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LINEAR VIBRATIONS OF FRAMES CARRYING A CONCENTRATED MASS

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Abstract - The free and forced in-plane and out-of-plane vibrations of frames are investigated. The beam has a straight and a curved part. It has a circular cross-section. A concentrated mass is also located at different points of the frame with different mass ratios. FEM is used to analyze the problem. The in-plane and out-of-plane natural frequencies, point and transfer receptances of the system are obtained to determine the sensitive and non-sensitive frequency intervals depending on the location and direction of the force.

Keywords- frame vibrations, FEM, concentrated mass.

1. INTRODUCTION

The curved beams are commonly used for many purposes in technology. They can be used as gears, electrical machines, pumps and turbines, ships, in horizontally curved continuous bridges or in the design of ribs, edge stiffeners in bridge deck slabs and stiffened shell characteristics of turbomachinary and rockets, etc. The governing equations were presented together with their solutions in the book by Love [1]. Den Hartog [2] used the Rayleigh-Ritz method to obtain the in-plane lowest frequencies of circular curved fixed-fixed beams. Volterra and Morell [3-4] and Ojalvo et al. [5] calculated the natural frequencies of in-plane and out-of-plane vibration of circular arches based on classical beam theory by excluding rotary inertia and shear deformation. Pestel and Leckie [6] compared the natural frequencies of curved beams obtained by different methods. Veletsos et al. [7] studied free in-plane vibrations. Pandalai and Sathyamoorthy [8] obtained the modal equations of large amplitude vibrations of beams, plates, rings, and shells using Lagrange equations. Bickford and Storm [9] sought an exact solution for in-plane and out-of-plane vibrations of arbitrarily shaped curved bars, including the effects of shear deformation and rotary inertia, by a vector/transfer matrix approach, using exact solutions of the differential equations by the transfer matrix method. The large amplitude free vibrations of horizontally curved beams were investigated by Mukhopadhyay and Sheikh [10] and numerical solutions were obtained using FEM. The effect of shear deformation on displacements and the effect of rotary inertia and shear deformation on the natural frequencies were investigated by Krishnan and Suresh [11] by using four degree-of-freedom linear beam elements. Kawakami et al. [12] presented an approximate method to study the analysis for planar free vibrations of horizontally curved beams with arbitrary shapes and variable cross-sections. Kang et al. [13] applied the differential quadrature method to calculate the eigenvalues of planar vibration of circular arches, based on the Bresse-Timoshenko beam theory in which both rotary inertia and shear deformation were taken into account. Wang and Sang [14] set up the displacements for a curved beam to derive the equations for the out-of-plane motion of the beam via Timoshenko beam theory. A systematic method for analyzing the out-of-plane dynamic behaviors of non-circular curved beams was presented by taking into account the effects of shear deformation, rotary inertia, and viscous damping as demonstrated in [15]. Without considering the shear deformation, the rotary inertia and the warping effects, the governing differential equations for the out-of-plane vibrations of curved non-uniform beams of constant radius were derived via Hamilton principle by Lee and Chao [16]. There are many systems comprised of straight and curved members used in practice. But there are very few studies in the literature. Yuan and Dickinson [17] presented an approximate approach using artificial springs to calculate the natural frequencies of a straight-curved beam frame using the Rayleigh-Ritz method. Kashimoto et al. [18] presented the dynamic stress concentration problem of an inhomogeneous rod of infinite length, consisting of two infinite straight portions and one finite portion of arbitrary curvature using transfer matrix method. They obtained natural frequency values for only curved part. Wang [19-20] set up the displacements which are two bending slopes and one twist angle, for a curved frame to derive the governing equations of a T-type curved frame via the same beam theory. An analytical method for both in-plane motion and out-of-plane motion of a curved hollow shaft was presented for two types of shaft structures, which are a curved hollow shaft and a fixed-fixed straight-curved-straight-hollow shaft by considering torsion and bending. The author found that the first in-plane modal frequency of a structure was greater than the first out-of-plane modal frequency of the same structure. Petrolito and Legge [21] developed a general nonlinear analysis method for structural frames with curved members to calculate the complete load-deflection response. Ercoli et al. [22] had an analytical and experimental investigation on vibrating arches clamped at one end and carrying a concentrated mass at the other end. Ercoli et al. [23] developed previous work by using an intermediate support. Laura et al. [24] investigated the in-plane vibrations of an elastically cantilevered circular arc with a tip mass. Cortinez et al. [25] calculated the inextensional natural frequencies of a fixed-free straight-curved beam system having a concentrated mass at the end of the curved member for in-plane vibrations by excluding rotatory inertia, used the Rayleigh-Schmidt technique, and compared the results with the results of Dunkerley's approach and FEM.

