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The Average Lower 2-Domination Number of Wheels Related Graphs and an Algorithm

Tufan Turaci

Department of Mathematics, Faculty of Science, Karabük University, Karabük 78050, Turkey; tufanturaci@karabuk.edu.tr; Tel.: +90-370-433-8374 (ext. 1229)

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Abstract: The problem of quantifying the vulnerability of graphs has received much attention nowadays, especially in the field of computer or communication networks. In a communication network, the vulnerability measures the resistance of the network to disruption of operation after the failure of certain stations or communication links. If we think of a graph as modeling a network, the average lower 2-domination number of a graph is a measure of the graph vulnerability and it is defined by $\gamma_{2av}(G) = \frac{1}{|V(G)|}\sum_{v\in V(G)} \gamma_{2v}(G)$, where the lower 2-domination number, denoted by $\gamma_{2v}(G)$, of the graph G relative to v is the minimum cardinality of 2-domination set in G that contains the vertex v. In this paper, the average lower 2-domination number of wheels and some related networks namely gear graph, friendship graph, helm graph and sun flower graph are calculated. Then, we offer an algorithm for computing the 2-domination number and the average lower 2-domination number of any graph G.

Keywords: graph vulnerability; connectivity; network design and communication; domination number; average lower 2-domination number

MSC: 05C40; 05C69; 68M10; 68R10

1. Introduction

Graph theory has seen an explosive growth due to interaction with areas like computer science, operation research, etc. In particular, it has become one of the most powerful mathematical tools in the analysis and study of the architecture of a network. The most common networks are telecommunication networks, computer networks, road and rail networks and other logistic networks [1]. In a communication network, the measures of vulnerability are essential to guide the designers in choosing a suitable network topology. They have an impact on solving difficult optimization problems for networks [2,3].

The graph vulnerability relates to the study of a graph when some of its elements (vertices or edges) are removed. The measures of graph vulnerability are usually invariants that measure how a deletion of one or more network elements changes properties of the network [4]. In the literature, various measures have been defined to measure the robustness of a network and a variety of graph theoretic parameters have been used to derive formulas to calculate network vulnerability. The best known measure of reliability of a graph is its connectivity. The connectivity is defined to be the minimum number of vertices whose deletion results in a disconnected or trivial graph [5].

The connectivity of a graph G is denoted by $k(G)$ and it is defined as follows:

$$k(G) = \min\{|S| : \ S \subset V \ and \ w(G-S) > 1\}$$

where $w(G-S)$ is the number of components of the graph $G-S$.

The toughness [6], the integrity [7], the domination number [8], the bondage number [9,10], the edge eccentric connectivity number [11], etc., have been proposed for measuring the vulnerability of networks. Recently, some average vulnerability parameters like the average lower independence number [12,13], the average lower domination number [13–17], the average connectivity number [18], the average lower connectivity number [19] and the average lower bondage number [4] have been defined.

Let $G = (V(G), E(G))$ be a simple undirected graph of order n. We begin by recalling some standard definitions that we need throughout this paper. For any vertex $v \in V(G)$, the open neighborhood of v is $N_G(v) = \{u \in V| \quad uv \in E(G)\}$ and closed neighborhood of v is $N_G[v] = N_G(v) \cup \{v\}$. The degree of vertex v in G denoted by $d_G(v)$, that is, the size of its open neighborhood [8]. The minimum degree of graph G is denoted by $\delta(G)$. A set $S \subseteq V(G)$ is a dominating set if every vertex in $V(G) - S$ is adjacent to at least one vertex in S. The minimum cardinality taken over all dominating sets of G is called the domination number of G and denoted by $\gamma(G)$ [8]. Another domination concept is 2-domination number. A 2-dominating set of a graph G is a set $D \subseteq V(G)$ of vertices of graph G such that every vertex of $V(G) - D$ has at least two neighbors in D. The 2-domination number of a graph G, denoted by $\gamma_2(G)$, is the minimum cardinality of a 2-dominating set of the graph G [8,20–22].

In 2004, Henning introduced the concept of average domination and average independence in [13]. Moreover, the average lower domination and average lower independence number are the theoretical vulnerability parameters for a network that modeled a graph [12,15]. The average lower domination number of a graph G, denoted by $\gamma_{av}(G)$, is defined as follows:

$$\gamma_{av}(G) = \frac{1}{|V(G)|} \sum_{v \in V(G)} \gamma_v(G) \tag{1}$$

where the lower domination number, denoted by $\gamma_v(G)$, is the minimum cardinality of a dominating set of the graph G that contains the vertex v [13,16]. In [15], an algorithm is given for computing the average lower domination number of any graph G.

