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Some Variants of Normal Čech Closure Spaces via Canonically Closed Sets

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Abstract: New generalizations of normality in Čech closure space such as π-normal, weakly π-normal and κ-normal are introduced and studied using canonically closed sets. It is observed that the class of κ-normal spaces contains both the classes of weakly π-normal and almost normal Čech closure spaces.

Keywords: closure space; canonically closed; weakly normal; almost normal; π-normal; weakly π-normal; κ-normal

MSC: 04A05; 54D15

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1. Introduction and Preliminaries

It is evident from the literature that topological structures which are more general than the classical topology are more suitable for the study of digital topology, image processing, network theory, pattern recognition and related areas. Various generalized structures such as closure spaces, generalized closure spaces, Čech closure spaces, generalized topologies (GT), weak structures (WS), Generalized neighborhood systems (GNS) etc. were introduced and studied in the past (see [1–5]). However, recently Čech closure spaces attracted the attention of researchers due to its possibility of application in other applied fields discussed above. Usefulness of this Čech closure setting in variety of allied fields such as digital topology, computer graphics, image processing and pattern recognition are available in the literature [6–9]. Čech closure space was defined by Čech [1], are obtained from Kuratowski [10] closure operator by omitting the idempotent condition. In this setting Galton [11] studied the motion of an object in terms of a function giving its position at each time and systematically investigated what a continuous motion looks like. J. Šlapal [6] observed that this structure is more suitable than others for application in digital topology because Čech closure spaces are well-behaved with respect to connectedness. Allam et al. [12,13] introduced a new method for generating closure spaces via a binary relation which was subsequently used by G. Liu [14] to establish a one-to-one correspondence between quasi discrete closures and reflexive relation. Furthermore, J. Šlapal and John L. Pfaltz [15] studied network structures via associated closure operators. Higher separation axioms in Čech closure space was introduced by Barbel M. R. Stadler and F. Peter Stadler [16] in 2003 and discussed the concept of Urysohn functions, normal, regular, completely normal etc. in the form of neighborhood. In 2018 Gupta and Das [17] introduced higher separation axioms via relation. Since normality is an important topological property, many weak variants of normality introduced and studied in the past to properly study normality in general topology (See [18–22]). In the present paper, we introduced some variants of normality in Čech closure space as π-normal, weakly π-normal and κ-normal using canonically closed sets. It is observed that some characterizations of normality and almost normality which holds in topological spaces may not hold in

Čech closure spaces. Further relation between newly defined notions and already defined notions was also investigated.

A closure space is a pair (X, cl), where X is any set and closure $cl : P(X) \rightarrow P(X)$ is a function associating with each subset $A \subseteq X$ to a subset $cl(A) \subseteq X$, called the closure of A, such that $cl(\emptyset) = \emptyset$, $A \subseteq cl(A)$, $cl(A \cup B) = cl(A) \cup cl(B)$. With any closure cl for a set X there is associated the interior operation int_{cl}, usually denoted by int, which is a single-valued relation on $P(X)$ ranging in $P(X)$ such that for each $A \subseteq X$, $int_{cl}(A) = X - cl(X - A)$. The set $int_{cl}(A)$ is called the interior of A in (X, cl). In a closure space (X, cl), a set A is closed if $cl(A) = A$ and open if its complement is closed i.e., if $cl(X - A) = (X - A)$. In other words, a set is open if and only if $int(A) = A$. Additionally, from closure axioms we have $cl(A \cap B) \subseteq cl(A) \cap cl(B)$ and $int(A) \cup int(B) \subseteq int(A \cup B)$. In a Čech closure space a canonically closed (regularly closed) set is a closed set A of X such that $cl(int(A)) = A$ and a canonically open (regularly open) set is an open set U of X such that $int(cl(U)) = U$.

