mathematics

MDPI

Article

Adomian Decomposition and Fractional Power Series Solution of a Class of Nonlinear Fractional Differential Equations

Pshtiwan Othman Mohammed [1,*], José António Tenreiro Machado [2], Juan L. G. Guirao [3,4,*] and Ravi P. Agarwal [5]

1 Department of Mathematics, College of Education, University of Sulaimani, Sulaimani 46001, Kurdistan Region, Iraq
2 Institute of Engineering, Polytechnic of Porto, Department of Electrical Engineering, Rua Dr. António Bernardino de Almeida, 431, 4249-015 Porto, Portugal; jtm@isep.ipp.pt
3 Department of Applied Mathematics and Statistics, Technical University of Cartagena, Hospital de Marina, 30203 Cartagena, Spain
4 Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
5 Department of Mathematics, Texas A & M University-Kingsville, Kingsville, TX 78363, USA; Ravi.Agarwal@tamuk.edu
* Correspondence: pshtiwansangawi@gmail.com (P.O.M.); juan.garcia@upct.es (J.L.G.G.)

Abstract: Nonlinear fractional differential equations reflect the true nature of physical and biological models with non-locality and memory effects. This paper considers nonlinear fractional differential equations with unknown analytical solutions. The Adomian decomposition and the fractional power series methods are adopted to approximate the solutions. The two approaches are illustrated and compared by means of four numerical examples.

Keywords: Adomian decomposition method; fractional power series method; nonlinear fractional differential equations

Citation: Mohammed, P.O.; Machado, J.A.T.; Guirao, J.L.G.; Agarwal, R.P. Adomian Decomposition and Fractional Power Series Solution of a Class of Nonlinear Fractional Differential Equations. *Mathematics* **2021**, *9*, 1070. https://doi.org/10.3390/math9091070

Academic Editors: M. Victoria Otero-Espinar and Christopher Goodrich

Received: 27 March 2021
Accepted: 30 April 2021
Published: 10 May 2021

1. Introduction and Preliminaries

Fractional calculus is a branch of mathematics that investigates the properties dealing with arbitrary order integral and differential operators (see [1–3]). Fractional differential equations are an excellent tool for the mathematical modeling of real world problems and dynamical systems, such as in engineering, physics, earthquake vibrations, biological and aerodynamics, chaotic and fractals, signal and image processing, artificial intelligence, and control theory (see [4–9]). In the last few years, researchers introduced various models involving fractional derivative and integral operators of arbitrary order (see [10]). Some recent contributions to the theory of fractional differential (or difference) equations and its applications can be found in [11–17] and references therein.

The nonlinear fractional differential equations (NFDEs) have been the topic of intense research and a number of articles with deal with their numerical solution. Numerical techniques are developed continuously because exact solutions are available in very few cases. Recently, a number of numerical procedures for solving NFDEs have been developed. Ganjiani [18] obtained numerical solutions of NFDEs by using the homotopy analysis method. Gaoa et al. [19] presented a new fractional numerical differentiation formula for NFDEs in the Caputo sense. Liu et al. [20] used the finite element approximation for NFDEs. Zayernouri and Karniadakis [21] presented fractional spectral collocation methods for NFDEs of variable order. Wang et al. [22] advanced with upper and lower control functions to construct an algorithm for NFDEs. Bekir et al. [23] developed (G′/G)-expansion method. Gao and Sun [24] presented a compact finite difference scheme for fractional subdiffusion equations using a compact finite difference technique. Duan et al. [25] used the Rach–Adomian–Meyers modified decomposition method to find approximate solutions of the initial value problem for NFDEs. Modanli et al. [26] used the residual power series method for solving a class of NFDEs with nonlocal conditions. Lu and Zheng [27] explored the full power of Adomian decomposition method to compute explicit closed form solutions of NFDEs with unprescribed initial conditions.

The effectiveness of the above methods was assessed by comparing the numerical schemes with the corresponding exact solutions. Nonetheless, when the exact solutions are unknown we need to adopt numerical techniques and compare the results provided by distinct alternative schemes.

In this paper, we study two numerical approaches for solving the following types of NFDEs:

$$\begin{aligned} &\frac{d\mathrm{Y}}{d\eta} + {}^{RL}D^{\frac{1}{2}}_{0+}\mathrm{Y}(\eta) - F(\mathrm{Y}(\eta)) = 0, \quad \eta \in [0, b_2],\ b_2 > 0, \\ &\mathrm{Y}(0) = c, \end{aligned} \tag{1}$$

where ${}^{RL}D^{\frac{1}{2}}_{0+}\mathrm{Y}(\eta)$ represents the $\frac{1}{2}$th left fractional integral of $\mathrm{Y}(\eta)$ in the Riemann–Liouville sense and $F(\mathrm{Y}(\eta))$ represents the nonlinear part of Equation (1). In the follow-up we consider four cases, namely $F(\mathrm{Y}(\eta)) = \left\{ \sin(\mathrm{Y}(\eta)), \cos(\mathrm{Y}(\eta)), e^{\mathrm{Y}(\eta)}, e^{-\mathrm{Y}(\eta)} \right\}$.

The article is structured as follows. Section 2 presents Adomian decomposition (ADM) and fractional power series methods (FPSM), and introduces some preliminary definitions and results of key importance in fractional calculus. Section 3 discusses appropriate conditions for the solutions of problem (1). Section 4 illustrates the results of the new technique for four examples. Finally, Section 5 highlights the main conclusions.

2. Preliminary

2.1. Fractional Calculus

We find in the literature a variety of definitions of fractional integrals and derivatives. Therefore, it is necessary to specify which definition is used. In this article, we adopt the left Riemann–Liouville (L–RL) fractional integral and derivative, which are defined below.

Definition 1 (see [1–3]). *For any L^1 function Y on $[b_1, b_2]$, the δth L–RL fractional integral of $\mathrm{Y}(\eta)$ is defined for $\mathrm{Re}(\delta) > 0$ as follows:*

$${}^{RL}\mathcal{I}^{\delta}_{b_1+}\mathrm{Y}(\eta) := \frac{1}{\Gamma(\delta)} \int_{b_1}^{\eta} (\eta - \xi)^{\delta - 1}\mathrm{Y}(\xi)\,\mathrm{d}\xi, \quad \eta \in [b_1, b_2]. \tag{2}$$

For any C^n function Y on $[b_1, b_2]$, the δth L–RL fractional derivative of $\mathrm{Y}(\eta)$ is defined for $n - 1 \le \mathrm{Re}(\delta) < n$ as follows:

$${}^{RL}D^{\delta}_{b_1+}\mathrm{Y}(\eta) := \frac{\mathrm{d}^n}{\mathrm{d}\eta^n}\, {}^{RL}\mathcal{I}^{n-\delta}_{b_1+}\mathrm{Y}(\eta), \quad \eta \in [b_1, b_2]. \tag{3}$$

These two definitions cover orders of differentiation throughout the complex plane, where we can write ${}^{RL}D^{-\delta}_{b_1+}\mathrm{Y}(\eta) = {}^{RL}\mathcal{I}^{\delta}_{b_1+}\mathrm{Y}(\eta)$. Consequently, the differentiation and integration are unified in a single operator which we may opt to call differintegration (see [28]).

Lemma 1 (see [1–3]). *If $\mu > -1, \eta > b_1$ and $\mathrm{Re}(\delta) > 0$, then the L–RL fractional integral of the power function satisfies the condition:*

$$\begin{aligned} {}^{RL}\mathcal{I}^{\delta}_{b_1+}(\eta - b_1)^{\mu} &= \frac{\Gamma(\mu + 1)}{\Gamma(\mu + 1 + \delta)}(\eta - b_1)^{\mu + \delta}, \\ {}^{RL}D^{\delta}_{b_1+}(\eta - b_1)^{\mu} &= \frac{\Gamma(\mu + 1)}{\Gamma(\mu + 1 - \delta)}(\eta - b_1)^{\mu - \delta}. \end{aligned}$$

2.2. The ADM and FPSM

The ADM proposes $\Upsilon(\eta)$ as a series expansion given by:

$$\Upsilon(\eta) := \sum_{n=0}^{\infty} \Upsilon_n(\eta), \quad n \geq 1. \tag{4}$$

If we denote $\mathcal{D} = \frac{d}{d\eta}$, then by applying $\mathcal{D}^{-1}(\cdot) = \int_0^\eta (\cdot) dt$ to both sides of Equation (1), we can deduce

$$\Upsilon(\eta) = c - \mathcal{D}^{-1}\left({}^{RL}D_{0+}^{\frac{1}{2}} \Upsilon(\eta) \right) + \mathcal{D}^{-1} F(\Upsilon(\eta)), \tag{5}$$

where c is an arbitrary constant. This expression can be rewritten as follows:

$$\sum_{n=0}^{\infty} \Upsilon_n(\eta) = c - {}^{RL}_{\ 0}\mathcal{I}^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} \Upsilon_n(\eta) \right) + \int_0^\eta \left(\sum_{n=0}^{\infty} A_n(t) \right) dt.$$

Consequently, we set the following recursion scheme:

$$\begin{aligned} \Upsilon_0(\eta) &= c, \\ \Upsilon_{n+1}(\eta) &= \int_0^\eta A_n(t) dt - {}^{RL}_{\ 0}\mathcal{I}^{\frac{1}{2}} \Upsilon_n(\eta), \quad n \geq 0, \end{aligned} \tag{6}$$

where the coefficients A_n are the Adomian polynomials (APs), defined by [29]:

$$F(\Upsilon(\eta)) := A_n = \left[\frac{1}{n!} \frac{d^n}{d\lambda^n} F\left(\sum_{i=0}^{\infty} \Upsilon_i(\eta)\, \lambda^i \right) \right]_{\lambda=0}, \quad n \geq 0. \tag{7}$$

Based on the structure of the ADM, we consider the solution $\Upsilon(\eta)$ as

$$\Upsilon(\eta) = \lim_{n \to \infty} \sigma_n(\eta),$$

where the $(n+1)-$term approximation of the solution is given by

$$\sigma_n(\eta) = \sum_{j=0}^{n} \Upsilon_j(\eta), \qquad \text{for } n \geq 1.$$

The FPSM can be represented in the following series:

$$\begin{aligned} \Upsilon(\eta) &= \sum_{n=0}^{\infty} \Upsilon_n(\eta)\, \eta^{\frac{n}{2}}, \\ \Upsilon_0(\eta) &= c. \end{aligned} \tag{8}$$

By using (8) and Lemma 1 in Equation (1), we can deduce

$$\sum_{n=1}^{\infty} \frac{n}{2} \Upsilon_n(\eta)\, \eta^{\frac{n}{2}-1} + \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n+1}{2}\right)} \Upsilon_n(\eta)\, \eta^{\frac{n-1}{2}} = F\left(\sum_{n=0}^{\infty} \Upsilon_n(\eta)\, \eta^{\frac{n}{2}} \right). \tag{9}$$

Remark 1. *The ADM and the power series method have been successfully applied to numerical problems and the results have shown a high degree of accuracy. For the convergence of these methods, one can refer to [30,31].*

3. Applying the ADM and FPSM for Specific $F(\Upsilon(\eta))$

For simplicity we use Υ_n instead of $\Upsilon_n(\eta)$ for each $n \geq 0$.

