A Perturbed Algorithm for Generalized Nonlinear Quasi-Variational Inclusions

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Abstract: In this paper, a perturbed iterative method for solving a generalized nonlinear quasi-variational inclusions, is presented and a convergence result which generalizes some known results in this field, is given.

1. INTRODUCTION

In 1994, Hassouni and Moudafi [4], have introduced a perturbed method for solving a new class of variational inclusions and presented a convergence result. In 1996, Samir Adly [2], has studied a perturbed iterative method in order to approximate a solution for a general class of variational inclusions and proved the convergence of the iterative algorithm by using some fixed point theorems.

The aim of this paper is, firstly to present a new iterative algorithm for solving a generalized nonlinear quasi-variational inclusions. Then we prove the convergence of this algorithm, by using the definition of multi-valued relaxed Lipschitz operators. Our result is more general than the one considered in [3,4,5,6,8,9,10] which motivated us for the present work.
2. PRELIMINARIES

Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let $\phi : H \to R \cup \{+\infty\}$ be a proper convex lower semicontinuous function and $\delta \phi$ be the subdifferential of $\phi$. Given a multivalued map $T : H \to 2^H$, where $2^H$ denotes the family of nonempty subsets of $H$, and $f, g, m : H \to H$ be single-valued maps, then we consider the following generalized nonlinear quasi-variational inclusions problem (GNQVIP):

\[(\text{GNQVIP}): \text{Find } x \in H, \ w \in T(x) \text{ such that } g(x) \in \text{dom} \ (\delta \phi) + m(x), \ \text{and} \]

\[< g(x) - f(w), \ y + m(x) - g(x) \geq \phi(g(x) - m(x)) - \phi(y), \ \forall y \in H. \ (2.1)\]

Inequality (2.1) is called generalized nonlinear quasi-variational inclusion.

It is clear that the generalized nonlinear quasi-variational inclusion (2.1), for the appropriate suitable choice of operators $T, f, g$ and $m$, includes many kinds of variational inequalities and quasi-variational inequalities of [4,6,8,9,10], as special cases.

3. ITERATIVE ALGORITHM

To begin with, let us show the equivalence of the generalized nonlinear quasi-variational inclusion (2.1) to a nonlinear equation.

**Lemma 3.1:** Elements $x \in H$ and $w \in T(x)$ are the solutions of (GNQVIP) if and only if $x$ and $w$ satisfy the following relation

\[g(x) = m(x) + J^\phi_\alpha (g(x) - m(x) - \alpha (g(x) - f(w))). \quad (3.1)\]

where $\alpha > 0$ is a constant and $J^\phi_\alpha := (I + \alpha \delta \phi)^{-1}$ is the so-called proximal mapping on $H$, $I$ stands for the identity operator on $H$.

**Proof:** From the definition of $J^\phi_\alpha$, we have

\[g(x) - m(x) - \alpha (g(x) - f(w)) \in g(x) - m(x) + \alpha \delta \phi (g(x) - m(x)),\]
and hence
\[ f(w) - g(x) \in \delta \phi(g(x) - m(x)). \]
This implies that \( g(x) \in \text{dom } (\delta \phi) + m(x) \) and by the definition of \( \delta \phi \), we have
\[ \phi(y) \geq \phi(g(x) - m(x)) + \langle f(w) - g(x), y + m(x) - g(x) \rangle, \quad \forall y \in H. \]
Thus \( x \) and \( w \) are solutions of (GNQVIP).

To obtain an approximate solution of (2.1), we can apply a successive approximation method to the problem of solving
\[ x = F(x) \tag{3.2} \]
where
\[ F(x) = x - g(x) + m(x) + J^\phi_\alpha(g(x) - \alpha(g(x) - f(w)) - m(x)). \]

Based on (3.1) and (3.2), we suggest the following iterative algorithm.

**ALGORITHM 3.1:** Given \( x_0 \in H \), compute \( x_{n+1} \) by the rule
\[ x_{n+1} = x_n - g(x_n) + m(x_n) + J^\phi_\alpha(g(x_n) - \alpha(g(x_n) - f(w_n)) - m(x_n)). \tag{3.3} \]
for each \( x \in N \), where \( \alpha > 0 \) is a constant.