Definition 1. *[23] A Čech closure space (X, cl) is said to be*

1. *normal if for every two disjoint closed sets $A = cl(A)$ and $B = cl(B)$ there exist disjoint open sets U and V containing $cl(A)$ and $cl(B)$ respectively.*
2. *almost normal if for every two disjoint closed sets $cl(A) = A$ and $cl(B) = B$ out of which one is canonically closed there exist disjoint open sets U and V containing $cl(A)$ and $cl(B)$ respectively.*
3. *weakly normal if for every two disjoint closed sets $cl(A) = A$ and $cl(B) = B$ there exists an open set U such that $A \subseteq U$ and $int(cl(U)) \cap B = \emptyset$.*

Remark 1. *The notion of normality defined above in the Definition 1 is different from the notion of normality defined in [1]. A closure space is said to be normal [1] if every pair of sets with disjoint closures are separated by disjoint neighborhoods. The disjoint sets considered by Čech for separation in the definition of normality are not necessarily closed sets and neighborhoods need not be open. Throughout the present paper, we have taken the notion of normality only in the sense of Definition 1.*

Lemma 1. *[1] If U and V are subsets of a closure space (X, cl) such that $U \subseteq V$ then $cl(U) \subseteq cl(V)$.*

Theorem 1. *[23] Suppose (X, cl) is a Čech closure space such that $int(cl(U))$ is canonically open for every open set U. Then (X, cl) is weakly normal and almost normal implies (X, cl) is normal.*

2. Variants of Normal Čech Closure Space

Definition 2. *Let (X, cl) be a Čech closure space then A is said to be π-closed if it is equal to the intersection of two canonically closed set.*

Example 1. *Let $X = \{a, b, c, d\}$ be the set and define $cl : P(X) \rightarrow P(X)$ as $cl(\{a\}) = \{a\}$, $cl(\{b\}) = cl(\{a, b\}) = \{a, b\}$, $cl(\{c\}) = cl(\{a, c\}) = cl(\{c, d\}) = cl(\{a, c, d\}) = \{a, c, d\}$, $cl(\{d\}) = \{d\}$, $cl(\{a, d\}) = \{a, d\}$, $cl(\{b, c\}) = cl(\{a, b, c\}) = cl(\{b, c, d\}) = cl(X) = X$, $cl(\{b, d\}) = cl(\{a, b, d\}) = \{a, b, d\}$, $cl(\emptyset) = \emptyset$. Here, the set $A = \{a\}$ is π-closed as it is the intersection of two canonically closed set i.e., $\{a, c, d\}$ and $\{a, b\}$ but $\{a\}$ is not canonically closed. In this Čech closure space, $cl(A) = \{d\} = A$ is closed but not π-closed as it is not equal to the intersection of two canonically closed set.*

The implications in Figure 1 are obvious from the definitions. However, none of these implications is reversible as shown in the above example.

canonically closed $\longrightarrow$ π-closed $\longrightarrow$ closed

Figure 1. Interrelation of types of closed sets.

Definition 3. *A Čech closure space (X, cl) is π-normal if for every two disjoint closed sets one of which is π-closed there exist two disjoint open sets U and V containing the closed set and the π-closed set respectively.*

It is obvious that in a Čech closure space (X, cl), every normal space is π-normal. However, the converse need not be true as shown below.

Example 2. *A Čech closure space which is π-normal but not normal.*
Let $X = Y \cup \{p, q\}$ be an infinite set, then any set $A \in P(X)$ is one of the following four types of sets:

Type-I: A is finite in X.
Type-II: A is infinite in Y such that $p \notin A$ and $q \notin A$.
Type-III: $(Y - A)$ is finite and A contains either p or q.
Type-IV: $(Y - A)$ is finite and A contains both p and q.