3.1. Case 1: $F(\Upsilon(\eta)) = \sin(\Upsilon(\eta))$

The first four APs for $\sin(\Upsilon(\eta))$ are given by [30]:

$$\begin{aligned}
A_0 &= \sin(\Upsilon_0),\\
A_1 &= \Upsilon_1\cos(\Upsilon_0),\\
A_2 &= \Upsilon_2\cos(\Upsilon_0) - \frac{1}{2}\Upsilon_1^2\sin(\Upsilon_0),\\
A_3 &= \Upsilon_3\cos(\Upsilon_0) - \Upsilon_1\Upsilon_3\sin(\Upsilon_0) - \frac{1}{3!}\Upsilon_1^3\cos(\Upsilon_0),\\
A_4 &= \Upsilon_4\cos(\Upsilon_0) - \left(\frac{1}{2}\Upsilon_2^2 + \Upsilon_1\Upsilon_3\right)\sin(\Upsilon_0) - \frac{1}{2}\Upsilon_1^2\Upsilon_2\cos(\Upsilon_0) + \frac{1}{4!}\Upsilon_1^4\sin(\Upsilon_0).
\end{aligned} \tag{10}$$

By applying the APs in (10) and Lemma 1 into (6), we can deduce the expressions of $\Upsilon_i, i = 0, 1, \ldots$. For the sake of parsimony we write only a few expressions as given in Appendix A. Consequently, the solution of the Equation (1) with a nonlinear part $F(\Upsilon(\eta)) = \sin(\Upsilon(\eta))$ is given by

$$\begin{aligned}
\Upsilon(\eta) = c &- \frac{2c}{\sqrt{\pi}}\eta^{1/2} + (c + \sin(c))\eta - \frac{4c}{3\sqrt{\pi}}(\cos(c) + \sin(c) + 1)\eta^{3/2}\\
&+ \left(c\cos(c) - \frac{c^2}{\pi}\sin(c) + \frac{1}{2}\sin(c) + \frac{1}{2}\sin(c)\cos(c) + \frac{c}{2}\right)\eta^2 + \cdots.
\end{aligned} \tag{11}$$

In view of the FPSM (8) with $F(\Upsilon(\eta)) = \sin(\Upsilon(\eta))$, we can deduce

$$\sum_{n=1}^{\infty}\frac{n}{2}\Upsilon_n(\eta)\,\eta^{\frac{n}{2}-1} + \sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n+1}{2}\right)}\Upsilon_n(\eta)\,\eta^{\frac{n-1}{2}} = \sin\left(\sum_{n=0}^{\infty}\Upsilon_n(\eta)\,\eta^{\frac{n}{2}}\right). \tag{12}$$

Calculating the right hand side of (12) yields:

$$P(\Upsilon(\eta)) = \sin\left(\Upsilon_0 + \Upsilon_1\eta^{\frac{1}{2}} + \Upsilon_2\eta + \Upsilon_3\eta^{\frac{3}{2}} + \Upsilon_4\eta^2 + \cdots\right).$$

Using the trigonometric identity $\sin(z+w) = \sin(z)\cos(w) + \cos(z)\sin(w)$, we have

$$\begin{aligned}
P(\Upsilon(\eta)) = \sin(\Upsilon_0)&\cos\left(\Upsilon_1\eta^{\frac{1}{2}} + \Upsilon_2\eta + \Upsilon_3\eta^{\frac{3}{2}} + \Upsilon_4\eta^2 + \cdots\right)\\
&+ \cos(\Upsilon_0)\sin\left(\Upsilon_1\eta^{\frac{1}{2}} + \Upsilon_2\eta + \Upsilon_3\eta^{\frac{3}{2}} + \Upsilon_4\eta^2 + \cdots\right).
\end{aligned}$$

Applying the Taylor expansions for

$$\cos\left(\Upsilon_1\eta^{\frac{1}{2}} + \Upsilon_2\eta + \Upsilon_3\eta^{\frac{3}{2}} + \cdots\right) \quad \text{and} \quad \sin\left(\Upsilon_1\eta^{\frac{1}{2}} + \Upsilon_2\eta + \Upsilon_3\eta^{\frac{3}{2}} + \cdots\right),$$

we obtain

$$\begin{aligned}
P(\Upsilon(\eta)) = \sin(\Upsilon_0)&\cos\left(1 - \frac{1}{2}\left(\Upsilon_1\eta^{\frac{1}{2}} + \Upsilon_2\eta + \cdots\right)^2 + \frac{1}{4!}\left(\Upsilon_1\eta^{\frac{1}{2}} + \Upsilon_2\eta + \cdots\right)^4 - \cdots\right)\\
&+ \cos(\Upsilon_0)\sin\left(\left(\Upsilon_1\eta^{\frac{1}{2}} + \Upsilon_2\eta + \cdots\right) - \frac{1}{3!}\left(\Upsilon_1\eta^{\frac{1}{2}} + \Upsilon_2\eta + \cdots\right)^3 + \cdots\right),
\end{aligned}$$

so that

$$\begin{aligned}
P(\Upsilon(\eta)) = \sin(\Upsilon_0)&\cos\left(1 - \frac{1}{2}\left(\Upsilon_1^2\eta + 2\Upsilon_1\Upsilon_2\eta^{\frac{3}{2}} + \cdots\right) + \frac{1}{4!}\Upsilon_1^4\eta^2 + \cdots\right)\\
&+ \cos(\Upsilon_0)\sin\left(\left(\Upsilon_1\eta^{\frac{1}{2}} + \Upsilon_2\eta + \Upsilon_3\eta^{\frac{3}{2}} + \Upsilon_4\eta^2 + \cdots\right) - \frac{1}{3!}\Upsilon_1^3\eta^{\frac{3}{2}} + \cdots\right).
\end{aligned}$$

This implies that

$$
\begin{aligned}
P(\mathrm{Y}(\eta)) = {} & \sin(\mathrm{Y}_0) + \cos(\mathrm{Y}_0)\mathrm{Y}_1\eta^{\frac{1}{2}} + \cos(\mathrm{Y}_0)\mathrm{Y}_2\eta - \sin(\mathrm{Y}_0)\frac{1}{2}\mathrm{Y}_1^2\eta + \cos(\mathrm{Y}_0)\mathrm{Y}_3\eta^{\frac{3}{2}} \\
& - \sin(\mathrm{Y}_0)\mathrm{Y}_1\mathrm{Y}_2\eta^{\frac{3}{2}} - \cos(\mathrm{Y}_0)\frac{1}{3!}\mathrm{Y}_1^3\eta^{\frac{3}{2}} + \cos(\mathrm{Y}_0)\mathrm{Y}_4\eta \\
& - \sin(\mathrm{Y}_0)\left(\frac{1}{2}\mathrm{Y}_2^2 + \mathrm{Y}_1\mathrm{Y}_3\right)\eta^2 - \frac{1}{2}\cos(\mathrm{Y}_0)\mathrm{Y}_1^2\mathrm{Y}_2\eta + \frac{1}{4!}\mathrm{Y}_1^4\sin(\mathrm{Y}_0)\eta^2 + \cdots. \quad (13)
\end{aligned}
$$

Substituting (13) into the right side of (12) and using the initial condition $\mathrm{Y}_0 = c$, we obtain

$$
\begin{aligned}
& \sum_{n=1}^{\infty} \frac{n}{2}\, \mathrm{Y}_n(\eta)\, \eta^{\frac{n}{2}-1} + \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n+1}{2}\right)} \mathrm{Y}_n(\eta)\, \eta^{\frac{n-1}{2}} \\
& = \sin(c) + \cos(c)\mathrm{Y}_1\eta^{\frac{1}{2}} + \cos(c)\mathrm{Y}_2\eta - \sin(c)\frac{1}{2}\mathrm{Y}_1^2\eta + \cos(c)\mathrm{Y}_3\eta^{\frac{3}{2}} \\
& - \sin(c)\mathrm{Y}_1\mathrm{Y}_2\eta^{\frac{3}{2}} - \cos(c)\frac{1}{3!}\mathrm{Y}_1^3\eta^{\frac{3}{2}} + \cos(c)\mathrm{Y}_4\eta \\
& - \sin(c)\left(\frac{1}{2}\mathrm{Y}_2^2 + \mathrm{Y}_1\mathrm{Y}_3\right)\eta^2 - \frac{1}{2}\cos(c)\mathrm{Y}_1^2\mathrm{Y}_2\eta + \frac{1}{4!}\mathrm{Y}_1^4\sin(c)\eta^2 + \cdots. \quad (14)
\end{aligned}
$$