In 2015, a new graph theoretical parameter namely the average lower 2-domination number was defined in [23,24]. The average lower 2-domination number of a graph G, denoted by $\gamma_{2av}(G)$, is defined as follows:

$$\gamma_{2av}(G) = \frac{1}{|V(G)|} \sum_{v \in V(G)} \gamma_{2v}(G) \tag{2}$$

where the lower 2-domination number, denoted by $\gamma_{2v}(G)$, is the minimum cardinality of a dominating set of the graph G that contains the vertex v [23,24].

If we think of a graph as modeling a network, then the average lower 2-domination number can be more sensitive for the vulnerability of graphs than the other known vulnerability measures of a graph [23]. We consider two connected simple graphs G and H in Figure 1, where $|V(G)| = |V(H)| = 10$ and $|E(G)| = |E(H)| = 17$. Graphs G and H have not only equal the connectivity but also equal the domination number, the average lower domination number and the 2-domination number such as $k(G) = k(H) = 1$, $\gamma(G) = \gamma(H) = 1$, $\gamma_{av}(G) = \gamma_{av}(H) = 19/5$ and $\gamma_2(G) = \gamma_2(H) = 5$. The results can be checked by readers. So, how can we distinguish between the graphs G and H?

When we compute $\gamma_{2av}(H)$ and $\gamma_{2av}(G)$, we get $\gamma_{2av}(H) = 51/10 = 5.1$ and $\gamma_{2av}(G) = 50/10 = 5$. So, the average lower 2-domination number may be used for distinguish between these two graphs G and H. Since $\gamma_{2av}(G) < \gamma_{2av}(H)$, we can say that the graph H is more vulnerable than the graph G. In other words, the graph G is tougher than the graph H [23,24].

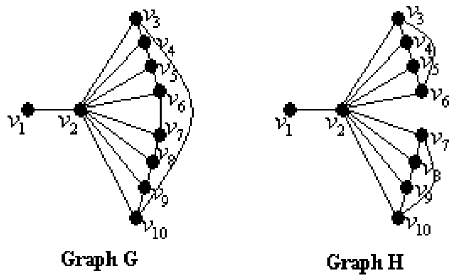

Figure 1. Graphs G and H.

The wheel graph has been used in different areas such as the wireless sensor networks, the vulnerability of networks, and so on. The wheel graph has many good properties. From the standpoint of the hub vertex, all elements, including vertices and edges, are in its one-hop neighborhood, which indicates that the wheel structure is fully included in the neighborhood graph of the hub vertex. Furthermore, wheel graphs are important for localizability because they are globally rigid in 2D space, which indicates an approach to identifying localizable vertices [25]. Moreover, the wheels and various related graphs have been studied for many reasons. The gear graphs, the friendship graph, the helm graphs and the sun flower graphs are among such graphs. The definitions of these graphs will be given in Section 3. In [26], Aytac and Odabas compute the residual closeness for wheels and related graphs. In [27], Javaid and Shokat give upper bounds for the cardinality of vertices in some wheel related graphs with a given partition dimension k.

Our aim in this paper is to study a new vulnerability parameter, called the average lower 2-domination number. In Section 2, well-known basic results are given for the average lower domination number, the average lower 2-domination number and the 2-domination number. In Section 3, we compute the average lower 2-domination numbers of wheels and some related graphs. Finally, an algorithm is proposed for computing the 2-domination number and the average lower 2-domination numbers of any given graph in Section 4.

2. Basic Results

In this section, well known basic results are given with regard to the average lower domination number, the average lower 2-domination number and the 2-domination number.

Theorem 1. [13] *Let G be any graph of order n with the domination number $\gamma(G)$, then*

$$\gamma_{av}(G) \leqslant \gamma(G) + 1 - \frac{\gamma(G)}{n}$$

with equality if and only if G has a unique $\gamma(G)$-set.