Define $cl : P(X) \to P(X)$ by

$$cl(A) = \begin{cases} A, & \text{if } A \text{ is of type-I;} \\ A \cup \{p, q\}, & \text{if } A \text{ is of type-II;} \\ A \cup \{p, q\}, & \text{if } A \text{ is of type-III;} \\ A, & \text{if } A \text{ is of type-IV.} \end{cases}$$

In this Čech closure space, type-I and type-IV sets are closed sets. A set U is open if U is an infinite set containing p and/or q whose complement is finite. Additionally, a finite set U in Y whose complement is infinite is an open set in X. In this space only two types of sets are canonically closed. i.e., (1) Every finite set in Y is canonically closed (2) a set containing both p and q whose complement is finite in Y is canonically closed. This space is π-normal but not normal because for two disjoint closed sets $A = C \cup \{p\}$ and $B = D \cup \{q\}$, where C and D are finite in Y, there does not exist disjoint open sets satisfying the condition of normal Čech closure space.

Example 3. *A space which is not π-normal.*
Let X be the set of integers defined by

$$cl(\{x\}) = \begin{cases} x, & \text{if } x \text{ is even ;} \\ \{x-1, x, x+1\}, & \text{if } x \text{ is odd .} \end{cases}$$

and $cl(A) = \bigcup_{x \in A} cl(x)$.

This Čech closure space is not π-normal because for the π-closed set $cl(A) = \{4\} = A$ and a closed set $cl(B) = \{0, 1, 2\} = B$ there does not exist disjoint open sets containing $cl(A)$ and $cl(B)$ respectively.

Following examples establish that the notion of weak normality defined earlier, and the notion of π-normality are independent notions.

Example 4. *A space which is weakly normal but not π-normal.*
Let X be the set of positive integers. Define $cl : P(X) \to P(X)$ as defined in Example 3. Here, the Čech closure space (X, cl) is weakly normal but not π-normal as shown in Example 3.

Example 5. *A space which is π-normal but not weakly normal.*
Let $X = \{a, b, c, d\}$ be the set and define $cl : P(X) \to P(X)$ as $cl(\{a\}) = \{a\}$, $cl(\{b\}) = \{b\}$,

$cl(\{c\}) = \{a,c,d\}$, $cl(\{d\}) = \{d\}$, $cl(\{a,b\}) = \{a,b\}$, $cl(\{a,c\}) = \{a,c,d\}$, $cl(\{a,d\}) = \{a,d\}$, $cl(\{b,c\}) = X$, $cl(\{b,d\}) = \{b,d\}$, $cl(\{c,d\}) = \{a,c,d\}$, $cl(\{a,b,c\}) = X$, $cl(\{a,b,d\}) = \{a,b,d\}$, $cl(\{a,c,d\}) = \{a,c,d\}$, $cl(\{b,c,d\}) = X$, $cl(X) = X$, $cl(\emptyset) = \emptyset$. Here, (X,cl) is a π-normal Čech closure space which fails to be weakly normal because for two disjoint closed sets $A = \{a\} = cl(A)$ and $B = \{d\} = cl(B)$ there does not exists an open set U such that $cl(A) \subseteq U$ and $int(cl(U)) \cap B = \emptyset$.

Theorem 2. *If (X,cl) is a π-normal Čech closure space then for every π-closed set $cl(A) = A$ and for every open set U containing $cl(A)$ there exists an open set V such that $cl(A) \subseteq V \subseteq cl(V) \subseteq U$.*

Proof. Let $cl(A) = A$ be a π-closed set and U be an open set containing $cl(A)$. Since, (X,cl) is π-normal, there exist disjoint open sets V and W such that $cl(A) \subseteq V$ and $(X-U) \subseteq W$ implies $V \subseteq (X-W)$. Thus, by Lemma 1, $cl(V) \subseteq cl(X-W)$ implies $W \subseteq X - cl(V)$. Therefore, $(X-U) \subseteq W \subseteq X - cl(V)$ and hence $cl(A) \subseteq V \subseteq cl(V) \subseteq U$. □

Theorem 3. *If (X,cl) is a π-normal Čech closure space then for every closed set $cl(A) = A$ and for every π-open set U containing $cl(A)$ there exists an open set V such that $cl(A) \subseteq V \subseteq cl(V) \subseteq U$.*