Equating the coefficients of like terms from both sides of (14), we can write

$$
\begin{aligned}
\mathrm{Y}_0 &= c, \\
\mathrm{Y}_1 &= -\frac{2c}{\sqrt{\pi}}, \\
\mathrm{Y}_2 &= c + \sin(c), \\
\mathrm{Y}_3 &= -\frac{4}{3\sqrt{\pi}}(c\cos(c) + \sin(c) + c), \\
\mathrm{Y}_4 &= \frac{1}{2}(c + \sin(c) + 2c\cos(c) + \sin(c)\cos(c)) - \frac{c^2}{\pi}\sin(c), \\
\mathrm{Y}_5 &= \frac{8}{15\pi^{3/2}}\Big(c^3\cos(c) + 2c^2\sin(c) - c\pi\cos^2(c) - 2\pi\sin(c)\cos(c) - 3c\pi\cos(c) \\
& \quad - \pi\sin(c) - c\pi\Big), \\
\mathrm{Y}_6 &= \frac{1}{6}\sin(c)\cos^2(c) + \frac{c}{2}\cos^2(c) - \frac{17c^2}{9\pi}\sin(c)\cos(c) + \frac{1}{2}\sin(c)\cos(c) + \frac{2c}{3}\cos(c) \\
& \quad - \frac{1}{6}\sin^3(c) - \frac{7c}{12}\sin^2(c) - \frac{5c^2}{12}\sin(c) - \frac{8c}{9\pi}\sin^2(c) - \frac{11c^2}{9\pi}\sin(c) - \frac{5c^3}{6\pi}\cos(c) \\
& \quad + \frac{2c^4}{9\pi^2}\sin(c) + \frac{1}{6}\sin(c) + \frac{c}{6},
\end{aligned}
$$

and similarly for other components. Hence, from the Equation (8), we have

$$\begin{aligned}
\Upsilon(\eta) &= c - \frac{2c}{\sqrt{\pi}}\eta^{1/2} + (c+\sin(c))\eta - \frac{4}{3\sqrt{\pi}}(c\cos(c)+\sin(c)+c)\eta^{3/2} \\
&+ \left[\frac{1}{2}(c+\sin(c)+2c\cos(c)+\sin(c)\cos(c)) - \frac{c^2}{\pi}\sin(c)\right]\eta^2 \\
&+ \left[\frac{8}{15\pi^{3/2}}\Big(c^3\cos(c)+2c^2\sin(c)-c\pi\cos^2(c)-2\pi\sin(c)\cos(c)-3c\pi\cos(c)\right. \\
&\left. - \pi\sin(c) - c\pi\Big)\right]\eta^{5/2} + \left[\frac{1}{6}\sin(c)\cos^2(c) + \frac{c}{2}\cos^2(c) - \frac{17c^2}{9\pi}\sin(c)\cos(c)\right. \\
&+ \frac{1}{2}\sin(c)\cos(c) + \frac{2c}{3}\cos(c) - \frac{1}{6}\sin^3(c) - \frac{7c}{12}\sin^2(c) - \frac{5c^2}{12}\sin(c) - \frac{8c}{9\pi}\sin^2(c) \\
&\left. - \frac{11c^2}{9\pi}\sin(c) - \frac{5c^3}{6\pi}\cos(c) + \frac{2c^4}{9\pi^2}\sin(c) + \frac{1}{6}\sin(c) + \frac{c}{6}\right]\eta^3 + \cdots .
\end{aligned} \tag{15}$$

3.2. *Case 2:* $F(\Upsilon(\eta)) = \cos(\Upsilon(\eta))$

The *APs* for $\cos(\Upsilon(\eta))$ are given by [30]:

$$\begin{aligned}
A_0 &= \cos(\Upsilon_0), \\
A_1 &= -\Upsilon_1\sin(\Upsilon_0), \\
A_2 &= -\Upsilon_2\sin(\Upsilon_0) - \frac{1}{2}\Upsilon_1^2\cos(\Upsilon_0), \\
A_3 &= -\Upsilon_3\sin(\Upsilon_0) - \Upsilon_1\Upsilon_3\cos(\Upsilon_0) + \frac{1}{3!}\Upsilon_1^3\sin(\Upsilon_0), \\
A_4 &= -\Upsilon_4\sin(\Upsilon_0) - \left(\frac{1}{2}\Upsilon_2^2 + \Upsilon_1\Upsilon_3\right)\cos(\Upsilon_0) + \frac{1}{2}\Upsilon_1^2\Upsilon_2\sin(\Upsilon_0) + \frac{1}{4!}\Upsilon_1^4\cos(\Upsilon_0).
\end{aligned} \tag{16}$$

By using the *APs* in (16) and Lemma 1 into (6), we can deduce the expressions $\Upsilon_i, i = \{0, 1, \ldots, 6\}$, as given in Appendix B. Consequently, the solution of the Equation (1) with a nonlinear part $F(\Upsilon(\eta)) = \cos(\Upsilon(\eta))$ is given by

$$\begin{aligned}
\Upsilon(\eta) &= c - \frac{2c}{\sqrt{\pi}}\eta^{1/2} + (c+\cos(c))\eta + \frac{4c}{3\sqrt{\pi}}(\sin(c)-\cos(c)-1)\eta^{3/2} \\
&\quad + \left(-c\sin(c) - \frac{c^2}{\pi}\cos(c) + \frac{1}{2}\cos(c) - \frac{1}{2}\sin(c)\cos(c) + \frac{c}{2}\right)\eta^2 + \cdots .
\end{aligned} \tag{17}$$

In view of the FPSM (8) with $F(\Upsilon(\eta)) = \cos(\Upsilon(\eta))$, we can deduce

$$\sum_{n=1}^{\infty}\frac{n}{2}\Upsilon_n(\eta)\,\eta^{\frac{n}{2}-1} + \sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n+1}{2}\right)}\Upsilon_n(\eta)\,\eta^{\frac{n-1}{2}} = \cos\left(\sum_{n=0}^{\infty}\Upsilon_n(\eta)\,\eta^{\frac{n}{2}}\right). \tag{18}$$

Proceeding as before (where $F(\mathrm{Y}(\eta)) = \sin(\mathrm{Y}(\eta))$), we have

$$
\begin{aligned}
\mathrm{Y}_0 &= c,\\
\mathrm{Y}_1 &= -\frac{2c}{\sqrt{\pi}},\\
\mathrm{Y}_2 &= c + \cos(c),\\
\mathrm{Y}_3 &= \frac{4}{3\sqrt{\pi}}(c\sin(c) - \cos(c) - c),\\
\mathrm{Y}_4 &= \frac{1}{2}(c + \cos(c) - 2c\sin(c) - \sin(c)\cos(c)) - \frac{c^2}{\pi}\cos(c),\\
\mathrm{Y}_5 &= \frac{8}{15\pi^{3/2}}\Big(-c^3\sin(c) + 2c^2\cos(c) - c\pi\sin^2(c) + 2\pi\sin(c)\cos(c) + 24c\pi\sin(c)\\
&\quad - \pi\cos(c) - c\pi\Big),\\
\mathrm{Y}_6 &= \frac{1}{9\pi}\Big(25c^2\sin(c)\cos(c) - 16c\cos^2(c) - 19c^2\cos(c)\Big) + \frac{2c^4}{9\pi^2}\cos(c) + \frac{5c^3}{6\pi}\sin(c)\\
&\quad + \frac{1}{6}\Big(\sin^2(c)\cos(c) + 3c\sin^2(c) - 3\sin(c)\cos(c) - 4c\sin(c)\\
&\quad + c + \cos(c) - \cos^3(c) - 2c\cos^2(c) - c^2\cos(c)\Big),
\end{aligned}
$$

and the same procedure for other components, where we have used the trigonometric identity $\cos(z + w) = \cos(z)\cos(w) - \sin(z)\sin(w)$. Hence, from Equation (8), we can write

$$
\begin{aligned}
\mathrm{Y}(\eta) &= c - \frac{2c}{\sqrt{\pi}}\eta^{1/2} + (c + \cos(c))\eta + \frac{4}{3\sqrt{\pi}}(c\sin(c) - \cos(c) - c)\eta^{3/2}\\
&\quad + \left[\frac{1}{2}(c + \cos(c) - 2c\sin(c) - \sin(c)\cos(c)) - \frac{c^2}{\pi}\cos(c)\right]\eta^2\\
&\quad + \Big[\frac{8}{15\pi^{3/2}}\Big(-c^3\sin(c) + 2c^2\cos(c) - c\pi\sin^2(c) + 2\pi\sin(c)\cos(c) + 24c\pi\sin(c)\\
&\quad - \pi\sin(c) - c\pi\Big)\Big]\eta^{5/2} + \Big[\frac{1}{9\pi}\Big(25c^2\sin(c)\cos(c) - 16c\cos^2(c) - 19c^2\cos(c)\Big)\\
&\quad + \frac{2c^4}{9\pi^2}\cos(c) + \frac{5c^3}{6\pi}\sin(c) + \frac{1}{6}\Big(\sin^2(c)\cos(c) + 3c\sin^2(c) - 3\sin(c)\cos(c)\\
&\quad - 4c\sin(c) + c + \cos(c) - \cos^3(c) - 2c\cos^2(c) - c^2\cos(c)\Big)\Big]\eta^3 + \cdots.
\end{aligned}
\tag{19}
$$

3.3. *Case 3:* $F(\mathrm{Y}(\eta)) = e^{\mathrm{Y}}$

The *APs* for e^{Y} are given by [30]:

$$
\begin{aligned}
A_0 &= e^{\mathrm{Y}_0},\\
A_1 &= \mathrm{Y}_1 e^{\mathrm{Y}_0},\\
A_2 &= \left(\frac{1}{2}\mathrm{Y}_1^2 + \mathrm{Y}_2\right)e^{\mathrm{Y}_0},\\
A_3 &= \left(\frac{1}{3!}\mathrm{Y}_1^3 + \mathrm{Y}_1\mathrm{Y}_2 + \mathrm{Y}_3\right)e^{\mathrm{Y}_0},\\
A_4 &= \left(\mathrm{Y}_4 + \frac{1}{2}\mathrm{Y}_2^2 + \mathrm{Y}_1\mathrm{Y}_3 + \frac{1}{2}\mathrm{Y}_1^2\mathrm{Y}_2 + \frac{1}{4!}\mathrm{Y}_1^4\right)e^{\mathrm{Y}_0}.
\end{aligned}
\tag{20}
$$

