a graded 2-prime ideal of \(R\). However, \(P\) is not graded weakly prime since \(3 \in h(R)\) such that \(0 \neq 3 \cdot 6 \in P\) but \(3 \notin P\) and \(6 \notin P\). The next example introduces a graded weakly prime ideal which is not graded 2-prime.

Example 3. Consider \(R = \mathbb{Z}_6[i]\) and \(G = \mathbb{Z}_4\). Then \(R\) is \(G\)-graded by \(R_0 = \mathbb{Z}_6\), \(R_2 = i\mathbb{Z}_6\) and \(R_1 = R_3 = \{0\}\). Now, \(P = \{0\}\) is a graded weakly prime ideal of \(R\) which is not graded 2-prime since \(2, 3 \in h(R)\) such that \(2 \cdot 3 \in P\) but \(2^2 \notin P\) and \(3^2 \notin P\).

Definition 2. Let \(R\) be a graded ring and \(P\) be a proper graded ideal of \(R\). Then \(P\) is said to be a graded semi-primary ideal of \(R\) if whenever \(x, y \in h(R)\) such that \(xy \in P\), then \(x \in \text{Grad}(P)\) or \(y \in \text{Grad}(P)\).

Clearly, every graded 2-prime ideal is graded semi-primary; however, the converse need not be true.

Example 4. Assume that \(R\) is trivially \(\mathbb{Z}\)-graded ring. Let \(K\) be a field and \(R = K[X, Y]\) with \(\text{deg}X = 1 = \text{deg}Y\). Consider the graded ideal \(P = (X^3, XY, Y^3)\) of \(R\). It follows by (11), Exercise 4.28); \(P\) is a graded primary ideal of \(R\), and then \(P\) is a graded semi-primary ideal of \(R\). On the other hand, \(P\) is not graded 2-prime since \(XY \in P\) with \(X^2 \notin P\) and \(Y^2 \notin P\).
If \( P \) is a graded ideal of a \( G \)-graded ring \( R \), then \( \text{Grad}(P) \) need not to be a graded ideal of \( R \); see [(12), Exercises 17 and 13 on pp. 127–128]. However, it has been proved in [(13), Lemma 2.13] that if \( P \) is a graded ideal of a \( Z \)-graded ring \( R \), then \( \text{Grad}(P) \) is a graded ideal of \( R \).

**Lemma 1.** Let \( R \) be a \( Z \)-graded ring and \( P \) be a graded ideal of \( R \). Then \( P \) is a graded semi-primary ideal of \( R \) if and only if \( \text{Grad}(P) \) is a graded prime ideal of \( R \).

**Proof.** Suppose that \( P \) is a graded semi-primary ideal of \( R \). By [(13), Lemma 2.13], \( \text{Grad}(P) \) is a graded ideal of \( R \). Let \( x, y \in h(R) \) such that \( xy \in \text{Grad}(P) \). Then \( (xy)^n = x^n y^n \in P \) for some positive integer \( n \). Since \( P \) is graded semi-primary, we have \( x^{2n} \in P \) or \( y^{2n} \in P \), which implies that \( x \in \text{Grad}(P) \) or \( y \in \text{Grad}(P) \). Hence, \( \text{Grad}(P) \) is a graded prime ideal of \( R \). Conversely, let \( x, y \in h(R) \) with \( xy \in P \).

Then as \( P \subseteq \text{Grad}(P) \), \( xy \in \text{Grad}(P) \). Since \( \text{Grad}(P) \) is graded prime, we have \( x \in \text{Grad}(P) \) or \( y \in \text{Grad}(P) \). Hence, \( P \) is a graded semi-primary ideal of \( R \).

**Proposition 2.** Let \( R \) be a \( Z \)-graded principal ideal domain and \( P \) be a graded ideal of \( R \). Then \( P \) is a graded semi-primary ideal of \( R \) if and only if \( P \) is a graded primary ideal of \( R \).