Theorem 2. [13] *If $K_{1,n-1}$ is a star graph of order n, where $n \geqslant 3$, then $\gamma_{av}(K_{1,n-1}) = 2 - \frac{1}{n}$.*

Theorem 3. [13] *If P_n is a path graph of order n, then*

$$\gamma_{av}(P_n) = \begin{cases} \frac{n+2}{3} - \frac{2}{3n}, & if\ n \equiv 2 (\text{mod } 3); \\ \frac{n+2}{3}, & otherwise. \end{cases}$$

Theorem 4. [13] *If C_n is a cycle graph of order n, then $\gamma_{av}(C_n) = 2$.*

Theorem 5. [13] *If K_n is a complete graph of order n, then $\gamma_{av}(K_n) = 1$.*

Observation 1. *If W_n is a wheel graph of order $n+1$, then $\gamma_{av}(W_n) = \frac{2n+1}{n+1}$.*

Theorem 6. [28] *If K_n is a complete graph of order n, then $\gamma_2(K_n) = \min\{2, n\}$.*

Theorem 7. [28] *If P_n is a path graph of order n, then $\gamma_2(P_n) = \lfloor n/2 \rfloor + 1$.*

Theorem 8. [28] *If C_n is a cycle graph of order n, where $n \geqslant 3$, then $\gamma_2(C_n) = \lfloor (n+1)/2 \rfloor$.*

Theorem 9. [28] *If W_n is a wheel graph of order $n+1$, where $n \geqslant 3$, then*

$$\gamma_2(W_n) = \begin{cases} 2 & , \quad if\ n = 3,4; \\ \lfloor (n+1)/3 \rfloor \ +1 & , \ otherwise. \end{cases}$$

Theorem 10. [23] *Let G be any connected graph of order n. If $\gamma_2(G)$-set is unique, then*

$$\gamma_{2av}(G) = \gamma_2(G) + 1 - \frac{\gamma_2(G)}{n}$$

Theorem 11. [23] *Let G be any connected graph of order n. If $\delta(G) \geqslant 2$, then*

$$\gamma_{2av}(G) \leqslant \gamma_2(G) + 1 - \frac{\gamma_2(G)}{n}$$

Theorem 12. [23] *Let G be any connected graph of order $n \geqslant 2$. Then, $2 \leqslant \gamma_{2av}(G) \leqslant n - 1 + \frac{1}{n}$.*

Theorem 13. [23] *If P_n is a path graph of order n, then*

$$\gamma_{2av}(P_n) = \begin{cases} \lfloor n/2 \rfloor + 2 - \frac{\lfloor n/2 \rfloor + 1}{n} & , If\ n\ is\ odd; \\ \lfloor n/2 \rfloor + 1 & , \ If\ n\ is\ even. \end{cases}$$

Theorem 14. [23] *If C_n is a cycle graph of order n, then $\gamma_{2av}(C_n) = \lfloor (n+1)/2 \rfloor$.*

Theorem 15. [23] *If K_n is a complete graph of order n, then $\gamma_{av}(K_n) = 2$.*

Theorem 16. [23] *If $K_{1,n-1}$ is a star graph of order n, where $n \geqslant 3$, then $\gamma_{2av}(K_{1,n-1}) = n - 1 + \frac{1}{n}$.*

3. The Average Lower 2-Domination Number of Wheels Related Graphs

In this section, we have calculated the average lower 2-domination number of wheels and related graphs such as the wheel graph W_n, the gear graph G_n, the friendship graph f_n, the helm graph H_n and the sun flower graph Sf_n. Now, we recall the definitions of these graphs.

Definition 1. [26] The wheel graph W_n with n spokes is a graph that contains an *n-cycle* and one additional central vertex v_c that is adjacent to all vertices of the cycle. Wheel graph W_n has $(n+1)$-vertices and $2n$-edges.

Definition 2. [12] The gear graph G_n is a wheel graph with a vertex added between each pair adjacent graph vertices of the outer cycle. The gear graph G_n has $(2n+1)$-vertices and $3n$-edges.

Definition 3. [26] The friendship graph f_n is collection of n triangles with a common vertex. The friendship graph f_n has $(2n+1)$-vertices and $3n$-edges.

Definition 4. [27] The helm graph H_n is the graph obtained from an *n-wheel graph* by adjoining a pendant edge at each vertex of the cycle. The helm graph H_n has $(2n+1)$-vertices and $3n$-edges.