Using the *APs* in (20) and Lemma 1 into (6), we can deduce the expressions Y_i, $i = \{0, 1, \ldots, 6\}$, as presented in Appendix C. Consequently, the solution of the Equation (1) with a nonlinear part $F(\mathrm{Y}(\eta)) = e^{\mathrm{Y}}$ is given by

$$\mathrm{Y}(\eta) = c - \frac{2c}{\sqrt{\pi}}\eta^{1/2} + (c + e^c)\eta - \frac{4}{3\sqrt{\pi}}(c + (c+1)e^c)\eta^{3/2} + \left(\frac{1}{2}e^c + c + \frac{c^2}{\pi} + \frac{1}{2}\right)e^c\eta^2 + \cdots. \quad (21)$$

In view of the FPSM (8) with $F(\mathrm{Y}(\eta)) = e^{\mathrm{Y}}$, we can deduce

$$\sum_{n=1}^{\infty} \frac{n}{2}\,\mathrm{Y}_n(\eta)\,\eta^{\frac{n}{2}-1} + \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n+1}{2}\right)}\mathrm{Y}_n(\eta)\,\eta^{\frac{n-1}{2}} = \exp\left(\sum_{n=0}^{\infty} \mathrm{Y}_n(\eta)\,\eta^{\frac{n}{2}}\right). \quad (22)$$

Additionally, we have:

$$P(\mathrm{Y}(\eta)) = \exp\left(\sum_{n=0}^{\infty} \mathrm{Y}_n(\eta)\,\eta^{\frac{n}{2}}\right) = e^{\mathrm{Y}_0 + \mathrm{Y}_1\eta^{\frac{1}{2}} + \mathrm{Y}_2\eta + \mathrm{Y}_3\eta^{\frac{3}{2}} + \cdots}.$$

Since $e^{z+w} = e^z e^w$, for each $z, w \in \mathbb{R}$, we write

$$P(\mathrm{Y}(\eta)) = e^{\mathrm{Y}_0} e^{\left(\mathrm{Y}_1\eta^{\frac{1}{2}} + \mathrm{Y}_2\eta + \mathrm{Y}_3\eta^{\frac{3}{2}} + \mathrm{Y}_4\eta^2 + \cdots\right)}.$$

Using the Taylor expansion, we get

$$P(\mathrm{Y}(\eta)) = e^{\mathrm{Y}_0}\left[1 + \left(\mathrm{Y}_1\eta^{\frac{1}{2}} + \mathrm{Y}_2\eta + \mathrm{Y}_3\eta^{\frac{3}{2}} + e^{\mathrm{Y}_4\eta^2} + \cdots\right) + \frac{1}{2!}\left(\mathrm{Y}_1\eta^{\frac{1}{2}} + \mathrm{Y}_2\eta + \mathrm{Y}_3\eta^{\frac{3}{2}} + \mathrm{Y}_4\eta^2 + \cdots\right)^2 + \cdots\right].$$

We now group all terms with identical sum of subscripts, resulting

$$P(\mathrm{Y}(\eta)) = e^{\mathrm{Y}_0} + \mathrm{Y}_1\eta^{\frac{1}{2}}e^{\mathrm{Y}_0} + \left(\mathrm{Y}_2 + \frac{1}{2!}\mathrm{Y}_1^2\right)\eta e^{\mathrm{Y}_0} + \left(\mathrm{Y}_3 + \mathrm{Y}_1\mathrm{Y}_2 + \frac{1}{3!}\mathrm{Y}_1^3\right)\eta^{\frac{3}{2}}e^{\mathrm{Y}_0} + \left(\mathrm{Y}_4 + \mathrm{Y}_1\mathrm{Y}_3 + \frac{1}{2!}\mathrm{Y}_2^2 + \frac{1}{2!}\mathrm{Y}_1^2\mathrm{Y}_2 + \frac{1}{4!}\mathrm{Y}_1^4\right)\eta^2 e^{\mathrm{Y}_0} + \cdots. \quad (23)$$

Substituting (23) into the right side of (22) and using the initial condition $\mathrm{Y}_0 = c$, we obtain

$$\sum_{n=1}^{\infty} \frac{n}{2}\,\mathrm{Y}_n(\eta)\,\eta^{\frac{n}{2}-1} + \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n+1}{2}\right)}\mathrm{Y}_n(\eta)\,\eta^{\frac{n-1}{2}} = e^{\mathrm{Y}_0} + \mathrm{Y}_1\eta^{\frac{1}{2}}e^{\mathrm{Y}_0} + \left(\mathrm{Y}_2 + \frac{1}{2!}\mathrm{Y}_1^2\right)\eta e^{\mathrm{Y}_0} + \left(\mathrm{Y}_3 + \mathrm{Y}_1\mathrm{Y}_2 + \frac{1}{3!}\mathrm{Y}_1^3\right)\eta^{\frac{3}{2}}e^{\mathrm{Y}_0} + \left(\mathrm{Y}_4 + \mathrm{Y}_1\mathrm{Y}_3 + \frac{1}{2!}\mathrm{Y}_2^2 + \frac{1}{2!}\mathrm{Y}_1^2\mathrm{Y}_2 + \frac{1}{4!}\mathrm{Y}_1^4\right)\eta^2 e^{\mathrm{Y}_0} + \cdots. \quad (24)$$

Equating the coefficients of like terms from both sides of (24), we can write

$$
\begin{aligned}
Y_0 &= c,\\
Y_1 &= -\frac{2c}{\sqrt{\pi}},\\
Y_2 &= c+e^c,\\
Y_3 &= \frac{-4}{3\sqrt{\pi}}(ce^c+e^c+c),\\
Y_4 &= \frac{c^2}{\pi}e^c+\frac{1}{2}e^{2c}+ce^c+\frac{c}{2}+\frac{1}{2}e^c,\\
Y_5 &= -\frac{8}{15\pi^{3/2}}\left(c^3e^c+\frac{5c\pi}{2}e^{2c}+\frac{3c^2\pi}{2}e^c+2\pi\, e^{2c}+3c\pi\, e^c+2c^2e^c+c\pi+\pi\, e^c\right),\\
Y_6 &= \frac{17c^2}{9\pi}e^{2c}+\frac{1}{3}e^{3c}+\frac{13c}{12}e^{2c}+\frac{2c}{3}e^c+\frac{1}{2}e^{2c}+\frac{5c^2}{12}e^c\\
&\quad+\frac{8c}{9\pi}e^{2c}+\frac{11c^2}{9\pi}e^c+\frac{5c^3}{6\pi}e^c+\frac{2c^4}{9\pi^2}e^c+\frac{c}{6}+\frac{1}{6}e^c,
\end{aligned}
\tag{25}
$$

and similarly for other components. Substituting (25) in (8), we get

$$
\begin{aligned}
Y(\eta) &= c-\frac{2c}{\sqrt{\pi}}\eta^{1/2}+(c+e^c)\eta-\frac{4}{3\sqrt{\pi}}(ce^c+e^c+c)\eta^{3/2}\\
&\quad+\left(\frac{c^2}{\pi}e^c+\frac{1}{2}e^{2c}+ce^c+\frac{c}{2}+\frac{1}{2}e^c\right)\eta^2\\
&\quad-\frac{8}{15\pi^{3/2}}\left(c^3e^c+\frac{5c\pi}{2}e^{2c}+\frac{3c^2\pi}{2}e^c+2\pi\, e^{2c}+3c\pi\, e^c+2c^2e^c+c\pi+\pi\, e^c\right)\eta^{5/2}\\
&\quad+\Big(\frac{17c^2}{9\pi}e^{2c}+\frac{1}{3}e^{3c}+\frac{13c}{12}e^{2c}+\frac{2c}{3}e^c+\frac{1}{2}e^{2c}+\frac{5c^2}{12}e^c+\frac{8c}{9\pi}e^{2c}+\frac{11c^2}{9\pi}e^c\\
&\quad+\frac{5c^3}{6\pi}e^c+\frac{2c^4}{9\pi^2}e^c+\frac{c}{6}+\frac{1}{6}e^c\Big)\eta^3+\cdots.
\end{aligned}
\tag{26}
$$

3.4. Case 4: $F(Y(\eta)) = e^{-Y(\eta)}$

The *APs* for $e^{-Y(\eta)}$ are given by [30]:

$$
\begin{aligned}
A_0 &= e^{-Y_0},\\
A_1 &= -Y_1e^{-Y_0},\\
A_2 &= \left(\frac{1}{2}Y_1^2-Y_2\right)e^{-Y_0},\\
A_3 &= \left(-\frac{1}{3!}Y_1^3+Y_1Y_2-Y_3\right)e^{-Y_0},\\
A_4 &= \left(-Y_4+\frac{1}{2}Y_2^2+Y_1Y_3-\frac{1}{2}Y_1^2Y_2+\frac{1}{4!}Y_1^4\right)e^{-Y_0}.
\end{aligned}
\tag{27}
$$

By using the *APs* in (27) and Lemma 1 into (6), we can deduce the expressions $Y_i, i=\{0,1,\dots,6\}$, as presented in Appendix D. Consequently, the solution of the Equation (1) with a nonlinear part $F(Y(\eta)) = e^{-Y(\eta)}$ is given by

$$
\begin{aligned}
Y(\eta) &= c-\frac{2c}{\sqrt{\pi}}\eta^{1/2}+(c+e^{-c})\eta-\frac{4}{3\sqrt{\pi}}((c-1)e^c-c)\eta^{3/2}\\
&\quad+\left(\frac{-1}{2}e^{-c}+\frac{c^2}{\pi}+\frac{1}{2}-c\right)e^{-c}\eta^2+\cdots.
\end{aligned}
\tag{28}
$$

In view of the FPSM (8) with $F(\Upsilon(\eta)) = e^{-\Upsilon(\eta)}$, we can deduce

$$\sum_{n=1}^{\infty} \frac{n}{2}\Upsilon_n(\eta)\,\eta^{\frac{n}{2}-1} + \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n+1}{2}\right)}\Upsilon_n(\eta)\,\eta^{\frac{n-1}{2}} = \exp\left(-\sum_{n=0}^{\infty}\Upsilon_n(\eta)\,\eta^{\frac{n}{2}}\right). \tag{29}$$