**Proof.** Suppose that \( P \) is a graded semi-primary ideal of \( R \). Then by Lemma 1, \( \text{Grad}(P) \) is a graded prime ideal, and then as \( R \) is a principal ideal domain, \( \text{Grad}(P) \) is a graded maximal ideal of \( R \), which implies that \( P \) is a graded primary ideal of \( R \) by [(5), Proposition 1.11]. The converse is clear.

**Definition 3.** Let \( R \) be a graded ring. Then \( x \in h(R) \) is said to be a homogeneous reducible element of \( R \) if \( x = yz \) for some non-unit elements \( y, z \in h(R) \). Otherwise, \( x \) is called a homogeneous irreducible element of \( R \).

**Lemma 2.** Let \( R \) be a graded principal ideal domain. Then the set of all graded primary ideals of \( R \) is \( \{0\} \cup \{Rx^n : x \text{ is a homogeneous irreducible element of } R, n \in \mathbb{N}\} \).

**Proof.** \( \{0\} \) is a graded prime ideal of \( R \) as \( R \) is a domain. For a homogeneous irreducible element \( x \) of \( R \) and \( n \in \mathbb{N} \), the graded ideal \( Rx^n \) is a power of a graded maximal ideal of \( R \). We have \( \text{Grad}(Rx^n) = Rx \) is a graded maximal ideal of \( R \). So, by [(5), Proposition 1.11] we have that \( Rx^n \) is a graded primary ideal of \( R \). On the other hand, a nonzero graded primary ideal of \( R \) should have the form \( Rx \) for some nonzero \( x \in h(R) \), and \( x \) cannot be a unit since a graded primary ideal is proper. Since \( R \) is a unique factorization domain, we can write \( x \) as a product of homogeneous irreducible elements of \( R \). If \( x \) is divisible by two homogeneous irreducible elements \( a \) and \( b \) of \( R \) which are not associated, then \( Ra \) and \( Rb \) are graded maximal ideals of \( R \), and they both are graded minimal prime ideals of \( Rx \), which contradicts [(5), Corollary 1.9]. Therefore, \( Rx \) is generated by a positive power of some homogeneous irreducible element of \( R \).

**Proposition 3.** Let \( R \) be a graded principal ideal domain and \( P \) be a graded ideal of \( R \). Then \( P \) is a graded 2-prime ideal of \( R \) if and only if \( P \) is a graded primary ideal of \( R \).

**Proof.** We show that \( P \) is a graded 2-prime ideal of \( R \) if and only if either \( P = Rx^n \), for some positive integer \( n \) and a homogeneous irreducible element \( x \) of \( R \) or \( x = 0 \); and so the result follows from Lemma 2. Suppose that \( P \) is a nonzero graded 2-prime ideal of \( R \). Since \( R \) is a principal ideal domain, there exists \( s \in h(R) \) such that \( P = Rs \). If \( s \) is irreducible, then \( n = 1 \) and we are done. Suppose that \( s \) is not an irreducible element. Since \( R \) is a unique factorization domain, we have \( s \) can be written in the form \( s = x_1^{n_1} x_2^{n_2} \ldots x_m^{n_m} \), where \( m, n_1, n_2, \ldots, n_m \) are positive integers, \( m > 1 \) and \( x_i \)'s are homogeneous irreducible elements of \( R \) such that \( x_i \) and \( x_j \) are not associated for \( i \neq j \). Let \( a = x_1^{n_1} \) and \( b = x_2^{n_2} \ldots x_m^{n_m} \). Then \( ab \in P \). As \( P \) is graded 2-prime, then either \( a^2 \in P \) or \( b^2 \in P \).
If $a^2 = x_1^{2n_1} \in P = Rs$, then there exists $y \in h(R)$ such that $a^2 = x_1^{2n_1} = ys = yx_1^{n_1} x_2^{n_2} \ldots x_m^{n_m}$, and then $x_1^{n_1} = yx_2^{n_2} \ldots x_m^{n_m}$, which implies that $x_1$ divides $x_j$ for some $2 \leq j \leq m$. Since $x_1$ is an irreducible element of $R$, we have $x_1$ and $x_j$ are associated, which is a contradiction.

