AN ANALYSIS OF THE SYMMETRIES AND CONSERVATION LAWS OF THE CLASS OF ZAKHAROV-KUZNETSOV EQUATIONS

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Abstract- In this paper, we study and classify the conservation laws of the Zakharov-Kuznetsov equations. It is shown that these can be obtained by studying the interplay between symmetry generators and ‘multipliers’. This is, particularly, useful for the higher-order multipliers. As a final note, we include Drinfeld-Sokolov-Wilson system to demonstrate the usefulness of the approach to systems of pdes.

Keywords- Conservation laws, symmetry generators.

1. INTRODUCTION AND BACKGROUND

The class of Zakharov-Kuznetsov equations with power law nonlinearity

\[ u_t + au^n u_x + b(u_{xxx} + u_{xyy}) = 0, \]  

(1)

has recently been a subject of extensive study in plasma physics, for e.g., [2, 3, 4, 5]. There are some interesting detailed accounts given in these references. However, none, it seems, categorizes analytic, exact or invariant solutions or studies the underlying conservation laws that are related to or independent of the symmetry properties of the equation. In this paper, an attempt at an analysis of both these aspects of the equation are done.

We include Drinfeld-Sokolov-Wilson system to demonstrate the usefulness of the approach to systems of pdes too.

The use of symmetry properties of a given system of partial differential equations to construct or generate new conservation laws from known conservation laws has been investigated [7, 8].

In this paper, we apply the recently established notion [1] that the symmetry invariance properties of the multipliers lead to a large class of conserved flows that would not be provided by variational techniques or the standard methods especially the higher-order multipliers.
Consider an \( r \)-th order system of partial differential equations (PDEs) of \( n \) independent variables \( x = (x^1, x^2, \ldots, x^n) \) and \( m \) dependent variables \( u = (u^1, u^2, \ldots, u^m) \)

\[
G^\mu(x, u, u^{(1)}, \ldots, u^{(r)}) = 0, \quad \mu = 1, \ldots, \tilde{m},
\]

where \( u^{(1)}, u^{(2)}, \ldots, u^{(r)} \) denote the collections of all first, second, \ldots, \( r \)-th order partial derivatives, that is, \( u^a_i = D_i(u^a), u^a_{ij} = D_jD_i(u^a), \ldots \) respectively, with the total differentiation operator with respect to \( x^i \) given by

\[
D_i = \frac{\partial}{\partial x^i} + u^a_i \frac{\partial}{\partial u^a} + u^a_{ij} \frac{\partial}{\partial u^a_j} + \ldots, \quad i = 1, \ldots, n,
\]

where the summation convention is used whenever appropriate.

A current \( \Phi = (\Phi^1, \ldots, \Phi^n) \) is conserved if it satisfies

\[
D_i \Phi^i = 0
\]

along the solutions of (2).

It can be shown that every admitted conservation law arises from multipliers \( Q_\mu(x, u, u^{(1)}, \ldots) \) such that

\[
Q_\mu G^\mu = D_i \Phi^i
\]

holds identically (i.e., off the solution space) for some current \( \Phi^i \) ‘modulo a curl’. When the PDE system is variational, multipliers are variational symmetries. There is a determining system for finding multipliers (and hence conservation laws) for any given PDE system. Then, the conserved density

is determined by a homotopy formula like

\[
\int_0^1 u \Lambda(t, x, \lambda u, \lambda u_x, \lambda u_{xx}, \ldots) d\lambda, \quad \text{where} \quad \Lambda = \frac{\delta}{\delta u} \Phi^i \text{ and } \frac{\delta}{\delta u} \text{ is the Euler operator (see [6] for details).}
\]

Our method resorts mainly to the following theorem [1].