Proceeding as in Case 3, we obtain

$$\begin{aligned}
\Upsilon_0 &= c,\\
\Upsilon_1 &= -\frac{2c}{\sqrt{\pi}},\\
\Upsilon_2 &= c + e^{-c},\\
\Upsilon_3 &= \frac{4}{3\sqrt{\pi}}\left(ce^{-c} - e^{-c} - c\right),\\
\Upsilon_4 &= \frac{c}{2} + \frac{1}{2}e^{-c} - \frac{c^2}{\pi}e^{-c} - \frac{1}{2}e^{-2c} - ce^{-c},\\
\Upsilon_5 &= \frac{8}{15\pi^{3/2}}\Big(2\pi\, e^{-2c} + 3c\pi\, e^{-c} - c^3e^{-c} - \frac{5c\pi}{2}e^{-2c}\\
&\quad - \frac{3c^2\pi}{2}e^{-c} - 2c^2e^{-c} - c\pi - \pi\, e^{-c}\Big),\\
\Upsilon_6 &= -\frac{17c^2}{9\pi}e^{-2c} + \frac{1}{3}e^{-3c} + \frac{13c}{12}e^{-2c} - \frac{2c}{3}e^{-c} - \frac{1}{2}e^{-2c} + \frac{5c^2}{12}e^{-c}\\
&\quad + \frac{8c}{9\pi}e^{-2c} + \frac{11c^2}{9\pi}e^{-c} - \frac{5c^3}{6\pi}e^{-c} + \frac{2c^4}{9\pi^2}e^{-c} + \frac{c}{6} + \frac{1}{6}e^{-c}.
\end{aligned} \tag{30}$$

Substituting (30) into (8), we get

$$\begin{aligned}
\Upsilon(\eta) &= c - \frac{2c}{\sqrt{\pi}}\eta^{1/2} + \left(c + e^{-c}\right)\eta + \frac{4}{3\sqrt{\pi}}\left(ce^{-c} - e^{-c} - c\right)\eta^{3/2}\\
&\quad + \left(\frac{c}{2} + \frac{1}{2}e^{-c} - \frac{c^2}{\pi}e^{-c} - \frac{1}{2}e^{-2c} - ce^{-c}\right)\eta^2 + \frac{8}{15\pi^{3/2}}\Big(2\pi\, e^{-2c} + 3c\pi\, e^{-c}\\
&\quad - c^3e^{-c} - \frac{5c\pi}{2}e^{-2c} - \frac{3c^2\pi}{2}e^{-c} - 2c^2e^{-c} - c\pi - \pi\, e^{-c}\Big)\eta^{5/2}\\
&\quad + \Big(\frac{1}{3}e^{-3c} - \frac{17c^2}{9\pi}e^{-2c} + \frac{13c}{12}e^{-2c} - \frac{2c}{3}e^{-c} - \frac{1}{2}e^{-2c} + \frac{5c^2}{12}e^{-c} + \frac{8c}{9\pi}e^{-2c}\\
&\quad + \frac{11c^2}{9\pi}e^{-c} - \frac{5c^3}{6\pi}e^{-c} + \frac{2c^4}{9\pi^2}e^{-c} + \frac{c}{6} + \frac{1}{6}e^{-c}\Big)\eta^3 + \cdots.
\end{aligned} \tag{31}$$

Remark 2. *By comparing the pairs of Equations* (11) *and* (15), (17) *and* (19), (21) *and* (26), *and* (28) *and* (31), *we can find some common sub-expressions. This comes with no surprise given the mathematical relationship between the four selected functions.*

4. Test Examples

In this section, we solve four illustrative examples to show the accuracy and efficiency of the obtained results. In this section, h represents the step size, so that $\eta_i = ih$. Additionally, absolute error is the absolute of the error between the two approximate solutions, that is, Absolute Error $= |\sigma_5(\eta) - \Upsilon(\eta)|$.

Example 1. *Consider the following NFDE:*

$$\frac{d\Upsilon}{d\eta} + {}^{RL}D_{0+}^{\frac{1}{2}}\Upsilon(\eta) = e^{\Upsilon(\eta)}, \qquad \Upsilon(0) = -3. \tag{32}$$

Table 1 compares the results for $c = -3$ when using the ADM ($\sigma_5 = \sum_{j=0}^{5} \mathrm{Y}_j$) and the FPSM for $h = 0.001$. Figure 1 shows the numerical solution of Equation (32) for various values of h. It can be concluded that the numerical results obtained by ADM and FPSM are in good agreement when h is small enough.

Table 1. Comparison of the numerical solutions and absolute errors for Equation (32).

η	ADM (σ_5)	FPSM	Absolute Error
0.001	−2.895830485459265	−2.895830485959273	$5.000080349759628 \times 10^{-10}$
0.002	−2.854309691415448	−2.854309695415539	$4.000091369249503 \times 10^{-9}$
0.003	−2.823068788065474	−2.823068801565858	$1.350038347780469 \times 10^{-8}$
0.004	−2.797138899820482	−2.797138931821545	$3.200106313272499 \times 10^{-8}$
0.005	−2.774596809639852	−2.774596872142201	$6.250234863003357 \times 10^{-8}$
0.006	−2.754456876224310	−2.754456984228803	$1.080044929047119 \times 10^{-7}$
0.007	−2.736133991439165	−2.736134162946946	$1.715077808661647 \times 10^{-7}$
0.008	−2.719247187482035	−2.719247443494563	$2.560125276218628 \times 10^{-7}$
0.009	−2.703532023937481	−2.703532388456558	$3.645190767009865 \times 10^{-7}$
0.010	−2.688796094003734	−2.688796594031533	$5.000277987221580 \times 10^{-7}$

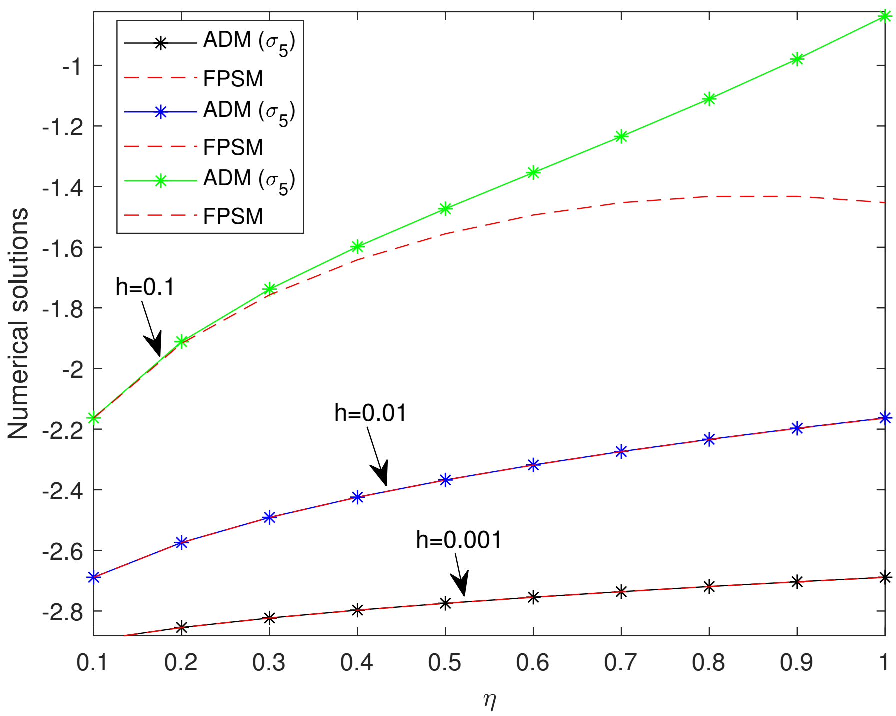

Figure 1. The ADM (σ_5) and FPSM solutions of Equation (32), $\mathrm{Y}(\eta)$ vs η, for $h = \{0.1, 0.01, 0.001\}$.

Example 2. *Consider the following NFDE:*

$$\frac{d\mathrm{Y}}{d\eta} + {}^{RL}D^{\frac{1}{2}}_{0+}\mathrm{Y}(\eta) = e^{-\mathrm{Y}(\eta)}, \qquad \mathrm{Y}(0) = e^{-2}. \tag{33}$$

Table 2 compares the results for $c = e^{-2}$ when using the ADM and FPSM for $h = 0.001$. Figure 2 shows the numerical solution of Equation (33) for various values of h. It can be concluded that the numerical results obtained by the new methods are in good agreement for small value of h.

Table 2. Comparison of the numerical solutions and absolute errors for Equation (33).

η	ADM (σ_5)	FPSM	Absolute Error
0.001	0.131493773315120	0.131493763159134	$1.015598596887379 \times 10^{-8}$
0.002	0.130463574078089	0.130463533585157	$4.049293203522986 \times 10^{-8}$
0.003	0.129887409961439	0.129887319172856	$9.078858251299060 \times 10^{-8}$
0.004	0.129542977152579	0.129542816356528	$1.607960508631656 \times 10^{-7}$
0.005	0.129344587858063	0.129344337608454	$2.502496097567253 \times 10^{-7}$
0.006	0.129248502746662	0.129248143878728	$3.588679343413670 \times 10^{-7}$
0.007	0.129228859614488	0.129228373258195	$4.863562931001209 \times 10^{-7}$
0.008	0.129268886151545	0.129268253743375	$6.324081692210548 \times 10^{-7}$
0.009	0.129356986562976	0.129356189856445	$7.967065303038368 \times 10^{-7}$
0.010	0.129484757528826	0.129483778603963	$9.789248623659486 \times 10^{-7}$