If $b^2 = (x_2^{n_2} \ldots x_m^{n_m})^2 \in P = Rs$, then there exists $z \in h(R)$ such that $(x_2^{n_2} \ldots x_m^{n_m})^2 = zs = zx_1^{n_1} x_2^{n_2} \ldots x_m^{n_m}$, which implies that $x_1$ divides $(x_2^{n_2} \ldots x_m^{n_m})^2$. Since $R$ is a principal ideal domain, $x_1$ divides $x_j$ for some $2 \leq j \leq m$, which is a contradiction. Conversely, suppose that $P = Rs$ for some a homogeneous irreducible element $x$ of $R$ and a positive integer $n$. Assume that $a, b \in h(R)$ such that $ab \in P$. Then $a = yx^k$ and $b = zv^l$ for some $y, z \in h(R)$ such that $j + k \geq n$. Assume that $2k < n$ and $2j < n$. Then $2k + 2j < 2n$, which is a contradiction since $j + k \geq n$. Hence, $a^2 \in P$ or $b^2 \in P$, and so $P$ is a graded 2-prime ideal of $R$.  

By combining Proposition 2 and Proposition 3, we state the following result.

**Corollary 1.** Let $R$ be a $\mathbb{Z}$-graded principal ideal domain and $P$ be a graded ideal of $R$. Then the following are equivalent.

1. $P$ is a graded semi-primary ideal of $R$.
2. $P$ is a graded 2-prime ideal of $R$.
3. $P$ is a graded primary ideal of $R$.

**Proposition 4.** Let $R$ be a $\mathbb{Z}$-graded ring and $P$ be a graded ideal of $R$. If $P$ is a graded 2-prime ideal of $R$, then $Grad(P)$ is a graded prime ideal of $R$. Moreover, $Grad(P)$ is the smallest graded prime ideal of $R$ containing $P$.

**Proof.** Since $P$ is a graded 2-prime ideal of $R$, we have $P$ is a graded semi-primary ideal of $R$, and then by Lemma 1, $Grad(P)$ is a graded prime ideal of $R$. Moreover, one can easily prove that every graded prime ideal of $R$ containing $P$ should also contain $Grad(P)$.

**Proposition 5.** Let $R$ be a graded ring and $P$ be a graded ideal of $R$. If $P$ is a graded prime ideal of $R$, then $P^2$ is a graded 2-prime ideal of $R$.

**Proof.** By ([5], Lemma 2.1) $P^2$ is a graded ideal of $R$. Let $x, y \in h(R)$ such that $xy \in P^2$. Then as $P^2 \subseteq P$, $xy \in P$. Since $P$ is graded prime, then either $x \in P$ or $y \in P$, and then either $x^2 \in P$ or $y^2 \in P^2$. Hence, $P^2$ is a graded 2-prime ideal of $R$.

4. Graded 2-Prime Ideals over Graded Ring Homomorphisms, Cartesian Product of Graded Rings and Factor Rings

In this section, we study graded 2-prime ideals over graded ring homomorphisms, cartesian product of graded rings and factor rings. Let $R$ and $S$ be two $G$-graded rings. A ring homomorphism $f : R \to S$ is said to be graded homomorphism if $f(R_g) \subseteq S_g$ for all $g \in G$ (see [2]).

**Lemma 3.** ([14], Lemma 3.11) Suppose that $f : R \to S$ is a graded epimorphism of graded rings. If $P$ is a graded ideal of $R$ with $Ker(f) \subseteq P$, then $f(P)$ is a graded ideal of $S$.

**Proposition 6.** Suppose that $f : R \to S$ is a graded epimorphism of graded rings. If $P$ is a graded 2-prime ideal of $R$ with $Ker(f) \subseteq P$, then $f(P)$ is a graded 2-prime ideal of $S$.