In this study, it is aimed to investigate the linear free and forced, in-plane and out-of-plane vibrations of frames carrying a point mass. The frame is fixed at the left end of the straight beam while the right end of the curved part is free. The frame is modeled by FEM. Rotary inertia and extensional effects are included for curved part while shear effects are excluded for all parts of the frame. The in-plane vibrations are analyzed where the longitudinal and flexural vibrations are coupled for each beam. Cubic functions are used for bending and elongation which means four degrees of freedom for each node of the frame. The natural frequencies are obtained for different additional mass locations. Then the out-of-plane vibrations are investigated. The flexural and torsional vibrations are coupled in out-of-plane vibrations for both beams. The three degrees of freedom is used for each node for out-of plane vibrations. The changes in natural frequencies are investigated depending on the location and amount of additional mass. The point and transfer receptance curves are plotted for different cases.

2. THE FRAME SYSTEM, ENERGY EQUATIONS AND FINITE ELEMENT FORMULATION

The frame system used is shown in Figure 1. *X, Y*, and *Z* are global coordinates, and *u, v*, *w* are the longitudinal, transverse, and out-of-plane displacements for the straight part, and the tangential, radial and out-of-plane displacements for the circular part, respectively. The straight and curved parts have torsion Φ_s and Φ_c . Sub indices *c* and *s* denote curved and straight members, respectively. Curved beam lies in *X-Y* plane. L_T and *S* are the lengths of straight and curved members respectively. Cross-sections are circular for both members, *A* is the cross-sectional area. Modulus of elasticity *E*, modulus of shear *G,* mass moment of inertia *I*, and polar moment of inertia *J* are also equal for both beams. α is the arch angle of curved beam.

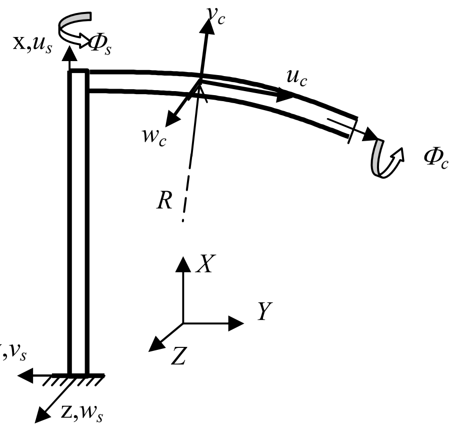


Figure 1. The frame

The in-plane (XY), out-of-plane (XZ) elastic and kinetic energies of the frame can be expressed as follows

$$U_{in} = \frac{1}{2}E\int_s \left[A\varepsilon_{c\,in}^2 + I\kappa_{c\,in}^2 \right] ds + \frac{1}{2}E\int_x \left[A\varepsilon_{s\,in}^2 + I\kappa_{s\,in}^2 \right] dx ,$$

$$T_{in} = \frac{1}{2}\rho\int_s \left[A(\dot{u}_{cin}^2 + \dot{v}_{cin}^2) + I\dot{\beta}_{cin}^2 \right] ds + \frac{1}{2}\rho\int_x \left[A\left(\dot{u}_{sin}^2 + \dot{v}_{sin}^2\right) + I\dot{\beta}_{sin}^2 \right] dx \tag{1}$$

$$U_{out} = \frac{1}{2}EI\int_L \kappa_{s\,out}^2 dx + \frac{1}{2}GJ\int_L \varphi_{s\,out}^2 dx + \frac{1}{2}EI\int_S \kappa_{c\,out}^2\, ds + \frac{1}{2}GJ\int_S \varphi_{c\,out}^2 ds$$

$$T_{out} = \frac{1}{2}\rho A\int_L \dot{w}_{sout}^2\, dx + \frac{1}{2}\rho J\int_L \dot{\Phi}_{sout}^2\, dx + \frac{1}{2}\rho A\int_S \dot{w}_{cout}^2\, ds + \frac{1}{2}\rho I\int_S \dot{\Psi}_{out}^2\, ds + \frac{1}{2}\rho J\int_S \dot{\Phi}_{out}^2\, ds$$