Definition 5. [27] The sun flower graph Sf_n the graph obtained from an *n-wheel graph* with central vertex v_c and *n-cycle* $v_0, v_1, v_2, \ldots, v_{n-1}$ and additional n vertices $w_0, w_1, w_2, \ldots, w_{n-1}$ where w_i is joined by edges to v_i, v_{i+1} for $i \in \{0, 1, \ldots, n-1\}$ where $(i+1)$ is taken modulo n. The sun flower graph Sf_n has $(2n+1)$-vertices and $4n$-edges.

We display the graphs W_4, G_4, f_4, H_4 and Sf_4 in Figure 2.

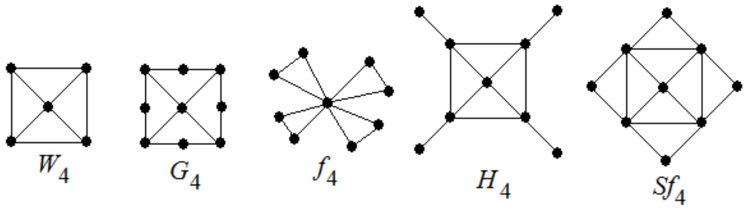

Figure 2. Graphs W_4, G_4, f_4, H_4 and Sf_4.

Theorem 17. *If W_n is a wheel graph of order $n+1$, where $n \geqslant 5$, then $\gamma_{2av}(W_n) = 1 + \lceil n/3 \rceil$.*

Proof. The $\gamma_2(W_n)$- set of a graph W_n, $n \geqslant 5$, is a set with the vertex v_c and $\lceil n/3 \rceil$ vertices from the set $V(W_n)\ \{v_c\}$. So, $\gamma_2(W_n) = 1 + \lceil n/3 \rceil$. Thus, $\gamma_2(W_n) = 1 + \lceil n/3 \rceil$ is obtained for every vertex $v \in V(W_n)$. As a result, we get $\gamma_{2av}(W_n) = 1 + \lceil n/3 \rceil$.

Remark 1. *Let W_3 and W_4 be wheels graph with order 3 and 4, respectively. Then, $\gamma_{2av}(W_3) = 2$ and $\gamma_{2av}(W_4) = 11/5$.*

Remark 2. *If W_{2n} is a wheel graph of order $2n+1$, then $\gamma_{2av}(W_{2n}) = 1 + \lceil 2n/3 \rceil$.*

Theorem 18. *If G_n is a gear graph of order $2n+1$, then $\gamma_{2av}(G_n) = \frac{2n^2+2n+1}{2n+1}$.*

Proof. We partition the vertices of graph G_n into three subsets V_1, V_2 and V_3 as follows:

$$V_1 = \left\{ v_c \in V(G_n) \middle| \ d_{G_n}(v_c) = n \right\}$$

$$V_2 = \left\{ v_i \in V(G_n) \middle| \ d_{G_n}(v_i) = 3, \quad i \in \{1, 2, \ldots, n\} \right\}$$

$$V_3 = \left\{ v_i \in V(G_n) \middle| \ d_{G_n}(v_i) = 2, \quad i \in \{n+1, n+2, \ldots, 2n\} \right\}$$

When the $\gamma_{2av}(G_n)$ is calculated for all vertices v in the graph G_n, each vertex satisfies one of the three cases below.

Case 1. Let v_c be the vertex of V_1. The center vertex v_c is adjacent to n vertices in V_2. Thus, all vertices of V_2 are 1-dominated. By the definition of gear graphs, the whole vertex set V_2 (or V_3) is taken to $\gamma_2(G_n)$-set, then $\gamma_{2v_c}(G_n) = n+1$ is obtained.

Case 2. Let v_i be the vertex of V_2. Clearly every vertex of the graph G_n is 2-dominated by the vertices of V_2. As a result, we have $\gamma_{2v_i}(G_n) = n$, where $i \in \{1, 2, \ldots, n\}$.

Case 3. Let v_i be the vertex of V_3. The $\gamma_2(G_n)$-set including vertex v_i is similar to $\gamma_2(G_n)$-set in the Case 1. So, we have $\gamma_{2v_i}(G_n) = n+1$, where $i \in \{n+1, n+2, \ldots, 2n\}$.