To perturb scheme (3.3), first, we add in the righthand side of (3.3), an error \( e_n \) to take into account a possible inexact computation of the proximal point and we consider an other perturbation by replacing in (3.3) \( \phi \) by \( \phi_n \), where the sequence \( \{\phi_n\} \) approximates \( \phi \). Finally, we obtain the perturbed algorithm which generates from any starting point \( x_0 \) in \( H \) a sequence \( \{x_n\} \) by the rule
\[ x_{n+1} = x_n - g(x_n) + m(x_n) + J^\phi_n(g(x_n) - \alpha(g(x_n) - f(w_n)) - m(x_n)) + e_n \tag{3.4} \]
our algorithm (3.4) is more general than the algorithms considered by Hassouni and Moudafi [4], Noor [6] and Siddiqi and Ansari [8].
4. CONVERGENCE THEORY

We need the following concepts and result to prove the main result of this paper.

**DEFINITION 4.1:** A mapping $g : H \to H$ is said to be

(i) **Strongly monotone** if there exists $r > 0$ such that

$$< g(x_1) - g(x_2), x_1 - x_2 > \geq r \| x_1 - x_2 \|^2, \quad \forall \ x_1, x_2 \in H,$$

(ii) **Lipschitz continuous** if there exists $s > 0$ such that

$$\| g(x_1) - g(x_2) \| \leq s \| x_1 - x_2 \|, \quad \forall \ x_1, x_2 \in H.$$

**DEFINITION 4.2:** Let $f : H \to H$ be a map. Then a multivalued map $T : H \to 2^H$ is said to be relaxed Lipschitz with respect to $f$ if for given $k \leq 0$,

$$< f(w_1) - f(w_2), x_1 - x_2 > \leq k \| x_1 - x_2 \|^2, \quad \forall \ w_1 \in T(x_1) \text{ and } w_2 \in T(x_2),$$

and $\forall \ x_1, x_2 \in H$.

The multivalued map $T$ is called Lipschitz continuous if for $m \geq 1$,

$$\| w_1 - w_2 \| \leq m \| x_1 - x_2 \|, \quad \forall \ w_1 \in T(x_1) \text{ and } w_2 \in T(x_2), \text{ and } \forall \ x_1, x_2 \in H.$$

**Lemma 4.1** [1]: Let $\phi$ be a proper convex lower semicontinuous function. Then $J_\alpha^\phi = (I + \alpha \delta \phi)^{-1}$ is nonexpansive, that is

$$\| J_\alpha^\phi(x) - J_\alpha^\phi(y) \| \leq \| x - y \|, \quad \forall \ x, y \in H.$$

Now we prove the following main result of this paper.

**THEOREM 4.1:** Let $g : H \to H$ be strongly monotone and Lipschitz continuous with corresponding constants $r > 0$ and $s > 0$; $f : H \to H$ be Lipschitz continuous with constant $t > 0$, and $m : H \to H$ be Lipschitz continuous with constant $\mu > 0$. Let $T : H \to 2^H$ be relaxed Lipschitz with respect to $f$ and Lipschitz continuous with corresponding constants $k \leq 0$ and $m \geq 1$. Assume

$$\lim_{n \to +\infty} \| J_\alpha^{\phi_n}(y) - J_\alpha^{\phi}(y) \| = 0, \quad \text{for all } y \in H \text{ and } \lim_{n \to +\infty} \| e_n \| = 0,$$
then the sequences \{x_n\} and \{w_n\}, generated by (3.4) with \(x_0 \in H\) and \(w_0 \in T(x_0)\), and

\[
|\alpha - \frac{1 - k + p[1 - 2(p + \mu)]}{1 - 2k + 2m^2 - p^2}| < \frac{\sqrt{[1 - k + p(1 - 2(p + \mu))]^2 - 4(p + \mu)(1 - (p + \mu))(1 - 2k + 2m^2 - p^2)}}{1 - 2k + 2m^2 - p^2},
\]  

(4.1)

where \(1 - k > p(2(p + \mu) - 1) + \sqrt{4(p + \mu)(1 - (p + \mu))(1 - 2k + 2m^2 - p^2)}\), for \(p = \sqrt{1 - 2r + s^2}\), converges strongly to \(x\) and \(w\), respectively, the solution of (2.1).

**PROOF:** Using (3.2), we can write

\[
x = x - g(x) + m(x) + J^\phi_\alpha(g(x)) - \alpha(g(x) - f(w)) - m(x)
\]

(4.2)

Denoting \(h(x) = g(x) - \alpha(g(x) - f(w)) - m(x)\)

and \(h(x_n) = g(x_n) - \alpha(g(x_n) - f(w_n)) - m(x_n)\),

then we have

\[
\| x_{n+1} - x \| \leq \| x_n - x - (g(x_n) - g(x)) \| + \| m(x_n) - m(x) \| + \| J^\phi_\alpha(h(x_n)) - J^\phi_\alpha(h(x)) \| + \| e_n \|
\]

(4.3)

On the other hand, by introducing the term \(J^\phi_\alpha(h(x))\), we get

\[
\| J^\phi_\alpha(h(x_n)) - J^\phi_\alpha(h(x)) \| \leq \| h(x_n) - h(x) \| + \| J^\phi_\alpha(h(x_n)) - J^\phi_\alpha(h(x)) \|
\]

Since \(J^\phi\) is nonexpansive.