**Theorem 0.1** If \( \Phi^i \) is a conserved current with multiplier \( Q_\mu \) then \( \Phi^i_X := pr \tilde{X} \Phi^i \) is also a conserved current and has multiplier \( Q^X_\mu := Q_\mu'(P) + \tilde{R}^*(Q_\mu) \) where \( \tilde{R}^* \) is the adjoint of the operator \( \tilde{R} \). In the case of a point symmetry, this becomes

\[
\Phi^i_X = pr X \Phi^i + 2 \Phi^i D_i \xi^i \text{ modulo curls and } Q^X_\mu = pr X Q_\mu + Q_\mu D_i \xi^i + \tilde{R}^*(Q_\mu) \quad \text{where} \quad \tilde{R} = \tilde{R} + \xi^i D_i \text{ ie } pr X G^\mu = R(G^\mu).
\]

**2. RESULTS**

The symmetry and conservation laws structure splits into two cases (i) \( n \neq 1 \) and (ii) \( n = 1 \).
(i) It can be shown that the point symmetry generators of (1) for this case is a four-dimensional Lie algebra spanned with basis are time and space translations $X_1 = \partial_t$, $X_2 = \partial_x$, $X_3 = \partial_y$ and scaling $X_4 = \frac{1}{3} x \partial_x + \frac{1}{3} y \partial_y + t \partial_t - \frac{2}{3n} u \partial_u$.

In this section we construct multipliers that have the form determined by the ray invariance condition in Theorem 1.1, viz., $XQ = (\lambda + R)Q$, where $R$ is determined by the action of $X$ on the PDE and ‘some’ divergence term.

As a first case, we consider $X_2$ for which $R = 0$. The invariants of the equation $XQ = \lambda Q$ are given by the system

$$\frac{dx}{1} = \frac{dt}{0} = \frac{du}{0} = \frac{d\epsilon t}{0} = \frac{d\epsilon u}{0} = \frac{d\mu t}{0} = \frac{d\mu u}{0} = \cdots = \frac{d\nu xx}{0} = \cdots = \frac{dQ}{\lambda Q}$$

so that, for e.g.,

$$Q = e^{\lambda x} f(t, u, \theta, \epsilon, \mu, \nu, \eta),$$

where $\theta = u_x$, $\epsilon = u_t$, $\mu = u_{xx}$, $\nu = u_{xt}$, $\nu = u_{tt}$ and $\eta = u_{xxx}$. Since the Euler operator annihilates a total divergence, i.e., $\frac{\delta}{\delta u} D_i \Phi^i = 0$, we require

$$\frac{\delta}{\delta u} (Q(1)) = 0$$

wherein we impose the form of $Q$ to be as in (6). The greater the order of the assumed derivative of $Q$, the more cumbersome the expansion of the left hand side of (7). We have extensively employed the use of software to expand (7), separate the resultant by monomials and solve the overdetermined system of PDEs - this would otherwise be impossible and the interesting forms of $Q$ and, hence, the conserved flows would be lost (to some extent, the finer details can be seen for the KdV equation in [6]). In summary, we obtain the multipliers

$$Q_1 = u_{yy} + u_{xx} + \frac{a}{b(n+1)} u^{n+1}, \quad Q_2 = u, \quad Q_3 = f(y)$$

where $f(y)$ is an arbitrary function of $y$ and each giving rise to corresponding conserved flows. We study the association of the conserved flows with symmetry by studying the action of the symmetries on the multipliers $Q_i$. Firstly, notice that there are no first-order (in derivatives) multipliers but we do have a second-order multiplier $Q_1$. This action is enumerated below as

$$X_i(Q_1) = 0, \quad i = 1, 2, 3$$

$$X_4(Q_1) = -\frac{2}{3}(1 + \frac{1}{n})Q_1,$$
\[ X_i(Q_2) = 0, \quad i = 1, 2, 3 \]
\[ X_4(Q_2) = -\frac{2}{3n}Q_2, \quad (9) \]
\[ X_i(Q_3) = 0, \quad i = 1, 2 \]
\[ X_3(Q_3) = f'(y) = g(y), \quad (10) \]
\[ X_4(Q_3) = \frac{1}{3}yf'(y) = h(y). \]

Thus, \( Q_1 \) and \( Q_2 \) are strictly invariant under \( X_i \) for \( i = 1, 2, 3 \) but ray invariant under \( X_4 \) with \( \lambda = -\frac{2}{3}(1 + \frac{1}{n}) \) and \( -\frac{2}{3n} \), respectively. \( Q_3 \) is strictly invariant under \( X_1 \) and \( X_2 \), ray invariant under \( X_3 \) but not invariant with respect to \( X_4 \). The strict invariant condition is synonymous with the association of the symmetry \( X_j \) with the resultant conserved vector from the multiplier \( Q_k \).