Proof. Let $cl(A) = A$ be a closed set and U be a π-open set containing $cl(A)$ implies $(X-U)$ is a π-closed set which is disjoint from the closed set A. Since, (X,cl) is π-normal, there exist disjoint open sets V and W such that $cl(A) \subseteq V$ and $(X-U) \subseteq W$. Thus, $V \subseteq (X-W)$ implies $cl(V) \subseteq cl(X-W) = (X-W)$, and so, $W \subseteq (X - cl(V))$. Therefore, $(X-U) \subseteq W \subseteq (X - cl(V))$ and hence $cl(A) \subseteq V \subseteq cl(V) \subseteq U$. □

Definition 4. *[24] A Čech closure space (X,cl) is said to be regular if for a closed set $cl(A) = A$ and a point $x \notin cl(A)$ there exist disjoint open sets U and V such that $x \in U$ and $cl(A) \subseteq V$.*

Definition 5. *[1] A Čech closure space is said to be*

1. *T_1 if for two distinct points x and y, we have $x \notin cl(\{y\})$ and $y \notin cl(\{x\})$.*
2. *T_2 if any two distinct points x and y are separated.*

Remark 2. *In a Čech closure space, every normal T_1 space is regular and T_2. but if we replace normal by π-normal then the result need not be true. Consider the space defined in Example 2 which is π-normal and T_1 but neither T_2 nor regular. The space is not T_2 because disjoint points 'p' and 'q' cannot be separated and is not regular because for closed set $A = C \cup \{p\}$ where C is finite in Y and a point 'q' there does not exist disjoint open sets satisfying the required condition.*

Definition 6. *[24] A Čech closure space is said to be almost regular if for canonically closed set $cl(int(A)) = A$ and a point $x \notin cl(int(A))$ there exist disjoint open sets U and V such that $x \in U$ and $cl(int(A)) \subseteq V$.*

Theorem 4. *In a Čech closure space, every π-normal T_1 space is almost regular.*

Proof. let $cl(int(A)) = A$ be a canonically closed set and $x \notin cl(int(A))$ be a point. Since the space is a T_1 Čech closure space, the singleton set $\{x\}$ is closed. As every canonically closed set is π-closed, by π-normality there exist disjoint open sets U and V such that $cl(int(A) \subseteq U$ and $\{x\} \subseteq V$. Hence (X,cl) is an almost regular Čech closure space. □

Definition 7. *A Čech closure space is said to be weakly π-normal if for two disjoint π-closed sets there exist disjoint open sets separating them.*

Definition 8. *A Čech closure space is said to be κ-normal if for two disjoint canonically closed sets A and B there exist disjoint open sets U and V containing A and B respectively.*

From the definitions it is observed that every π-normal space is weakly π-normal, every weakly π-normal space as well as every almost normal space is κ-normal. Thus, the implications in Figure 2 are obvious but none of them is reversible which is exhibited below by Examples.

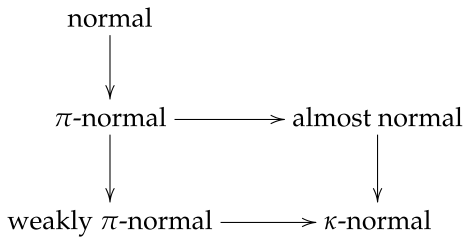

Figure 2. Interrelation of variants of normality.

Example 6. *A Čech closure space which is weakly π-normal but not π-normal.*
Let $X = \{a, b, c, d\}$ be a set and define $cl : P(X) \to P(X)$ as $cl(\{a\}) = cl(\{b\}) = cl(\{a, b\}) = \{a, b\}$, $cl(\{c\}) = \{c\}$, $cl(\{d\}) = cl(\{c, d\}) = \{b, c, d\}$, $cl(\{a, c\}) = cl(\{b, c\}) = cl(\{a, b, c\}) = \{a, b, c\}$, $cl(\{a, d\}) = cl(\{b, d\}) = cl(\{a, b, d\}) = cl(\{a, c, d\}) = cl(\{b, c, d\}) = cl(X) = X$, $cl(\emptyset) = \emptyset$. This space is vacuously weakly π-normal but not π-normal because for the π-closed set $\{a, b\}$ and a closed set $\{c\}$, there does not exist disjoint open sets containing $\{a, b\}$ and $\{c\}$.