Hence,

\[
\| J^\phi_\alpha(h(x_n)) - J^\phi_\alpha(h(x)) \| \leq (1 - \alpha) \| x_n - x - (g(x_n) - g(x)) \| + \\
\| (1 - \alpha)(x_n - x) + \alpha(f(w_n) - f(w)) \| + \| m(x_n) - m(x) \| + \\
\| J^\phi_\alpha(h(x)) - J^\phi_\alpha(h(x)) \|
\]

(4.4)
From (4.3) and (4.4), we get

\[
\| x_{n+1} - x \| \leq (2 - \alpha) \| x_n - x - (g(x_n) - g(x)) \| + 2 \| m(x_n) - m(x) \| + \\
\| (1 - \alpha)(x_n - x) + \alpha(f(w_n) - f(w)) \| + \| J_{\alpha}^{\phi_n}(h(x)) - J_{\alpha}^{\phi}(h(x)) \| + \| e_n \| 
\]

(4.5)

By Lipschitz continuity and strong monotonicity of \( g \), we obtain

\[
\| x_n - x - (g(x_n) - g(x)) \|^2 \leq (1 - 2r + s^2) \| x_n - x \|^2 
\]

(4.6)

Since \( T \) is Lipschitz continuous and relaxed Lipschitz with respect to \( f \), and \( f \) is Lipschitz continuous, we have

\[
\| (1 - \alpha)(x_n - x) + \alpha(f(w_n) - f(w_{n-1})) \|^2 = (1 - \alpha)^2 \| x_n - x \|^2 + 2\alpha(1 - \alpha) < f(w_n) - f(w), x_n - x > + \alpha^2 \| f(w_n) - f(w) \|^2 \leq \\
(1 - \alpha)^2 \| x_n - x \|^2 + 2\alpha(1 - \alpha)k \| x_n - x \|^2 + \alpha^2 \| x_n - x \|^2 \\
((1 - \alpha)^2 + 2\alpha(1 - \alpha)k + \alpha^2 t^2 m^2) \| x_n - x \|^2 
\]

(4.7)

Again, since \( m \) is Lipschitz continuous, we have

\[
\| m(x_n) - m(x) \| \leq \mu \| x_n - x \| 
\]

(4.8)

By combining (4.5) to (4.8), we finally obtain

\[
\| x_{n+1} - x \| \leq [(2 - \alpha)p + 2\mu + {(1 - \alpha)^2 + 2\alpha(1 - \alpha)k + \alpha^2 t^2 m^2}^{1/2}] \| x_n - x \|, 
\]

where \( p = (1 - 2r + s^2)^{1/2} \). Therefore

\[
\| x_{n+1} - x \| \leq \theta \| x_n - x \| + \| J_{\alpha}^{\phi_n}(h(x)) - J_{\alpha}^{\phi}(h(x)) \| + \| e_n \|, 
\]

where \( \theta = (2 - \alpha)p + 2\mu + {(1 - \alpha)^2 + 2\alpha(1 - \alpha)k + \alpha^2 t^2 m^2}^{1/2} \). It follows from (4.1) that \( \theta < 1 \).

By setting \( \epsilon_n = \| J_{\alpha}^{\phi_n}(h(x)) - J_{\alpha}^{\phi}(h(x)) \| + \| e_n \| \), we can write

\[
\| x_{n+1} - x \| \leq \theta \| x_n - x \| + \epsilon_n 
\]
Hence
\[ \| x_{n+1} - x \| \leq \theta^{n+1} \| x_0 - x \| + \sum_{j=1}^{n} \theta^j \epsilon_{n+1-j} \]

By the assumption of Theorem, \( \lim_{n \to \infty} \epsilon_n = 0 \). Hence the sequence \( \{x_n\} \) strongly converges to \( x \) (e.g.; see, Ortega and Rheinboldt [7]). Now the Lipschitz continuity of \( T \) implies that the sequence \( \{w_n\} \) strongly converges to \( w \). This completes the proof of the Theorem.

REFERENCES