In these equations $(\dot{\ })$ denotes differentiation with respect to time *t*. In-plane strain, net cross-sectional rotation and curvature change of the curved and straight member, out-of-plane curvature change and torsion in Equation (1) are as follows

$$\varepsilon_{cin} = \frac{\partial u_{cin}}{\partial s} + \frac{v_{cin}}{R} ,\ \varepsilon_{sin} = \frac{\partial u_{sin}}{\partial x} ,\ \beta_{cin} = \frac{\partial v_{cin}}{\partial s} - \frac{u_{cin}}{R} ,\ \beta_{sin} = \frac{\partial v_{sin}}{\partial x} ,$$

$$\kappa_{cin}=\frac{\partial\beta_{cin}}{\partial s}=\frac{\partial^2 v_{cin}}{\partial s^2}-\frac{1}{R}\frac{\partial u_{cin}}{\partial s},\kappa_{sin}=\frac{\partial\beta_{sin}}{\partial x}=\frac{\partial^2 v_{sin}}{\partial x^2}\ \kappa_{cout}=\frac{\Phi_{cout}}{R}-\frac{\partial^2 w_{cout}}{\partial s^2},\kappa_{sout}=\frac{\partial^2 w_{sout}}{\partial x^2}, \tag{2}$$
$$\varphi_{cout}=\frac{\partial\Phi_{cout}}{\partial s}+\frac{1}{R}\frac{\partial w_{cout}}{\partial s},\ \varphi_{sout}=\frac{\partial\Phi_{sout}}{\partial s},\ \Psi_{cout}=\partial w_{cout}/\partial s$$

Finite element method [26] is used for vibration analysis of the frame. Four degrees of freedom for in-plane vibrations, and three degrees of freedom for out-of-plane vibrations are good enough to obtain the results as demonstrated by [27]. One can take the element displacement vector as

$$[V]^T_{(cin,\sin)e}=\left[u_{(cin,\sin)1}\quad\alpha_{(cin,\sin)1}\quad v_{(cin,\sin)1}\quad\theta_{(cin,\sin)1}\quad u_{(cin,\sin)2}\quad\alpha_{(cin,\sin)2}\quad v_{(cin,\sin)2}\quad\theta_{(cin,\sin)2}\right]$$
$$[V]^T_{(cout,sout)e}=\left[\Phi_{(cout,sout)1}\quad w_{(cout,sout)1}\quad\Psi_{(cout,sout)1}\quad\Phi_{(cout,sout)2}\quad w_{(cout,sout)2}\quad\Psi_{(cout,sout)2}\right] \tag{3}$$

where $\alpha_{\sin}=\frac{\partial u_{\sin}}{\partial x}$, $\theta_{\sin}=\frac{\partial v_{\sin}}{\partial x}$, $\alpha_{cin}=\frac{\partial u_{cin}}{\partial s}$, $\theta_{cin}=\frac{\partial v_{cin}}{\partial s}$, $\Psi_{cout}=\frac{\partial w_{cout}}{\partial s}$, $\Psi_{sout}=\frac{\partial w_{sout}}{\partial x}$

The stiffness and inertia matrices for each finite element can be obtained. In in-plane vibrations (occurs in XY plane), the stiffness matrix of curved beam is a combination of elongation and bending part while the inertia matrix is that of the translation and rotation part of the circular member. In out-of-plane (occurs in XZ plane) vibrations, the flexural and torsional vibrations are coupled for both beams. The stiffness matrix is a combination of bending and torsion and inertia matrix is a combination of that of out-of-plane translation, rotary inertia, and torsion of the curved member. The concentrated mass appears in the inertia matrix. Since the coordinate systems used for two members of the frame are different, the coordinates of straight and curved beams should be transformed to each other to analyze the frame as a single system. The support condition is fixed-free.