Example 7. *A Čech closure space which is weakly π-normal but not almost normal.*
Let $X = \{a, b, c, d\}$ be the set and define $cl : P(X) \to P(X)$ as $cl(\{a\}) = \{a\}$, $cl(\{b\}) = cl(\{a, b\}) = \{a, b\}$, $cl(\{c\}) = cl(\{a, c\}) = cl(\{c, d\}) = cl(\{a, c, d\}) = \{a, c, d\}$, $cl(\{d\}) = \{d\}$, $cl(\{a, d\}) = \{a, d\}$, $cl(\{b, d\}) = cl(\{a, b, d\}) = \{a, b, d\}$, $cl(\{b, c\}) = cl(\{a, b, c\}) = cl(\{b, c, d\}) = cl(X) = X$, $cl(\emptyset) = \emptyset$. Clearly, (X, cl) is a Čech closure space which is vacuously weakly π-normal but not almost normal because for the canonically closed set $cl(int(A)) = \{a, b\} = A$ and the closed set $cl(B) = \{d\} = B$ there does not exist disjoint open sets containing A and B respectively.

Example 8. *A Čech closure space which is κ-normal but not almost normal.*
The Čech closure space defined in Example 7 is vacuously κ-normal but not almost normal as shown in Example 7.

Example 9. *A Čech closure space which is κ-normal.*
Let $X = Y \cup \{p, q\}$ be an infinite set. Define $cl : P(X) \to P(X)$ as in Example 2. Here, the closure space (X, cl) is κ-normal as for two disjoint canonically closed sets there exist disjoint open sets containing them.

Example 10. *A Čech closure space which is not κ-normal.*
Let X be the set of integers and define $cl : P(X) \to P(X)$ as shown in Example 3. This Čech closure space (X, cl) is not κ-normal because for two disjoint canonically closed sets $A = \{0, 1, 2\} = cl(int(A))$ and $B = \{4, 5, 6\} = cl(int(B))$ there does not exist disjoint open sets containing them.

Theorem 5. *If (X, cl) is a weakly π-normal Čech closure space then for every π-closed set A and for every π-open set U containing A there exists an open set V such that $A \subseteq V \subseteq cl(V) \subseteq U$.*

Proof. Let $cl(A) = A$ be a π-closed set and U be a π-open set containing $cl(A)$. Since, (X, cl) is weakly π normal, there exist disjoint open sets V and W such that $cl(A) \subseteq V$ and $(X - U) \subseteq W$. Thus, $V \subseteq X - W$ implies $cl(V) \subseteq cl(X - W) = (X - W)$. Therefore, $A \subseteq V \subseteq cl(V) \subseteq U$. □

Theorem 6. *If (X, cl) is a κ-normal Čech closure space then for every canonically closed set $cl(int(A)) = A$ and for every canonically open set $int(cl(U)) = U$ containing $cl(int(A))$ there exists an open set V such that $cl(int(A)) \subseteq V \subseteq cl(V) \subseteq int(cl(U))$.*

Proof. Proof of this theorem is similar to the proof of Theorem 5. □

Theorem 7. *Suppose (X, cl) is a Čech closure space such that $int(cl(U))$ is canonically open for every open set U. Then (X, cl) is weakly normal and κ-normal implies (X, cl) is almost normal.*

Proof. let $cl(int(A)) = A$ be a canonically closed set and $cl(B) = B$ be a closed set disjoint from canonically closed set $cl(int(A)) = A$. Since, (X, cl) is a weakly normal Čech closure space, there exists an open set U such that $A \subseteq U$ and $int(cl(U)) \cap B = \emptyset$. Since $int(cl(U))$ is canonically open, $X - int(cl(U))$ is canonically closed containing $cl(B)$. Thus, by κ-normality there exist disjoint open sets P and Q such that $cl(int(A)) \subseteq P$ and $cl(B) \subseteq X - (int(cl(U))) \subseteq Q$. Hence (X, cl) is an almost normal Čech closure space. □