**Proof.** By Lemma 3, $f(P)$ is a graded ideal of $S$. Let $a, b \in h(S)$ such that $ab \in f(P)$. Since $f$ is surjective, there exist $x, y \in h(R)$ such that $f(x) = a$ and $f(y) = b$, and then $f(xy) = ab \in f(P)$, which implies that $xy \in P$ since $Ker(f) \subseteq P$. Since $P$ is graded 2-prime, we have either $x^2 \in P$ or $y^2 \in P$, and then either $a^2 = (f(x))^2 = f(x^2) \in f(P)$ or $b^2 = (f(y))^2 = f(y^2) \in f(P)$. Hence, $f(P)$ is a graded 2-prime ideal of $S$. 

Lemma 4. Let $P$ be an ideal of a $G$-graded ring $R$ and $K$ be an ideal of a $G$-graded ring $S$. Then $P \times K$ is a graded ideal of $R \times S$ if and only if $P$ is a graded ideal of $R$ and $K$ is a graded ideal of $S$.

Proof. Clearly, $P \times K$ is an ideal of $R \times S$. If $(x, y) \in P \times K$. Then $x \in P$ and $y \in K$, and since $P, K$ are graded, $x_g \in P$ and $y_g \in K$ for all $g \in G$, which implies that $(x, y)_g = (x_g, y_g) \in P \times K$ for all $g \in G$. Thus, $P \times K$ is a graded ideal of $R \times S$. Conversely, let $x \in P$. Then $(x, 0_g) \in P \times K$, and since $P \times K$ is graded, then $(x_g, 0) = (x, 0_g) \in P \times K$ for all $g \in G$ and $x_g \in P$ for all $g \in G$. Therefore, $P$ is a graded ideal of $R$. Similarly, $K$ is a graded ideal of $S$. □

Proposition 7. Let $R$ and $S$ be $G$-graded rings. Then $P$ is a graded 2-prime ideal of $R$ if and only if $P \times S$ is a graded 2-prime ideal of $R \times S$.

Proof. Suppose that $P$ is a graded 2-prime ideal of $R$. By Lemma 4, $P \times S$ is a graded ideal of $R \times S$. Let $(x, y), (z, w) \in h(R \times S)$ such that $(x, y)(z, w) \in P \times S$. Then $x, z \in h(R)$ and $xz \in P$. Since $P$ is graded 2-prime, we have either $x^2 \in P$ or $z^2 \in P$, and then either $(x, y)^2 \in P \times S$ or $(z, w)^2 \in P \times S$. Hence, $P \times S$ is a graded 2-prime ideal of $R \times S$. Conversely, by Lemma 4, $P$ is a graded ideal of $R$. Let $x, y \in h(R)$ such that $xy \in P$. Then $(x, 1_S)(y, 1_S) \in h(R \times S)$ such that $(x, 1_S)(y, 1_S) \in P \times S$. Since $P \times S$ is graded 2-prime, we have either $(x, 1_S)^2 \in P \times S$ or $(y, 1_S)^2 \in P \times S$, and then either $x^2 \in P$ or $y^2 \in P$. Hence, $P$ is a graded 2-prime ideal of $R$. □

Similarly, one can prove the following:

Proposition 8. Let $R$ and $S$ be $G$-graded rings. Then $K$ is a graded 2-prime ideal of $S$ if and only if $R \times K$ is a graded 2-prime ideal of $R \times S$.

Lemma 5. ([3], Lemma 4.1) Let $R$ be a graded ring and $P$ be a graded ideal of $R$. Then $(P : a) = \{x \in R : xa \in P\}$ is a graded ideal of $R$ for all $a \in h(R)$.

Proposition 9. Let $R$ be graded ring and $P$ be a graded 2-prime ideal of $R$. Then $(P : a^2)$ is a graded 2-prime ideal of $R$ for all $a \in h(R) - Grad(P)$. In particular, $Grad((P : a^2)) = Grad(P)$.