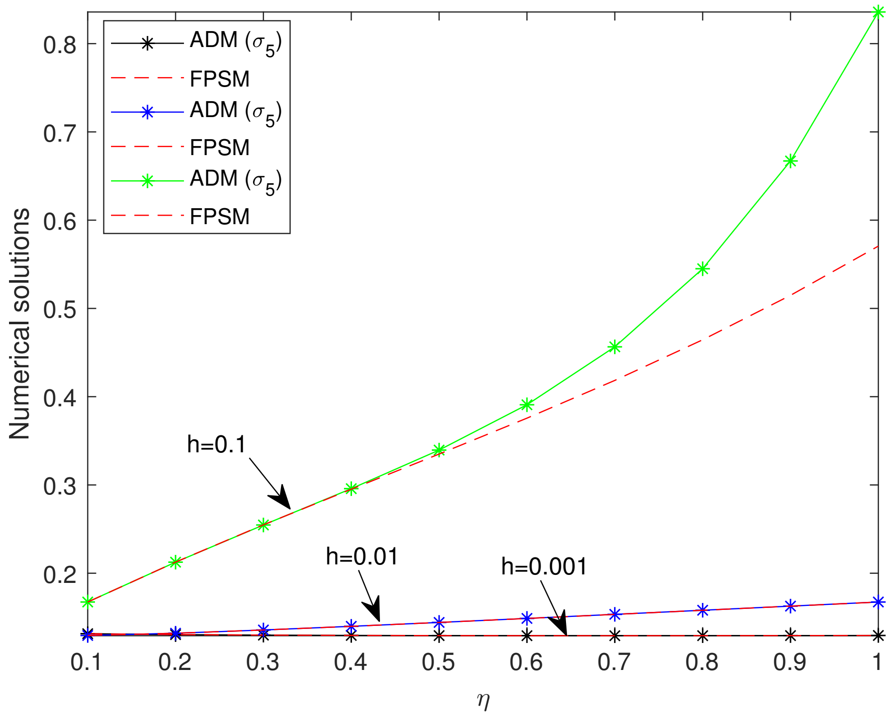

Figure 2. The ADM (σ_5) and FPSM solutions of Equation (33), $Y(\eta)$ vs η, for $h = \{0.1, 0.01, 0.001\}$.

Example 3. *Consider the following NFDE:*

$$\frac{dY}{d\eta} + {}^{RL}D_{0+}^{\frac{1}{2}} Y(\eta) = \sin(Y(\eta)), \qquad Y(0) = \frac{\pi}{6}. \tag{34}$$

Table 3 compares the results for $c = \frac{\pi}{6}$ when using the ADM and FPSM for $h = 0.001$. Figure 3 shows the numerical solution of Equation (34) for various values of h. Additionally, we can see that the numerical results are in good agreement for h to be small enough.

Table 3. Comparison of the numerical solutions and absolute errors for Equation (34).

η	ADM (σ_5)	FPSM	Absolute Error
0.001	0.505905033461083	0.505905042904738	$9.443654747087749 \times 10^{-9}$
0.002	0.499128746084825	0.499128799795063	$5.371023825695431 \times 10^{-8}$
0.003	0.494136214151394	0.494136362826111	$1.486747165935221 \times 10^{-7}$
0.004	0.490062464318850	0.490062770750714	$3.064318633283492 \times 10^{-7}$
0.005	0.486573127883647	0.486573665199666	$5.373160190114135 \times 10^{-7}$
0.006	0.483497046192112	0.483497896765728	$8.505736160957511 \times 10^{-7}$
0.007	0.480732731511055	0.480733986206116	$1.254695060848299 \times 10^{-6}$
0.008	0.478214179380159	0.478215936988369	$1.757608209640438 \times 10^{-6}$
0.009	0.475895625311021	0.475897992113969	$2.366802947961766 \times 10^{-6}$
0.010	0.473743807625935	0.473746897042911	$3.089416975798898 \times 10^{-6}$

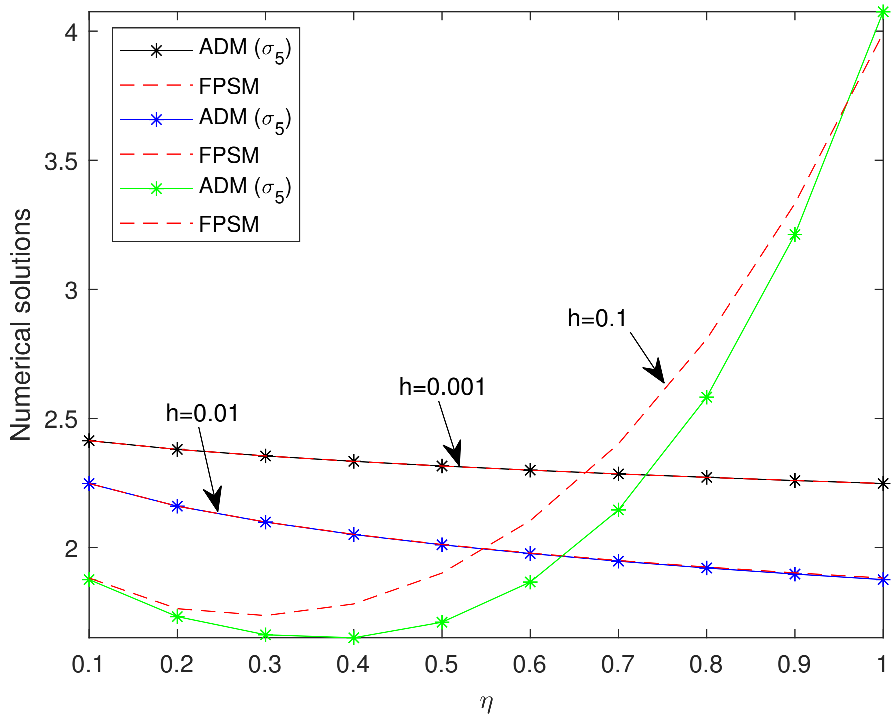

Figure 3. The ADM (σ_5) and FPSM solutions of Equation (34), $\Upsilon(\eta)$ vs η, for $h = \{0.1, 0.01, 0.001\}$.

Example 4. *Consider the following NFDE:*

$$\frac{d\Upsilon}{d\eta} + {}^{RL}D_{0+}^{\frac{1}{2}}\Upsilon(\eta) = \cos(\Upsilon(\eta)), \qquad \Upsilon(0) = \frac{\pi}{2}. \tag{35}$$

The comparison of the results for $c = \frac{\pi}{2}$ and $h = 0.001$ is tabulated in Table 4 using the ADM and FPSM. Figure 4 illustrates the numerical solution of Equation (35) for $h = \{0.1, 0.01, 0.001\}$.

Table 4. Comparison of numerical solutions and absolute errors for Equation (35).

η	ADM (σ_5)	FPSM	Absolute Error
0.001	1.516316429541230	1.516316429795210	$2.539795040945592 \times 10^{-10}$
0.002	1.494668256266018	1.494668258273179	$2.007161326034179 \times 10^{-9}$
0.003	1.478420422611530	1.478420429323466	$6.711936784142836 \times 10^{-9}$
0.004	1.464967274055237	1.464967289843333	$1.578809571967099 \times 10^{-8}$
0.005	1.453299536931999	1.453299567562721	$3.063072151476831 \times 10^{-8}$
0.006	1.442899597793140	1.442899650407696	$5.261455604532728 \times 10^{-8}$
0.007	1.433459973351498	1.433460056448405	$8.309690668895087 \times 10^{-8}$
0.008	1.424780323615165	1.424780447034934	$1.234197690713756 \times 10^{-7}$
0.009	1.416721493995723	1.416721668907195	$1.749114721949496 \times 10^{-7}$
0.010	1.409182168676689	1.409182407564684	$2.388879947190503 \times 10^{-7}$

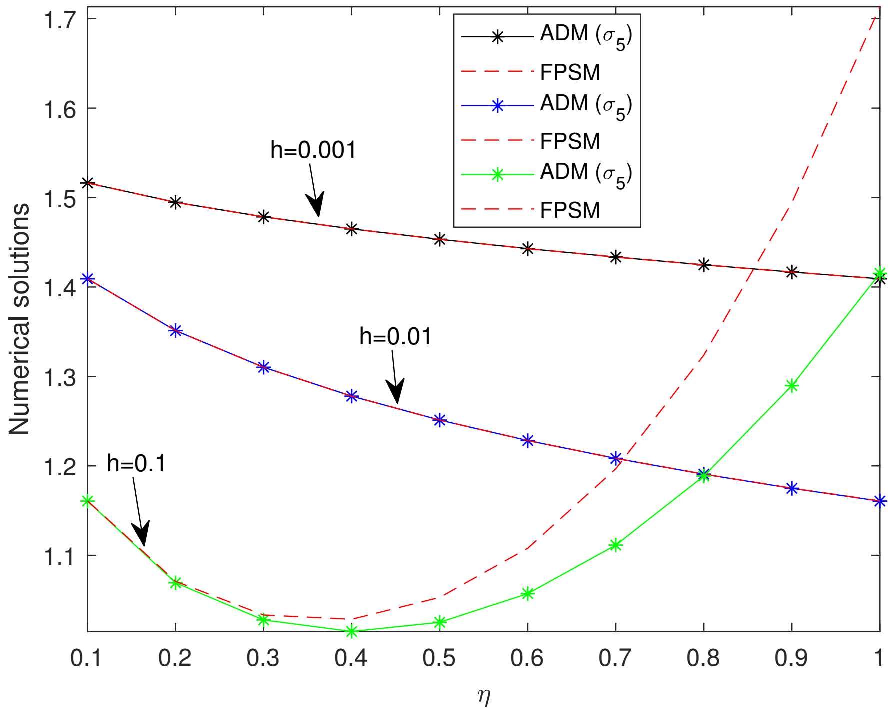

Figure 4. The ADM (σ_5) and FPSM solutions of Equation (35), $Y(\eta)$ vs η, for $h = \{0.1, 0.01, 0.001\}$.

5. Conclusions

The aim of the current study was to investigate two numerical algorithms, namely the performance of the ADM and FPSM. The methods were first presented and analyzed in detail and then used to solve the NFDE with unknown analytical solution. The ADM and FPSM were compared by means of four numerical examples. The results for both strategies are in good agreement, especially when h is small enough.