By Cases 1, 2 and 3, we have:

$$\gamma_{2av}(G_n) = \frac{1}{|V(G_n)|} \left(\sum_{v \in V_1} \gamma_{2v}(G_n) + \sum_{v \in V_2} \gamma_{2v}(G_n) + \sum_{v \in V_3} \gamma_{2v}(G_n) \right) \tag{3}$$

$$= \frac{1}{2n+1} \left(n + 1 + \sum_{v \in V_2}^{n} n + \sum_{v \in V_3}^{n} (n+1) \right) \tag{4}$$

$$= \frac{2n^2 + 2n + 1}{2n+1}. \tag{5}$$

Theorem 19. *If f_n is a friendship graph of order $2n+1$, then $\gamma_{2av}(f_n) = n+1$.*

Proof. By the definition of the friendship graph and 2-domination number, a $\gamma_2(f_n)$-set must include the vertex v_c whose degree is *2n*. Thus, *2n*-vertices are 1-dominated by the vertex v_c. Furthermore,

n-disjoint graphs K_2 are formed by these $2n$-vertices in the graph $f_n \backslash \{v_c\}$. When any vertex of each graph K_2 is taken to a $\gamma_2(f_n)$-set, $\gamma_2(f_n) = n+1$ is obtained. It is easy to see that $\gamma_{2v}(f_n) = n+1$ for every vertex $v \in V(f_n)$. Thus, we get $\gamma_{2av}(f_n) = n+1$.

Theorem 20. *If H_n is a helm graph of order $2n+1$, then $\gamma_{2av}(H_n) = \frac{2n^2+4n+1}{2n+1}$.*

Proof. Since the $\gamma_2(H_n)$-set is unique in the graph H_n, we have $\gamma_{2av}(H_n) = n+2-((n+1)/(2n+1))$ by the Theorem 10. As a result, $\gamma_{2av}(H_n) = \frac{2n^2+4n+1}{2n+1}$ is obtained.

Theorem 21. *If Sf_n is a sun flower graph of order $2n+1$, then $\gamma_{2av}(Sf_n) = \frac{2n^2+2n+1}{2n+1}$.*

Proof. The proof follows directly from the Theorem 18.

It is point out that the gear graph G_n is tougher than the friendship graph f_n and the helm graph H_n, where $|V(G_n)| = |V(f_n)| = |V(H_n)|$ and $|E(G_n)| = |E(f_n)| = |E(H_n)|$. Similarly, the wheel graph W_{2n} is tougher than the sun flower graph Sf_n, where $|V(W_{2n})| = |V(Sf_n)|$ and $|E(W_{2n})| = |E(Sf_n)|$. Readers can see that these results are shown in Figures 3 and 4.

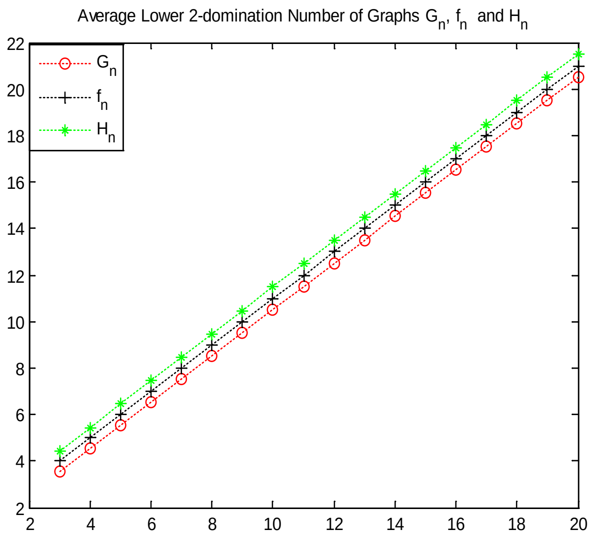

Figure 3. Values of $\gamma_{2av}(G_n)$, $\gamma_{2av}(f_n)$ and $\gamma_{2av}(H_n)$.

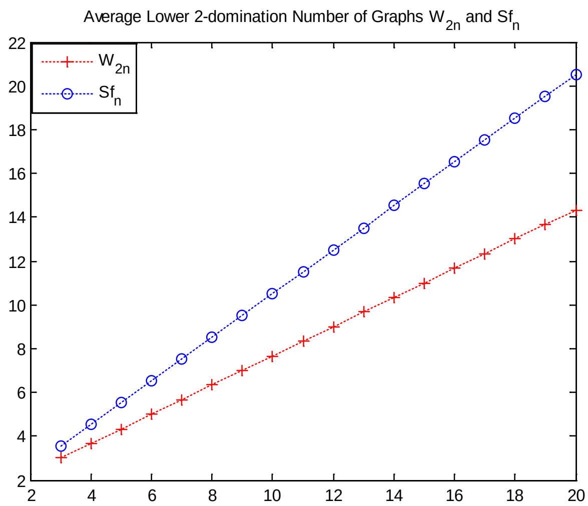

Figure 4. Values of $\gamma_{2av}(W_{2n})$ and $\gamma_{2av}(Sf_n)$.