Proof. Let $a \in h(R) - Grad(P)$. Then $a^2 \in h(R)$, and then by Lemma 5, $(P : a^2)$ is a graded ideal of $R$. Please note that if $a^2 \in P$ (which means that $a \in Grad(P)$), then $(P : a^2) = R$, but we need it to be proper. Therefore, we assume that $a \notin Grad(P)$. Let $x \in (P : a^2)$. Then $xg \in (P : a^2)$ for all $g \in G$. Therefore, for any $g \in G$, $x_g a^2 \in P$. Since $P$ is graded 2-prime and $a \notin Grad(P)$, we have $x_g^2 \in P$, and then $x_g \in Grad(P)$, and so $x = \sum_{g \in G} x_g \in Grad(P)$.

Hence, $P \subseteq (P : a^2) \subseteq Grad(P)$, and then $Grad(P) \subseteq Grad((P : a^2)) \subseteq Grad(Grad(P)) = Grad(P)$. Thus, $Grad((P : a^2)) = Grad(P)$, which means that $(P : a^2)$ is proper. Now, let $x, y \in h(R)$ such that $xy \in (P : a^2)$. Then $x g, y \in h(R)$ such that $x_g a^2 = (xa)(ya) \in P$. Since $P$ is graded 2-prime, then either $(xa)^2 \in P$ or $(ya)^2 \in P$, and either $x^2 \in (P : a^2)$ or $y^2 \in (P : a^2)$. Hence, $(P : a^2)$ is a graded 2-prime ideal of $R$. □

Lemma 6. ([3], Lemma 3.2) Let $R$ be a graded ring, $K$ be a graded ideal of $R$ and $P$ be an ideal of $R$ such that $K \subseteq P$. Then $P$ is a graded ideal of $R$ if and only if $P/K$ is a graded ideal of $R/K$.

Proposition 10. Let $R$ be a graded ring, $K$ be a graded ideal of $R$ and $P$ be an ideal of $R$ such that $K \subseteq P$. Then $P$ is a graded 2-prime ideal of $R$ if and only if $P/K$ is a graded 2-prime ideal of $R/K$.

Proof. Suppose that $P$ is a graded 2-prime ideal of $R$. By Lemma 6, $P/K$ is a graded ideal of $R/K$. Let $x + K, y + K \in h(R/K)$ such that $(x + K)(y + K) \in P/K$. Then $x, y \in h(R)$ such that $xy \in P$. As $P$ is graded 2-prime, then either $x^2 \in P$ or $y^2 \in P$, and then
either \((x + K)^2 \in P/K\) or \((y + K)^2 \in P/K\). Hence, \(P/K\) is a graded 2-prime ideal of \(R/K\). Conversely, by Lemma 6, \(P\) is a graded ideal of \(R\). Let \(x, y \in h(R)\) such that \(xy \in P\). Then \(x + K, y + K \in h(R/K)\) such that \((x + K)(y + K) \in P/K\). Since \(P/K\) is graded 2-prime, we have either \((x + K)^2 \in P/K\) or \((y + K)^2 \in P/K\), and then either \(x^2 \in P\) or \(y^2 \in P\). Hence, \(P\) is a graded 2-prime ideal of \(R\). \(\square\)

The next example shows that the intersection of two graded 2-prime ideals need not be graded 2-prime.

**Example 5.** Let \(R = \mathbb{Z}[i]\) and \(G = \mathbb{Z}_2\). Then \(R\) is \(G\)-graded by \(R_0 = \mathbb{Z}\) and \(R_1 = i\mathbb{Z}\). One can prove that \(P = 2R\) and \(K = 3R\) are graded prime ideals of \(R\); and then \(P\) and \(K\) are graded 2-prime ideals of \(R\). On the other hand, \(2, 3 \in h(R)\) such that \(2 \cdot 3 \in P \cap K\) with \(2^2 \notin P \cap K\) and \(3^2 \notin P \cap K\). Hence, \(P \cap K\) is not a graded 2-prime ideal of \(R\).