3. FREE VIBRATIONS

The total energy in the system is constant as follows

$$\{\dot V\}^T[M]\{\dot V\}+\{V\}^T[K]\{V\}=0 \tag{4}$$

where $\{V\}$ denotes global displacement vector, $[K]$ and $[M]$ are global stiffness and inertia matrices. The solution of equation (4) is assumed as

$$\{V\}=\{\overline{V}\}\ e^{j\omega_n t} \tag{5}$$

where $j=\sqrt{-1}$, ω_n is natural frequency and $\{\overline{V}\}$ is displacement amplitude vector of all nodes. Then, one obtains the eigenvalue equation giving the natural frequencies for both vibrations

$$\left|\,[K]-\omega_n^2[M]\,\right|=0\,. \tag{6}$$

4. FORCED VIBRATIONS

The characteristics of forced vibrations are important due to the need of controlling vibration amplitudes. The point and transfer receptances of the frame by forcing the free end are calculated by including structural damping. The general equation of motion for a harmonically forced system is as follows

$$M\ddot Q+C\dot Q+KQ=f\exp(j\omega t) \tag{7}$$

where Q is the nodal displacement vector, C is viscous damping matrix and f is the vector formed by forces at the nodes. There are some mathematical models for the damping expressing energy losses in mechanical systems as demonstrated in [19]. The structural damping can be considered by replacing $K(1+i\eta)$ with the viscous damping. In this case, Equation (7) becomes

$$Q = \left[K - \omega^2 M + i\eta K\right]^{-1} f \exp(i\omega t) \quad (8)$$

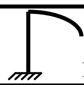
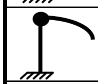
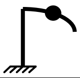
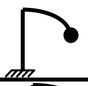
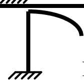
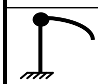
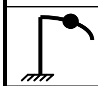
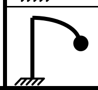
where η is the loss factor. Equation (8) can be written as

$$Q = \left[A_R + iA_I\right]^{-1} f \exp(i\omega t)\,,\ A_R = K - \omega^2 M\,,\ A_I = \eta K \quad (9)$$

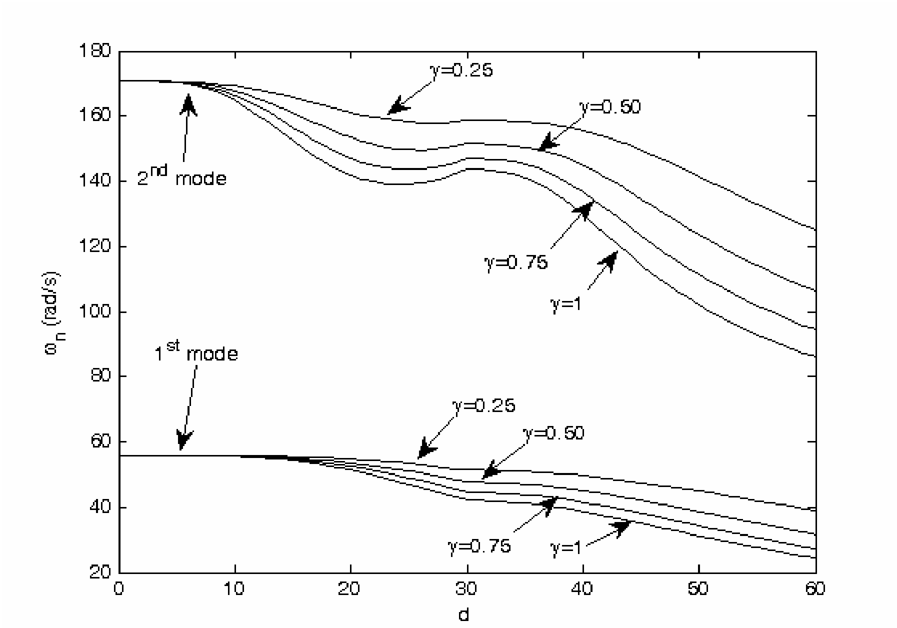
5. NUMERICAL RESULTS AND DISCUSSION

In this section, the results of free and forced planar vibration analysis will be given. The modulus of elasticity and shear of the frame are 200 GPa and 84 GPa respectively. The density of the material is 7800 kg/m^3. The lengths of curved and straight parts are 1 m each. The cross-sectional radius is r = 20 mm. In Table 1, the first five natural frequencies of in-plane and out-of-plane vibrations of frame are presented respectively for different mass and arch angles. The point mass is located at the connection point (position-a), at the middle of curved beam (position-b) and at free end (position-c), and the mass ratio (**γ**) is taken ¼ of the whole frame. Bending is the main behavior of the in-plane vibration modes. Torsion is effective as much as bending in out-of-plane vibration modes. The mass ratio (γ) is changed from ¼ to 1.