As an example, we note that the conserved flow corresponding to \( Q_1 \) for \( n = \frac{1}{2} \) is

\[
\Phi^x = \frac{1}{6b\sqrt{u}} \left( -2abu_y^2 + 4abu(2u_{yy} + 3u_{xx}) + b\sqrt{u} (3u_tu_x + b((u_{yy} + u_{xx})(u_{yy} + 3u_{xx}) - u_y(u_{yy} + u_{xyy})) \right.
\]
\[
+ u^{3/2}(12a^2 + b(-3u_{xt} + b(u_{yy} + u_{xxyy}))))),
\]

\[
\Phi^t = \frac{1}{6b\sqrt{u}} \left( 2abu_y^2 + 4abuu_{xy} + \sqrt{u} (3u_1u_y + b(-u_x(u_{yy} + u_{xxy})) \right.
\]
\[
+ 2(u_{xy}(u_{yy} + u_{xx}) + u_y(u_{xyy} + u_{xxx})) \right.
\]
\[
+ u^{3/2}(3u_{yt} + b(u_{yy} + u_{xxyy}))))),
\]

\[
\Phi^t = \frac{u(8a\sqrt{u} + 3b(u_{yy} + u_{xx}))}{6b}.
\]

(ii) For \( n = 1 \), we have an additional symmetry \( X_5 = at\partial_x + d_u \) and the calculations for the multipliers yield an additional one \( Q_4 = -atu + x \). The action of the \( X_i \’s \) (\( i = 1, \ldots, 5 \)) on \( Q_4 \) are as follows,

\[
X_1(Q_4) = -aQ_2,
\]
\[
X_2(Q_4) = 1,
\]
\[
X_3(Q_4) = 0,
\]
\[
X_4(Q_4) = \frac{1}{3}Q_4,
\]
\[
X_5(Q_4) = 0. \quad (12)
\]
For this case, therefore, \( Q_4 \) is strictly invariant under \( X_3 \) and \( X_5 \) so that the corresponding conserved vector is associated with \( X_3 \) and \( X_5 \). Also, as \( X_2(\text{Q}_4) = f(y) = Q_3 \) for \( f = \text{constant} \) which implies that the multiplier is obtainable by the symmetry action of \( X_2 \) on \( Q_4 \). Similarly, since \( X_1(\text{Q}_4) = -aQ_2 \), the action of \( X_1 \) on \( Q_4 \) yields \( Q_2 \) so that \( Q_2 \) as an independent multiplier can be dispensed with. That is, the conserved vector from \( Q_2 \) would not be in the basis set of conservation laws (see [8]).

The components of the conserved vector corresponding to the second-order multiplier \( Q_1 \) is

\[
\Phi^x = -\frac{1}{6u}(24abu_x^2 + au^3(3a + 4bu_{yy} + 6bu_{xx})) + 2bu(3u_tu_x + u_{yy}(6a + 2bu_{xy} + 3bu_{xx})) - b(u_{yy}u_x + 3u_xu_{xyy} - 2u_{xy}u_{xx} - 3u_{xx}^2 - 2u_xu_{xxy} + uy(u_{xyy} + u_{xxx})) + 2bu^2(-3u_xt + b(2u_{xxyy} + 3u_{xxyy} - u_{yxx})),
\]

\[
\Phi^y = -\frac{1}{3u}(12au_yu_x + au^3u_{xx}) + u(3u_uy - 6au_{xy} + b(u_{yy}u_{xx} + u_{xx}^2 + 3u_yu_{xx} + 3u_yu_{xxx} - u_x(u_{xyy} + u_{xxx})) - u_x(3u_uy + b(2u_{xxyy} + 3u_{xxyy} - u_{yxx})),
\]

\[
\Phi^t = -\frac{u(au^2 + 3b(u_{yy} + u_{xx}))}{3b},
\]