### 5. Graded 2-P-Rings

In this section, we study graded rings in which every graded 2-prime ideal is graded prime.

**Definition 4.** Let \(R\) be a graded ring. Then \(R\) is said to be a graded 2-P-ring if every graded 2-prime ideal of \(R\) is graded prime.

A graded ring \(R\) is said to be a graded local ring if it contains a unique graded maximal ideal, say \(X\), and it is denoted by \((R, X)_{gr}\), (see [2]).

**Lemma 7.** Let \((R, X)_{gr}\) be a graded local ring and \(P\) be a graded prime ideal of \(R\). Then \(PX\) is a graded 2-prime ideal of \(R\), and \(PX\) is a graded prime ideal of \(R\) if and only if \(PX = P\).

**Proof.** By ([5], Lemma 2.1), \(PX\) is a graded ideal of \(R\). Let \(x, y \in h(R)\) such that \(xy \in PX\). Then as \(PX \subseteq P\), \(xy \in P\). Since \(P\) is graded prime, then either \(x \in P\) or \(y \in P\). Suppose that \(x \in P\). Since \(P\) is proper as it is a graded prime, we have that \(x\) is not a unit element, and then \(x \in X\). Hence, \(x^2 \in PX\). Similarly, if \(y \in P\), then \(y^2 \in PX\). Hence, \(PX\) is a graded 2-prime ideal of \(R\). Now, suppose that \(PX\) is a graded prime ideal of \(R\) and \(a \in P\). Then \(a_g \in P\) for all \(g \in G\) as \(P\) is graded. Since \(P\) is proper, for any \(g \in G\), \(a_g\) is not a unit element, so \(a_g \in X\), and then \(a_g^2 \in PX\). Since \(PX\) is graded prime, we have \(a_g \in PX\), and then \(a = \sum_{g \in G} a_g \in PX\). Hence, \(PX = P\). \(\square\)

**Proposition 11.** Let \((R, X)_{gr}\) be a graded local ring. If \(R\) is a graded 2-P-ring, then \(R\) is a graded field.

**Proof.** Since \(X\) is graded maximal; we have that \(X\) is a graded prime ideal of \(R\). Apply Lemma 7 with \(P = X\), we conclude that \(X^2\) is a graded 2-ideal ideal of \(R\). Since \(R\) is graded 2-P, we have that \(X^2\) is a graded prime ideal of \(R\), and then by Lemma 7, \(X^2 = X\). We show that \(X = \{0\}\). Suppose that \(X \neq \{0\}\). Let \(\{x_1, ..., x_n\}\) be a minimal homogeneous generating set for \(X\). Since \(x_1 \in X^2\), there exist \(y_1, ..., y_n \in X\) such that \(x_1 = \sum_{i=1}^{n} y_i x_i\). Hence, 

\[
(1 - y_1)x_1 = y_2 x_2 + ... + y_n x_n.
\]

But since \(y_1 \in X\) and \(X\) is graded maximal, \(1 - y_1\) is a unit element of \(R\) with inverse, say \(u\). Hence, \(x_1 = \sum_{i=2}^{n} uy_i x_i\). Therefore, \(X\) is generated by \(\{x_2, ..., x_n\}\), which is a contradiction. Thus, \(X = \{0\}\), which implies that \(\{0\}\) is the only graded maximal ideal of \(R\). Therefore, \(R\) is a graded field. \(\square\)

First strongly graded rings have been introduced and studied in [15]; a \(G\)-graded ring \(R\) is said to be first strong if \(1 \in R_g \cdot R_{g^{-1}}\) for all \(g \in supp(R, G)\), where \(supp(R, G) = \{g \in G : R_g \neq \{0\}\}\). In fact, it has been proved that \(R\) is first strongly \(G\)-graded if and only
if \( \text{supp}(R, G) \) is a subgroup of \( G \) and \( R_g R_h = R_{gh} \) for all \( g, h \in \text{supp}(R, G) \). We introduce the following:

**Lemma 8.** Every \( G \)-graded field is first strongly graded.

**Proof.** Let \( R \) be a \( G \)-graded field. Suppose that \( g \in \text{supp}(R, G) \). Then \( R_g \neq \{0\} \), and then there exists \( 0 \neq x \in R_g \). Since \( R \) is a graded field, we conclude that there exists \( y \in h(R) \) such that \( xy = 1 \). Since \( y \in h(R) \), \( y \in R_h \) for some \( h \in G \), and then \( 1 = xy \in R_g R_h \subseteq R_{gh} \). Therefore, \( 0 \neq 1 \in R_{gh} \cap R_e \), which implies that \( gh = e \), i.e., \( h = g^{-1} \). Hence, \( 1 = xy \in R_g R_{g^{-1}} \), and thus \( R \) is first strongly graded. \( \square \)

**Corollary 2.** Let \( (R, X)_{gr} \) be a graded local ring. If \( R \) is a graded 2-P-ring, then \( R \) is first strongly graded.

**Proof.** Apply Proposition 11 and Lemma 8. \( \square \)

**Proposition 12.** Let \( (R, X)_{gr} \) be a graded local ring. If \( R \) is a graded 2-P-ring, then \( PX = K \), for every minimal graded prime ideal \( K \) over an arbitrary graded 2-prime ideal \( P \).

**Proof.** Let \( K \) be a minimal graded prime over a graded 2-prime ideal \( P \). Then \( P \) is graded prime and \( K = P \). By Lemma 7, \( PX \) is graded 2-prime and hence \( PX \) is graded prime. Again, by Lemma 7, \( PX = P \), as desired. \( \square \)

The next proposition shows that the converse of Proposition 12 is true when \( R \) is \( Z \)-graded.

**Proposition 13.** Let \( (R, X)_{gr} \) be a \( Z \)-graded local ring. If \( PX = K \), for every minimal graded prime ideal \( K \) over an arbitrary graded 2-prime ideal \( P \), then \( R \) is a graded 2-P-ring.

**Proof.** Let \( P \) be a graded 2-prime ideal of \( R \). By Proposition 4, \( K = \text{Grad}(P) \) is a graded prime ideal of \( R \) such that \( P \subseteq K \). Since \( PX = K \), we deduce that \( K = PX \subseteq P \cap X = P \), and so \( P = K \) is graded prime, as needed. \( \square \)

**Proposition 14.** Let \( R \) be a graded 2-P-ring. Then \( P^2 = P \), for every graded prime ideal \( P \) of \( R \).

**Proof.** Let \( P \) be a graded prime ideal of \( R \). By Proposition 5, \( P^2 \) is a graded 2-prime ideal of \( R \). Since \( R \) is a graded 2-P-ring, \( P^2 \) is a graded prime ideal of \( R \). Let \( x \in P \). Then \( x_{g} \in P \) for all \( g \in G \) as \( P \) is graded. Therefore, for any \( g \in G \), \( x_{g}^2 \in P^2 \), and since \( P^2 \) is graded prime; we have \( x_{g} \in P^2 \), and then \( x = \sum_{g \in G} x_{g} \in P^2 \). Therefore, \( P \subseteq P^2 \subseteq P \). Hence, \( P^2 = P \). \( \square \)

**Definition 5.** Let \( R \) be a graded ring, \( K \) be a graded ideal of \( R \) and \( P \) be a graded 2-prime ideal of \( R \). Then \( P \) is said to be a minimal graded 2-prime ideal over \( K \) if there is not a graded 2-prime ideal \( Q \) of \( R \) such that \( K \subseteq Q \subseteq P \). We denote the set of all minimal graded 2-prime ideals over \( K \) by \( 2-\text{Min}_{gr}(K) \). The set of all minimal graded prime ideals over \( K \) was denoted by \( \text{Min}_{gr}(K) \).