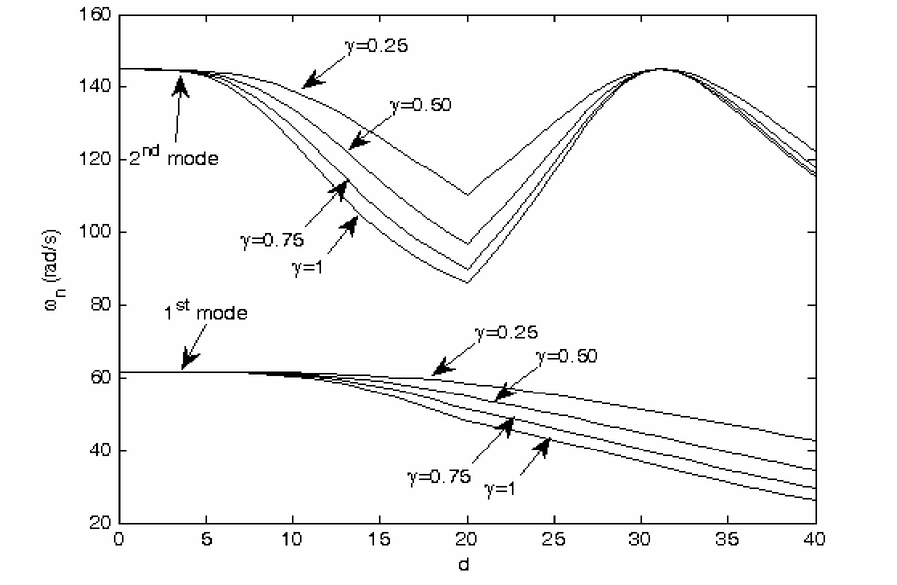
Table 1. The in-plane and out-of-plane natural frequencies of frame

		In-Plane-Vibrations (XY)					**Out-of-Plane Vibrations (XZ)**				
Arch angle	Position of Mass	ω_1 (r/s)	ω_2 (r/s)	ω_3 (r/s)	ω_4 (r/s)	ω_5 (r/s)	ω_1 (r/s)	ω_2 (r/s)	ω_3 (r/s)	ω_4 (r/s)	ω_5 (r/s)
30^0	 No Mass	55.94	170.73	765.12	1190.91	2502.67	61.73	144.89	720.66	1205.5	2535.50
	 (a)	51.36	158.61	764.70	1165.82	2500.75	58.50	110.46	720.04	981.60	2532.49
	 (b)	47.53	149.56	657.82	1061.86	2434.50	51.34	144.32	644.94	1104.45	2491.90
	 (c)	38.97	125.00	609.14	1093.99	2196.18	42.79	122.10	613.51	1081.63	2243.29
45^0	 No Mass	54.64	176.21	744.41	1197.34	2479.53	64.63	140.68	679.40	1197.39	2510.46
	 (a)	50.34	163.13	744.40	1165.45	2478.12	60.98	108.07	675.79	982.65	2502.86
	 (b)	46.40	154.42	646.67	1064.58	2400.30	53.49	140.16	604.61	1110.42	2473.36
	 (c)	38.00	129.57	583.32	1101.93	2170.53	45.27	117.56	569.17	1083.91	2208.93

All figures between 2 and 4 are drawn for 30° arch angle. In Figure 2 (a-b), the change of in-plane and out-of plane frequencies due to the location of concentrated mass are drawn respectively. The frame is divided into 60 finite elements for in-plane vibration, and 40 elements for out-of-plane vibration. **d** shows the place of concentrated mass from left hand side. For example if d is 30 in Figure 2a, that means the additional mass is on the connection point of the straight and curved beams (position-a in Table 1), and if d is 60 it means that the mass is on the free end (position-c).