Author Contributions: Conceptualization, P.O.M., J.L.G.G.; methodology, P.O.M., R.P.A.; software, P.O.M., R.P.A., J.L.G.G.; validation, P.O.M., R.P.A., J.A.T.M.; formal analysis, P.O.M., R.P.A.; investigation, P.O.M.; resources, P.O.M., J.L.G.G.; data curation, P.O.M., R.P.A.; writing—original draft preparation, R.P.A.; writing—review and editing, R.P.A., P.O.M., J.A.T.M.; visualization, R.P.A.; supervision, P.O.M., R.P.A., J.L.G.G. All authors have read and agreed to the final version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: This research has been partially supported by Ministerio de Ciencia, Innovacion y Universidades grant number PGC2018-0971-B-100 and Fundacion Seneca de la Region de Murcia grant number 20783/PI/18.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A. The First Six Expressions of Υ_i for Case 1

The first six expressions for $\Upsilon_i, i = 0, \ldots, 5$, for Case 1 in Section 3.1 are as follows:

$$
\begin{aligned}
\Upsilon_0 &= c, \\
\Upsilon_1 &= \sin(c)\eta - 2c\sqrt{\frac{\eta}{\pi}}, \\
\Upsilon_2 &= c\eta - \frac{4c}{3\sqrt{\pi}}(\cos(c) + \sin(c))\eta^{3/2} + \frac{1}{2}\sin(c)\cos(c)\eta^2,
\end{aligned}
$$

$$
\begin{aligned}
\Upsilon_3 = &-\frac{4c}{3\sqrt{\pi}}\eta^{3/2} + \left(c\cos(c) - \frac{c^2}{\pi}\sin(c) + \frac{1}{2}\sin(c)\right)\eta^2 \\
&+ \frac{4}{15\sqrt{\pi}}\left(3c\sin^2(c) - 4\sin(c)\cos(c) - 2c\cos^2(c)\right)\eta^{5/2} + \frac{1}{6}\cos(2c)\sin(c)\eta^3,
\end{aligned}
$$

$$
\begin{aligned}
\Upsilon_4 = &\frac{1}{2}c\eta^2 + \frac{8}{15\pi^{3/2}}\left(2c^2\sin(c) - 3c\pi\cos(c) - \pi\sin(c) + c^3\cos(c) + \frac{3}{2}c^2\sin(c)\right)\eta^{5/2} \\
&+ \left(\frac{c}{2}\cos^2(c) - \frac{7}{12}c\sin^2(c) + \frac{9\pi - 34c^2}{18\pi}\sin(c)\cos(c) - \frac{8c^2}{9\pi}\sin(c)\right)\eta^3 \\
&+ \frac{2}{105\sqrt{\pi}}\left(14\sin^3(c) - 24\sin(c)\cos^2(c) + 64c\sin^2(c)\cos(c) - 8c\cos^3(c)\right)\eta^{7/2} \\
&+ \left(\frac{1}{24}\sin(c)\cos^3(c) - \frac{5}{24}\sin^3(c)\cos(c)\right)\eta^4,
\end{aligned}
$$

$$
\begin{aligned}
\Upsilon_5 = &-\frac{8c}{15\sqrt{\pi}}\eta^{5/2} + \left(\frac{2c}{3}\cos(c) - \frac{5c^3}{6\pi}\cos(c) - \frac{11c^2}{9\pi}\sin(c) + \frac{2c^4}{9\pi^2}\sin(c) + \frac{2 - 5c^2}{12}\sin(c)\right)\eta^3 \\
&+ \frac{2}{105\pi^{3/2}}\left(48c^3\cos^2(c) - 50c^3\sin^2(c) + \left(\frac{9240}{63} - 92\pi\right)c^2\sin(c)\cos(c) + \frac{128}{3}c\sin^2(c)\right. \\
&\left. + 83c\pi\sin^2(c) - 48c\pi\cos^2(c) - 32\pi\sin(c)\cos(c)\right)\eta^{7/2} + \left(\frac{9c^2}{10\pi}\sin^3(c) - \frac{2}{9\pi}\sin^3(c)\right. \\
&- \frac{13}{48}\sin^3(c) + \frac{c}{6}\cos^3(c) - \frac{28c}{15\pi}\sin^2(c)\cos(c) - \frac{169c^2}{90\pi}\sin(c)\cos^2(c) - \frac{31c}{32\pi}\sin^2(c)\cos(c) \\
&\left. + \frac{1}{4}\sin(c)\cos^2(c)\right)\eta^4 + \frac{1}{\sqrt{\pi}}\left(\frac{32c}{945}\cos^4(c) + \frac{776}{945}\sin^3(c)\cos(c) + \frac{184c}{189}\sin^2(c)\cos^2(c)\right. \\
&\left. - \frac{44c}{135}\sin^4(c) - \frac{128}{945}\sin(c)\cos^3(c)\right)\eta^{9/2} \\
&+ \left(\frac{1}{24}\sin^5(c) + \frac{1}{120}\sin(c)\cos^4(c) - \frac{3}{20}\sin^3(c)\cos^2(c)\right)\eta^5.
\end{aligned}
$$

Appendix B. The First Six Expressions of Υ_i for Case 2

The first six expressions for $\Upsilon_i, i = 0, \ldots, 5$, for Case 2 in Section 3.2 are given by:

$$\begin{aligned}
\Upsilon_0 &= c, \\
\Upsilon_1 &= \cos(c)\eta - 2c\sqrt{\frac{\eta}{\pi}}, \\
\Upsilon_2 &= c\eta + \frac{4c}{3\sqrt{\pi}}(\sin(c) - \cos(c))\eta^{3/2} - \frac{1}{2}\sin(c)\cos(c)\eta^2, \\
\Upsilon_3 &= -\frac{4c}{3\sqrt{\pi}}\eta^{3/2} + \left(-c\sin(c) - \frac{c^2}{\pi}\cos(c) + \frac{1}{2}\cos(c)\right)\eta^2 \\
&\quad + \frac{4}{15\sqrt{\pi}}\left(3c\cos^2(c) + 4\sin(c)\cos(c) - 2c\sin^2(c)\right)\eta^{5/2} - \frac{1}{6}\cos(c)\cos(2c)\eta^3,
\end{aligned}$$

$$\begin{aligned}
\Upsilon_4 &= \frac{1}{2}c\eta^2 + \frac{8}{15\pi^{3/2}}\left(2c^2\cos(c) + 3c\pi\sin(c) - \pi\cos(c) - c^3\sin(c) + \frac{3}{2}c^2\cos(c)\right)\eta^{5/2} \\
&\quad + \left(\frac{c}{2}\sin^2(c) - \frac{7}{12}c\cos^2(c) + \frac{34c^2 - 9\pi}{18\pi}\sin(c)\cos(c) - \frac{8c^2}{9\pi}\cos(c)\right)\eta^3 \\
&\quad + \frac{2}{105\sqrt{\pi}}\left(4\sin^3(c) - 24\sin^2(c)\cos(c) - 64c\sin(c)\cos^2(c) + 8c\sin^3(c)\right)\eta^{7/2} \\
&\quad + \left(\frac{5}{24}\sin(c)\cos^3(c) - \frac{1}{24}\sin^3(c)\cos(c)\right)\eta^4,
\end{aligned}$$

$$\begin{aligned}
\Upsilon_5 &= -\frac{8c}{15\sqrt{\pi}}\eta^{5/2} + \left(\frac{5c^3}{6\pi}\sin(c) - \frac{2c}{3}\sin(c) - \frac{11c^2}{9\pi}\cos(c) + \frac{2c^4}{9\pi^2}\cos(c) + \frac{2-5c^2}{12}\cos(c)\right)\eta^3 \\
&\quad + \frac{2}{105\pi^{3/2}}\left(48c^3\sin^2(c) - 50c^3\cos^2(c) - \left(\frac{9240}{63} + 92\pi\right)c^2\sin(c)\cos(c) + \frac{128}{3}c\cos^2(c)\right. \\
&\quad \left. + 83c\pi\cos^2(c) - 48c\pi\sin^2(c) + 32\pi\sin(c)\cos(c)\right)\eta^{7/2} + \left(\frac{9c^2}{10\pi}\cos^3(c) - \frac{2}{9\pi}\cos^3(c)\right. \\
&\quad - \frac{13}{48}\cos^3(c) - \frac{c}{6}\sin^3(c) + \frac{28c}{15\pi}\sin(c)\cos^2(c) - \frac{169c^2}{90\pi}\sin^2(c)\cos(c) + \frac{31c}{32\pi}\sin(c)\cos^2(c) \\
&\quad \left. + \frac{1}{4}\sin^2(c)\cos(c)\right)\eta^4 + \frac{1}{\sqrt{\pi}}\left(-\frac{32c}{945}\sin^4(c) - \frac{776}{945}\sin(c)\cos^3(c) + \frac{184c}{189}\sin^2(c)\cos^2(c)\right. \\
&\quad \left. - \frac{44c}{135}\cos^4(c) + \frac{128}{945}\sin^3(c)\cos(c)\right)\eta^{9/2} \\
&\quad + \left(\frac{1}{24}\cos^5(c) + \frac{1}{120}\sin^4(c)\cos(c) - \frac{3}{20}\sin^2(c)\cos^3(c)\right)\eta^5.
\end{aligned}$$