**Proposition 15.** Let \( (R, X)_{gr} \) be a graded local ring, \( K \) be a graded prime ideal of \( R \) contains \( P \) and assume that \( (\text{Grad}(P))^2 \subseteq P \), for every graded 2-prime ideal \( P \) of \( R \). If \( PX = K \) for every graded ideal \( P \in 2-\text{Min}_{gr}(K^2) \) and \( K \in \text{Min}_{gr}(P) \), then \( P = K \) for every graded ideal \( P \in 2-\text{Min}_{gr}(K^2) \) with \( P \subseteq K \).

**Proof.** Let \( P \in 2-\text{Min}_{gr}(K^2) \) with \( P \subseteq K \). First, we show that \( K \in \text{Min}_{gr}(P) \). Assume that there exists a graded prime ideal \( Q \) of \( R \) such that \( P \subseteq Q \subseteq K \). Clearly, \( K^2 \subseteq P \subseteq Q \subseteq K \). Let \( x \in K \). Then \( x_{g} \in K \) for all \( g \in G \) as \( K \) is graded. Therefore, for any \( g \in G \), \( x_{g}^2 \in K^2 \), and hence, \( x_{g}^2 \in Q \). Since \( Q \) is graded prime, we have \( x_{g} \in Q \) for all \( g \in G \), and then \( x \in Q \).
Therefore, $K = Q$, as desired. Clearly, $PX \subseteq P \subseteq K$. Now, by assumption, $PX = K$, and so $P = K$. □

The next proposition shows that the converse of Proposition 15 is true when $R$ is $\mathbb{Z}$-graded.

**Proposition 16.** Let $(R, X)$ be a $\mathbb{Z}$-graded local ring, $K$ be a graded prime ideal of $R$ and assume that $(\text{Grad}(P))^2 \subseteq P$, for every graded 2-prime ideal $P$ of $R$. If $P = K$ for every graded ideal $P \in 2\text{-}\text{Min}_{gr}(K^2)$ with $P \subseteq K$, then $PX = K$ for every graded ideal $P \in 2\text{-}\text{Min}_{gr}(K^2)$ and $K \in \text{Min}_{gr}(P)$.

**Proof.** Let $P \in 2\text{-}\text{Min}_{gr}(K^2)$ and $K \in \text{Min}_{gr}(P)$. Since $\text{Grad}(P)$ is a graded prime ideal of $R$ by Proposition 4 and since $K \in \text{Min}_{gr}(P)$, we conclude that $P = K$. Hence, by assumption, $(\text{Grad}(P))^2 = K^2 \subseteq P \subseteq K$. Thus, by assumption, $P = K$. Since $K^2 \subseteq KX \subseteq P = K$ and $KX$ is graded 2-prime by Lemma 7, we conclude that $KX = PX = K$. □

6. Conclusions

In this article, we introduced the concept of graded 2-prime ideals as a generalization of graded prime ideals. We proved that graded 2-prime ideals and graded semi-prime ideals are completely different. Furthermore, we proved that graded 2-prime ideals and graded weakly prime ideals are totally different. Several properties of graded 2-prime ideals have been investigated. Graded rings in which every graded 2-prime ideal is graded prime have been investigated. Moreover, we introduced the concept of graded semi-primary ideals. We proved that every graded 2-prime ideal is graded semi-primary. However, we proved that the converse is not true in general. In fact, we proved that graded semi-primary, graded 2-prime and graded primary ideals are equivalent over $\mathbb{Z}$-graded principal ideal domain. As a proposal of further work on the topic, we are going to extend the concept of graded 2-prime ideals into graded 2-prime $R$-submodules. We will try to introduce and study the concept of graded 2-prime $R$-submodules as a generalization of graded prime $R$-submodules. Also, we will investigate the properties of graded 2-prime submodules and try to study them in special graded modules such as graded multiplication modules.

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