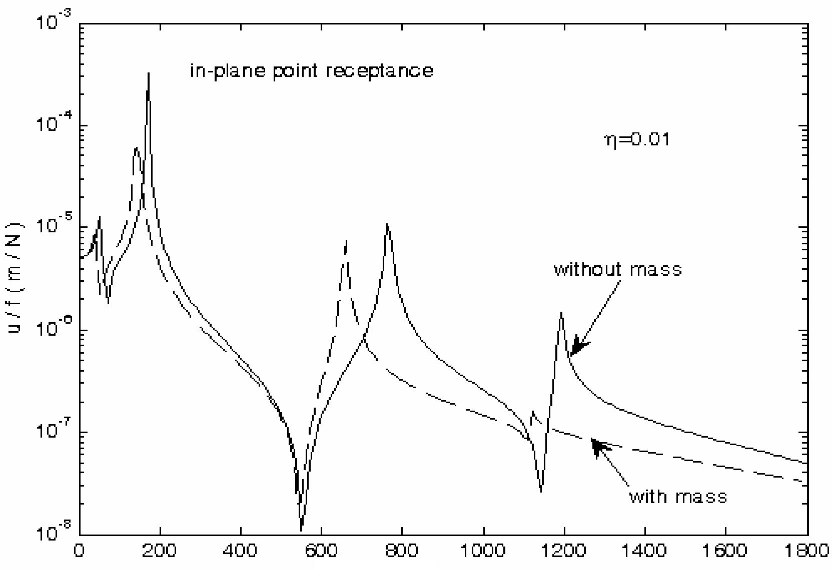
(a)



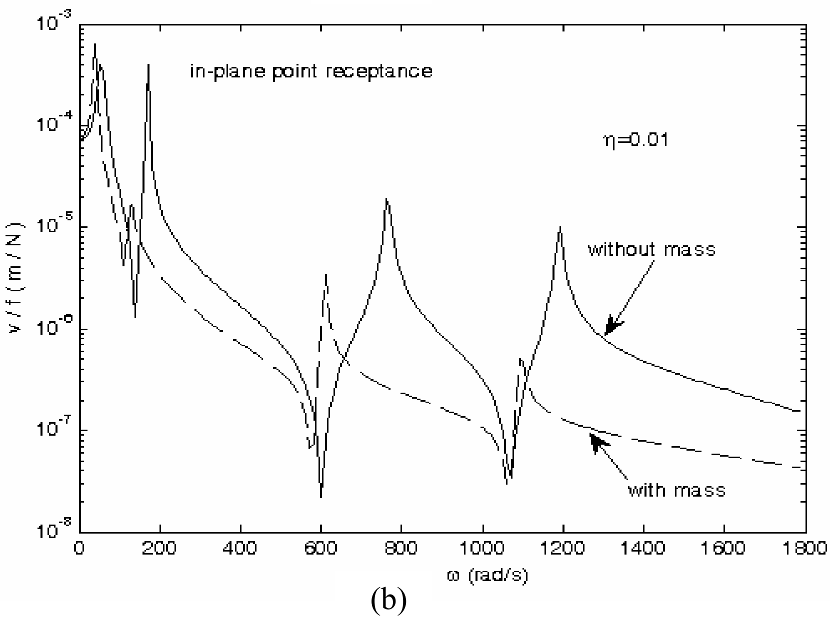
(b)

Figure 2. The variation of 1st and 2nd in-plane (a) and out-of-plane (b) natural frequencies due to the mass location and ratio

As it seen in Figure 2a, the moving the additional mass from left to right generally decrease the frequencies for all mass ratios. The increase in the mass also decreases the frequencies further. This effect becomes larger at the free end (position-c). Replacing the mass toward the center of the curved part (position-b) slightly increases the frequencies in the second mode. In Figure 2b (out-of-plane vibrations), all frequencies decrease again by adding mass and this effect becomes larger as it is replaced at the free end. In the second mode, there is an increase in frequencies when the mass is moved to right from the position-a. However, the natural frequencies take constant values around the position-b interestingly and decreases again further.