4. An Algorithm for Computing the Average Lower 2-Domination Number

In this section, the algorithm in [29] which finds the domination number and all the minimal dominating sets of a graph is improved. The improved algorithm also computes the 2-domination number and the average lower 2-domination number of a graph. The definitions used in the algorithm below are found in [29].

$i, j, n, \gamma_2, es, top,$ min : positive integer
f : element of $L(n)$
D, W : array n^+ of $L(n)$
γ_{2av}: real number
BEGIN
for $j \leftarrow 1$ to n do
begin
    $D[j] \leftarrow 0;$
    if $d_G[v_j] = 0$ then $D[j] \leftarrow D[j] + v_j$ end if;
    if $d_G[v_j] = 1$ then $D[j] \leftarrow D[j] + v_j$
    ELSE
    $D[j] \leftarrow D[j] + v_j$
        for $i \leftarrow 1$ to $n - 1$ do
        begin
            for $k \leftarrow i + 1$ to n do
            begin
                if $[(j \hat{=} i)$ and $(j \hat{=} k)$ and $(v_j E v_i)$ and $(v_k E v_j)]$
                then $D[j] \leftarrow D[j] + v_i v_k$ end if;
            end; {*for k*}
        end; {*for i*}
    end if;
end; {*for j*}
$f \leftarrow 1$
for $j \leftarrow 1$ to n do
begin
    $f \leftarrow f * D[j];$
end;
$\gamma_2 \leftarrow \min\limits_{x \in f} \{|x|\};$
$es \leftarrow 0;$
for $x \in f$ do
$es \leftarrow es + 1;$
$top \leftarrow 0;$
    for $i \leftarrow 1$ to n do
    begin
    $S \leftarrow 0;$
        for $j \leftarrow 1$ to es do
        begin
            if $v_i \in f[j]$ then $S \leftarrow S + 1$ end if;
            if $(S = 1)$ then min $\leftarrow |f[j]|$ end if;
            if $(|f[j]| <$ min$)$ then min $\leftarrow |f[j]|$ end if;
        end; {*for j*}
    $top \leftarrow top +$ min;
    end; {*for i*}
    $\gamma_{2av} \leftarrow top/n;$
END.

Example 1. Compute the 2-domination number and the average lower 2-domination number of graph G in Figure 5.

Firstly, we must find function f as follows:

$$f = (a)\,(b + ec)\,(c + bd)\,(d + ec)(e + bd)$$

Then, two mathematical logic functions are used as follows:

$(i)\quad x\,x = x$

$(ii)\quad x + xy = x$

Thus, we have

$$\begin{aligned} f &= (ab + aec)\,(c + bd)\,(d + ec)(e + bd) \\ &= (abc + abd + aec)\,(d + ec)(e + bd) \\ &= (abd + aec)\,(e + bd) \\ &= (abd + aec). \end{aligned}$$

Furthermore, we have $|f| = |abd + aec|$.

Clearly, the 2-domination sets $\{a, b, d\}$ and $\{a, e, c\}$ have been found by the algorithm. Thus, we get $\gamma_2(G) = \gamma_{2av}(G) = 3$.

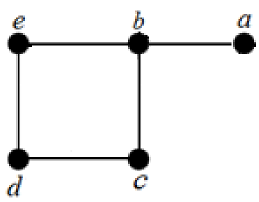

Figure 5. Graph G with 5-vertices and 5-edges.

5. Conclusions

Communication systems are often subjected to failures and attacks. A variety of measures have been proposed in the literature to quantify the robustness of networks and a number of graph theoretic parameters have been used to derive formulas for calculating network reliability. In this paper we have studied the average lower 2-domination number for graph vulnerability. The average lower 2-domination number can be more sensitive than the other measures of vulnerability like connectivity, domination number, average lower domination number and 2-domination number. We have also studied wheel graphs and wheels related graphs. Finally, an algorithm is proposed for computing the 2-domination number and the average lower 2-domination numbers of any given graph G.

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