Theorem 8. *Suppose (X, cl) is a T_1 Čech closure space such that $int(cl(U))$ is canonically open for every open set U. Then (X, cl) is weakly π-normal and weakly normal implies (X, cl) is almost regular.*

Proof. let $cl(int(A)) = A$ be a canonically closed set and $x \notin cl(int(A))$ be a point. Since (X, cl) is T_1, the singleton set $\{x\}$ is closed. By weak normality, there exists an open set U such that $A \subseteq U$ and $int(cl(U)) \cap \{x\} = \emptyset$. Since $int(cl(U))$ is canonically open, $X - int(cl(U))$ is canonically closed containing $\{x\}$. Thus, by weak π-normality, there exist disjoint open sets P and Q such that $cl(intA) \subseteq P$ and $\{x\} \subseteq X - (int(cl(U))) \subseteq Q$. Hence (X, cl) is an almost regular Čech closure space. □

It is clear from Example 11 that the T_1 axiom cannot be relaxed from the Theorem 8 as the space is weakly π-normal and weakly normal but not almost regular.

Example 11. *Let $X = \{a, b, c\}$ be the set and define $cl : P(X) \to P(X)$ as $cl(\{a\}) = \{a\}$, $cl(\{b\}) = \{a, b\}$, $cl(\{c\}) = \{a, c\}$, $cl(\{a, b\}) = \{a, b\}$, $cl(\{a, c\}) = \{a, c\}$, $cl(\{b, c\}) = cl(X) = X$, $cl(\emptyset) = \emptyset$. Clearly, (X, cl) is a Čech closure space which is weakly π-normal and weakly normal but not almost regular.*

Definition 9. *[24] A Čech closure space (X, cl) is said to be β-normal if for two disjoint closed sets $cl(A) = A$ and $cl(B) = B$ there exist disjoint open sets U and V whose closures are disjoint such that $cl(A \cap U) = cl(A)$ and $cl(B \cap V) = cl(B)$.*

Definition 10. *[24] A Čech closure space is extremally disconnected (E. D) if for every open set U, $cl(U)$ is open.*

Example 12. *A Space which is extremally disconnected.*
Let $X = \{a, b, c, d\}$ be the set. Define $cl : P(X) \to cl$ as $cl(\{a\}) = cl(\{a, c\}) = \{a, c\}$, $cl(\{b\}) = \{b\}$, $cl(\{c\}) = \{c\}$, $cl(\{d\}) = cl(\{b, d\}) = \{b, d\}$, $cl(\{a, b\}) = cl(\{a, b, c\}) = \{a, b, c\}$, $cl(\{b, c\}) = \{b, c\}$, $cl(\{c, d\}) = cl(\{b, c, d\}) = \{b, c, d\}$, $cl(\{a, d\}) = cl(\{a, b, d\}) = cl(\{a, c, d\}) = cl(X) = X$, $cl(\emptyset) = \emptyset$. In this space, closure of every open set is open. Thus, the space is extremally disconnected.

Theorem 9. *In an extremally disconnected Čech closure space (X, cl), every β-normal space is κ-normal.*

Proof. Let $cl(int(A)) = A$ and $cl(int(B)) = B$ be two disjoint canonically closed sets. Thus, $cl(int(A))$ and $cl(int(B))$ are two disjoint closed sets. We must show (X, cl) is κ-normal. Since (X, cl) is β-normal, there exist disjoint open sets U and V such that $cl(cl(A) \cap U) =$

$cl(A), cl(cl(B) \cap V) = cl(B)$ and $cl(U) \cap cl(V) = \emptyset$. Thus, $cl(A) = cl(cl(A) \cap U) \subseteq cl(U)$ and $cl(B) = cl(cl(B) \cap V) \subseteq cl(V)$. By extremally disconnectedness of (X, cl), $cl(U)$ and $cl(V)$ are two disjoint open sets containing $cl(A)$ and $cl(B)$ respectively. Hence (X, cl) is κ-normal. □

Example 13. *A Čech closure space which is κ-normal but not β-normal.*
Let $X = Y \cup \{p, q\}$ be an infinite set. Define $cl : P(X) \to P(X)$ as in Example 2. Here, the closure space (X, cl) is κ-normal but not β-normal because for two disjoint closed sets $cl(A) = C \cup \{p\}$ and $cl(B) = D \cup \{q\}$, where C and D are finite in Y, there does not exist disjoint open sets satisfying the condition of β-normal Čech closure space.