\( Q_2 \) is

\[
\Phi^x = \frac{1}{6}(2au^3 - b(u_x^2 + 3u_{xx}^2) + 2bu(u_{yy} + 3u_{xx})),
\]

\[
\Phi^y = -\frac{1}{4}b(u_yu_x - 2u_{xy}),
\]

\[
\Phi^t = \frac{1}{4}u^2,
\]

\( Q_3 \) is

\[
\Phi^x = \frac{1}{6}(2b(uf'' - f'u_y) + f(3au^2 + 2b(u_{yy} + 3u_{xx}))),
\]

\[
\Phi^y = -\frac{1}{3}b(f'u_x - 2f_{xy}),
\]

\[
\Phi^t = fu,
\]

and \( Q_3 \) is

\[
\Phi^x = \frac{1}{6}(3axu^2 - 2a^2tu^3 - 2abtu(u_{yy} + 3u_{xx}) + b(au_y^2 + 2uxu_{yy} - 6u_x + 3atu_x^2 + 6xu_{xx})),
\]

\[
\Phi^y = \frac{1}{3}b(u_y(-1 - atu_x) + 2(x - atu)u_{xy}),
\]

\[
\Phi^t = xu - \frac{1}{4}atu^2.
\]

**Notes.** - systems example.

The Drinfeld-Sokolov-Wilson system

\[
u_t + 2uv_x = 0
\]

\[
v_t - au_{xxx} + 3bu_xv + 3kv_x = 0
\]

admits a three-dimensional Lie point symmetry algebra spanned by

\[
X_1 = \partial_t, \quad X_2 = \partial_x, \quad X_3 = -2u\partial_u - 2v\partial_v + 3t\partial_t + x\partial_x
\]
with commutator table

\[
\begin{array}{ccc}
X_1 & X_2 & X_3 \\
X_1 & 0 & 0 & 3X_1 \\
X_2 & 0 & 0 & X_2 \\
X_3 & -3X_1 & -X_2 & 0 \\
\end{array}
\]

Its zero order multipliers were shown to be \((1, 0), (0, 1)\) and \((\frac{3}{2}bu, v)\) ([9])- the first two are of minimal interest as the corresponding conserved flow yield one equation in the system. Further detailed calculations, as done previously, shows, in fact, that a second-order multiplier exists, viz.,

\[ (Q^1, Q^2) = (a(b - k)u_{xx} + 3(k^2 + kb - 2b^2)u^2 - (k + 2b)v^2, 2(au_{xx} - kv - 2buv)). \]

The corresponding conserved density \((b \neq k)\) is

\[ \Phi^t = \frac{1}{b - k}[-\frac{1}{2}(b - k)u_x^2 - au_x^2 + b(k - 2b)u^3 - 2buv^2 + k^2u^3 - kv^2]. \]

The action of the \(X_i\)'s \((i = 1, 2, 3)\) on the \(Q_j\)'s \((j=1,2)\) are

\[
\begin{align*}
X_1(Q_j) &= X_2(Q_j) = 0, \\
X_3(Q^1) &= -4Q^1, \quad X_3(Q^2) = -4Q^2.
\end{align*}
\]

Thus, the conserved density \(\Phi^t\) and conserved flux are associated with \(X_1\) and \(X_2\) and not with \(X_3\) as the multiplier \((Q^1, Q^2)\) is ray invariant, as opposed to strictly invariant, under \(X_3\).

Further investigation can be done for various combinations of \(b\) and \(k\) like \(k = 2b\).

3. CONCLUSION

We have shown that pdes or systems of pdes may have multiplier that are higher-order (than two in derivatives) and lead to new and nontrivial conservation laws. A number of possible relationships between the multipliers and Lie point symmetry generators exist. These have consequences, inter alia, on the basis of conservation laws of the pde/systems of pdes.

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4. REFERENCES