Appendix C. The First Six Expressions of Υ_i for Case 3

The first six expressions for $\Upsilon_i, i = 0, \ldots, 5$, for Case 3 in Section 3.3 are:

$$\begin{aligned}
\Upsilon_0 &= c, \\
\Upsilon_1 &= e^c\eta - 2c\sqrt{\frac{\eta}{\pi}}, \\
\Upsilon_2 &= c\eta - \frac{4(c+1)}{3\sqrt{\pi}}e^c\eta^{3/2} + \frac{1}{2}e^{2c}\eta^2, \\
\Upsilon_3 &= -\frac{4c}{3\sqrt{\pi}}\eta^{3/2} + \left(c + \frac{c^2}{\pi} + \frac{1}{2}\right)e^c\eta^2 - \frac{4}{15\sqrt{\pi}}(4+5c)e^{2c}\eta^{5/2} + \frac{1}{3}e^{3c}\eta^3,
\end{aligned}$$

$$\Upsilon_4 = \frac{1}{2}c\eta^2 - \frac{8}{15\pi^{3/2}}\left(\pi + 3c\pi + 2c^2 + \frac{3}{2}c^2\pi + c^3\right)e^{c}\eta^{5/2} + \left(\frac{1}{2} + \frac{13c}{12} + \frac{8c}{9\pi} + \frac{17c^2}{9\pi}\right)e^{2c}\eta^3 + \frac{4}{105\sqrt{\pi}}(26 - 35c)e^{3c}\eta^{7/2} + \frac{1}{4}e^{4c}\eta^4,$$

$$\Upsilon_5 = -\frac{8c}{15\sqrt{\pi}}\eta^{5/2} + \left(\frac{1}{6} + \frac{2c}{3} + \frac{5c^2}{12} + \frac{11c^2}{9\pi} + \frac{5c^3}{6\pi} + \frac{2c^4}{9\pi^2}\right)e^{c}\eta^3 - \frac{2}{105\pi^{3/2}}\left(32\pi + 131\pi c + \frac{128c}{3} + \frac{8190c^2}{63} + 92c^2\pi - 98c^3\right)e^{2c}\eta^{7/2} + \left(\frac{2}{9\pi} + \frac{25}{48} + \frac{109c}{96} + \frac{28c}{15\pi} + \frac{25c^2}{9\pi}\right)e^{3c}\eta^4 + \frac{4}{\sqrt{\pi}}\left(\frac{226}{945} - \frac{c}{3}\right)e^{4c}\eta^{9/2} + \frac{1}{5}e^{5c}\eta^6.$$

Appendix D. The First Six Expressions of Υ_i for Case 4

The first six expressions for $\Upsilon_i, i = 0, \ldots, 5$, for Case 4 in Section 3.4:

$$\Upsilon_0 = c,$$
$$\Upsilon_1 = e^{-c}\eta - 2c\sqrt{\frac{\eta}{\pi}},$$
$$\Upsilon_2 = c\eta + \frac{4(c-1)}{3\sqrt{\pi}}e^{-c}\eta^{3/2} - \frac{1}{2}e^{-2c}\eta^2,$$
$$\Upsilon_3 = -\frac{4c}{3\sqrt{\pi}}\eta^{3/2} + \left(\frac{c^2}{\pi} + \frac{1}{2} - c\right)e^{-c}\eta^2 - \frac{4}{15\sqrt{\pi}}(4 - 5c)e^{-2c}\eta^{5/2} + \frac{1}{3}e^{-3c}\eta^3,$$

$$\Upsilon_4 = \frac{1}{2}c\eta^2 + \frac{8}{15\pi^{3/2}}\left(-\pi + 3c\pi - 2c^2 - \frac{3c^2}{2}\pi + c^3\right)e^{-c}\eta^{5/2} + \left(-\frac{1}{2} + \frac{13c}{12} + \frac{8c}{9\pi} - \frac{17c^2}{9\pi}\right)e^{-2c}\eta^3 + \frac{4}{105\sqrt{\pi}}(35c - 26)e^{-3c}\eta^{7/2} - \frac{1}{4}e^{-4c}\eta^4,$$

$$\Upsilon_5 = -\frac{8c}{15\sqrt{\pi}}\eta^{5/2} + \left(\frac{1}{6} - \frac{2c}{3} + \frac{5c^2}{12} + \frac{11c^2}{9\pi} - \frac{5c^3}{6\pi} + \frac{2c^4}{9\pi^2}\right)e^{-c}\eta^3 + \frac{2}{105\pi^{3/2}}\left(32\pi - 131\pi c - \frac{128c}{3} + \frac{8190c^2}{63} + 92c^2\pi - 98c^3\right)e^{-2c}\eta^{7/2} + \left(\frac{2}{9\pi} + \frac{25}{48} - \frac{109c}{96} - \frac{28c}{15\pi} + \frac{25c^2}{9\pi}\right)e^{-3c}\eta^4 + \frac{4}{\sqrt{\pi}}\left(\frac{226}{945} - \frac{c}{3}\right)e^{-4c}\eta^{9/2} + \frac{1}{5}e^{-5c}\eta^6.$$

References

1. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; North-Holland Mathematics Studies; Elsevier: Amsterdam, The Netherlands, 2006; Volume 204.
2. Hilfer, R. *Applications of Fractional Calculus in Physics*; World Scientific: Singapore, 2000.
3. Podlubny, I. *Fractional Differential Equations*; Academic Press: New York, NY, USA, 1999.
4. Mainardi, F. *Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models*; Imperial College Press: Singapore, 2010.
5. Magin, R. *Fractional Calculus in Bioengineering*; Begall House Publisher, Inc.: Connecticut, CT, USA, 2006.
6. Wu, J.; Yuan, J.; Gao, W. Analysis of fractional factor system for data transmission in SDN. *Appl. Math. Nonlinear Sci.* **2019**, *4*, 191–196. [CrossRef]
7. Şahin, R.; Yağcı, O. Fractional calculus of the extended hypergeometric function. *Appl. Math. Nonlinear Sci.* **2020**, *5*, 369–384. [CrossRef]
8. Balcı, M.A. Fractional interaction of financial agents in a stock market network. *Appl. Math. Nonlinear Sci.* **2020**, *5*, 317–336. [CrossRef]

9. Alqudah, M.A.; Mohammed, P.O.; Abdeljawad, T. Solution of singular integral equations via Riemann—Liouville fractional integrals. *Math. Prob. Eng.* **2020**, *2020*. [CrossRef]
10. Srivastava, H.M. Fractional-order derivatives and integrals: Introductory overview and recent developments. *Kyungpook Math. J.* **2020**, *60*, 73–116.
11. Srivastava, H.M.; Mohammed, P.O. A correlation between solutions of uncertain fractional forward difference equations and their paths. *Front. Phys.* **2020**, *8*. [CrossRef]
12. Mohammed, P.O.; Abdeljawad, T.; Jarad, F.; Chu, Y.-M. Existence and uniqueness of uncertain fractional backward difference equations of Riemann-Liouville type. *Math. Probl. Eng.* **2020**, *2020*. [CrossRef]
13. Günerhan, H.; Çelik, E. Analytical and approximate solutions of fractional partial differential-algebraic equations. *Appl. Math. Nonlinear Sci.* **2020**, *5*, 109–120. [CrossRef]
14. Kurt, A.; Şenol, M.; Tasbozan, O.; Chand, M. Two reliable methods for the solution of fractional coupled Burgers' equation arising as a model of Polydispersive sedimentation. *Appl. Math. Nonlinear Sci.* **2019**, *4*, 523–534. [CrossRef]
15. Touchent, K.A.; Hammouch, Z.; Mekkaoui, T. A modified invariant subspace method for solving partial differential equations with non-singular kernel fractional derivatives. *Appl. Math. Nonlinear Sci.* **2020**, *5*, 35–48. [CrossRef]
16. Onal, M.; Esen, A. A Crank-Nicolson approximation for the time fractional Burgers equation. *Appl. Math. Nonlinear Sci.* **2020**, *5*, 177–184. [CrossRef]
17. Srivastava, H.M.; Mohammed, P.O.; Guirao, J.L.G.; Hamed, Y.S. Some higher-degree lacunary fractional splines in the approximation of fractional differential equations. *Symmetry* **2021**, *13*, 422. [CrossRef]
18. Ganjiani, M. Solution of nonlinear fractional differential equations using homotopy analysis method. *Appl. Math. Model.* **2010**, *34*, 1634–1641. [CrossRef]
19. Gaoa, G.-H.; Sun, Z.-Z.; Zhang, H.-W. A new fractional numerical differentiation formula to approximate the Caputo fractional derivative and its applications. *J. Comput. Phys.* **2014**, *259*, 33–50. [CrossRef]
20. Liu, Q.; Liu, F.; Turner, I.; Anh, V. Finite element approximation for a modified anomalous subdiffusion equation. *Appl. Math. Model.* **2011**, *35*, 4103–4116. [CrossRef]
21. Zayernouri, M.; Karniadakis, G.E. Fractional spectral collocation methods for linear and nonlinear variable order FPDEs. *J. Comput. Phys.* **2015**, *293*, 312–338. [CrossRef]
22. Wang, C.; Zhang, H.; Wang, S. Positive Solution of a Nonlinear Fractional Differential Equation Involving Caputo Derivative. *Discret. Dyn. Nat. Soc.* **2012**, *2012*, 425408. [CrossRef]
23. Bekir, A.; Güner, Ö. Exact solutions of nonlinear fractional differential equations by (G′/G)-expansion method. *Chin. Phys. B* **2013**, *22*, 110202. [CrossRef]
24. Gao, G.; Sun, Z. A compact finite difference scheme for the fractional subdiffusion equations. *J. Comput. Phys.* **2011**, *230*, 586–595. [CrossRef]
25. Duan, J.-S.; Chaolu, T.; Rach, R. Solutions of the initial value problem for nonlinear fractional ordinary differential equations by the Rach-Adomian-Meyers modified decomposition method. *Appl. Math. Comput.* **2012**, *218*, 8370–8392. [CrossRef]
26. Modanli, M.; Abdulazeez, S.T.; Husien, A.M. A residual power series method for solving pseudo hyperbolic partial differential equations with nonlocal conditions. *Numer. Methods Partial. Differ. Equ.* **2021**, *37*, 2235–2243. [CrossRef]
27. Lu, T.-T.; Zheng, W.-Q. Adomian decomposition method for first order PDEs with unprescribed data. *Alex. Eng. J.* **2021**, *60*, 2563–2572. [CrossRef]
28. Ortigueira, M.D.; Machado, J.A.T.; da Costa, J.S. Which Differintegration? *IEE Proc. Vis. Image Signal Process.* **2005**, *152*, 846–850. [CrossRef]
29. Wazwaz, A.M. *Linear and Nonlinear Integral Equations*; Higher Education Press: Beijing, China; Springer: Berlin/Heidelberg, Germany; New York, NY, USA, 2011.
30. Wazwaz, A.M. A new algorithm for calculating Adomian polynomials for nonlinear operators. *Appl. Math. Comput.* **2000**, *111*, 33–51. [CrossRef]
31. Cherrault, Y. Convergence of Adomian's decomposition method. *Math. Comp. Model.* **1990**, *14*, 83–86. [CrossRef]