Example 14. *Let X be an infinite set. Define $cl : P(X) \to P(X)$ as defined in [1] by*

$$cl(A) = \begin{cases} A, & \text{if } A \text{ is finite;} \\ \\ X, & \text{otherwise.} \end{cases}$$

Here, (X, cl) is a Čech closure space which is T_1 almost normal but not regular because for closed set $cl(A) = A$ and a point disjoint from the closed set A there does not exist disjoint open sets separating them.

The following theorem directly follows from the Theorem 1.

Theorem 10. *Suppose (X, cl) is a weakly normal Čech closure space such that $int(cl(U))$ is canonically open for every open set U. Then following are equivalent:*

1. *(X, cl) is normal.*
2. *(X, cl) is π-normal.*
3. *(X, cl) is weakly π-normal.*
4. *(X, cl) is κ-normal.*
5. *(X, cl) is almost normal.*

3. Discussion and Conclusions

Closure space was first appeared in 1966 in the book "Topological Spaces" is popularly known as Čech closure space in the name of the author of the book E. Čech. After many decades of its introduction, it is now slowly becoming objects of increasing interest and importance. The purpose of this discussion is to discuss some important developments in this area in the last two decades. In 2003, some higher separation axioms including completely regular and completely normal spaces are studied in closure setting by Stadler et al. [16]. In 2008, Dimitrije Andrijević and others [25] considered families of subset of a closure space equipped with different Vietoris-like topologies and studied properties such as connectedness and compactness of the space and its hyperspaces. Subsequently in 2010, they generalized the notions of the compact-open and graph topology to the set of functions between two Čech closure spaces [26]. Additionally, they investigated how the separation properties (T_0, T_1 and regular) of the initial spaces are related to those of function spaces.

Recently, in 2021, Antonio Rieser [27] studied homotopy theory on the category of Čech closure spaces, whose objects are sets endowed with a Čech closure operator and whose morphisms are the continuous maps between them. They introduced some new classes of Čech closure structures on metric spaces, graphs, and simplicial complexes.

Another approach of generating closure spaces via a binary relation was also adopted by many researchers to address various issues in mathematics and other allied fields (see [12–15]). In [17], we have introduced and studied some new separation axioms on closure spaces generated through binary relations.

Apart from this, Junsheng Qiao [28] shown that the category of Čech closure spaces can be embedded in the category of stratified L-Čech closure spaces as a coreflective full subcategory. Perfilieva et al. [29] investigated the relationship between L-Fuzzy Čech closure spaces and L-Fuzzy co-topological spaces from the categorical viewpoint. Relational variants of categories related to L-Fuzzy closure spaces was studied in [30].

In this paper, we have defined and investigated few variants of normality in Čech closure spaces using canonically closed sets. Normality is an important topological property, and its importance is due to its behaviour as it behaves differently from other separation axioms for subspaces and products. Additionally, the class of normal spaces are more general than the important class of compact Hausdorff spaces. Normality involves separation of closed sets by open sets. On the other hand, in digital image processing a picture needs to be segmented into subsets where relationship of these subset from other neighboring subsets and adjoining points plays a prominent role for the processing of images. Such types of relationships between sets/points are either geometrical or topological. Geometrical relation involves position of points whereas topological relation involves concepts such as adjacency, neighborhood, separation, connectedness and compactness. So, the possibility of application of the notions defined in this paper in digital topology and digital image processing cannot be ruled out.

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