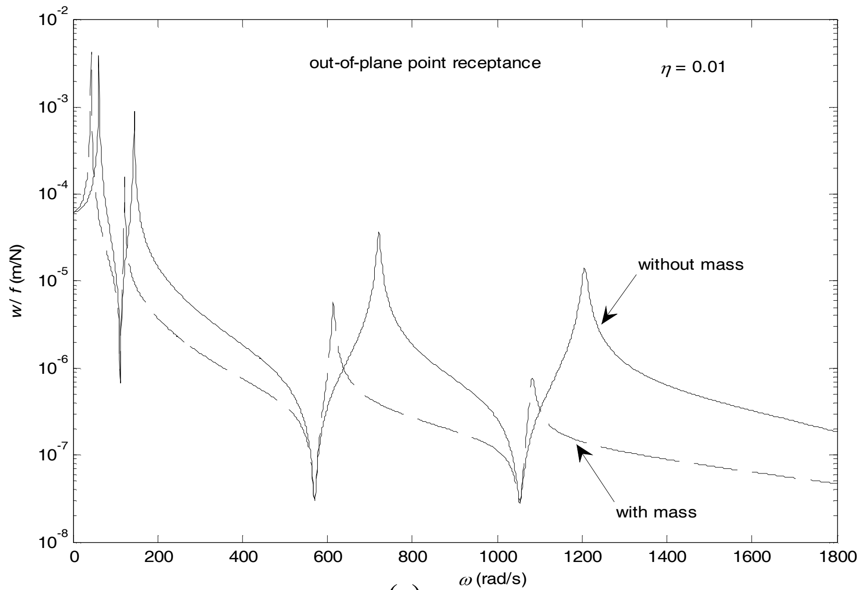
(a)



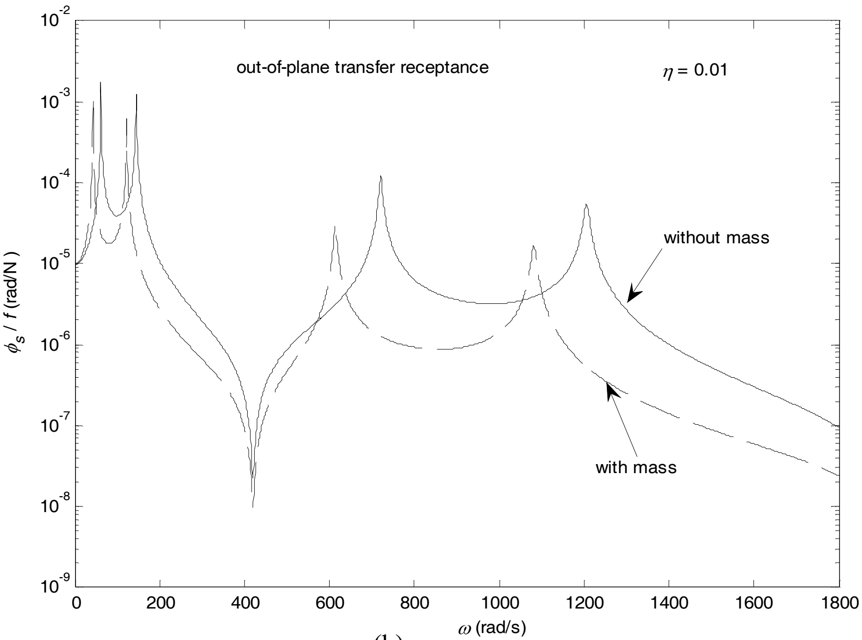
(b)

Figure 3. The in-plane point receptance amplitudes a) *u/f*, b) *v/f* (γ =0.25 , η=0.01)

Finally, the forced vibrations of the frame are investigated. In Figure 3a and b, the in-plane point receptance amplitudes of tangential and radial directions at the free end are shown, in Figure 4a and b, the out-of-plane point and transfer receptance amplitudes are shown, respectively, the force is applied at the free end and transfer receptance of the connection point between straight and curved part is obtained. The loss factor is assumed as 0.01 and the mass ratio is 0.25, and located at the free end. The last two figures can be used to understand the behavior of traffic or lighting poles under wind forces.



(a)



(b)

(b)**Figure 4. The out-of-plane receptance amplitudes a) *w/f*, b) Φ_s/f (γ =0.25 , η=0.01)**

6. CONCLUSIONS

The linear, in-plane and out-of-plane vibrations of frames having a straight and a curved part are investigated. The frame also carries a point mass. The rotary inertia effects and extensibility are included for curved part while the shear effects are excluded for all parts of the frame. The vibrations are analyzed using FEM, and the natural frequencies are obtained for different additional mass location. The frequencies decrease mostly by adding point mass. Finally, in-plane and out-of-plane receptances are obtained when the external force acted at the free end of the frame. The changes of point and transfer receptance amplitudes are plotted and the effect of additional mass is observed. The sensitive and non-sensitive frequency intervals are determined.

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