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On a Problem Arising in Application of the Re-Quantization Method to Construct Asymptotics of Solutions to Linear Differential Equations with Holomorphic Coefficients at Infinity

Maria Korovina [1,*], Ilya Smirnov [1,*] and Vladimir Smirnov [2,3]

1 Lomonosov Moscow State University, 119991 Moscow, Russia
2 Moscow Aviation Institute (National Research University), 125993 Moscow, Russia; vl-smirnov@mail.ru
3 Russian University of Transport (MIIT), 127994 Moscow, Russia
* Correspondence: betelgeuser@yandex.ru (M.K.); ismirnov@cs.msu.ru (I.S.); Tel.: +7-926-457-2230 (I.S.)

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Abstract: The re-quantization method—one of the resurgent analysis methods of current importance—is developed in this study. It is widely used in the analytical theory of linear differential equations. With the help of the re-quantization method, the problem of constructing the asymptotics of the inverse Laplace–Borel transform is solved for a particular type of functions with holomorphic coefficients that exponentially grow at zero. Two examples of constructing the uniform asymptotics at infinity for the second- and forth-order differential equations with the help of the re-quantization method and the result obtained in this study are considered.

Keywords: re-quantization method; asymptotic; holomorphic coefficient; Laplace–Borel transform

1. Introduction

This study is devoted to the application of the re-quantization method to constructing asymptotics of solutions to ordinary differential equations with holomorphic coefficients in the neighborhood of infinity. Namely, the following ordinary differential equations with holomorphic coefficients are considered:

$$\left(\frac{d}{dx}\right)^n u(x)+a_{n-1}(x)\left(\frac{d}{dx}\right)^{n-1}u(x)+\ldots+a_i(x)\left(\frac{d}{dx}\right)^i u(x)+...+a_0(x)u(x)=0. \quad (1)$$

Here, the coefficients $a_i(x)$ are regular at infinity, which means that there is such a circle exterior $|x|>a$ that the functions $a_i(x), i=0,1,...,n-1$ can be expanded in it into convergent power series $a_i(x)=\sum\limits_{j=0}^{\infty}\frac{b_i^j}{x^j}$. The aim of our study is to construct the asymptotics of the solution to Equation (1) for $x\to\infty$. The complexity of the problem of constructing asymptotics of solutions of linear differential equations in the vicinity of an infinitely-distant point is conditioned by the fact that this point, generally speaking, is an irregular singular point.

This problem can be reduced to that of constructing the asymptotics of the solution in a neighborhood of zero for linear differential equations with a second-order beak-like singularity. The problem of constructing the asymptotics of solutions to ordinary differential equations with holomorphic coefficients is a classical problem of the analytic theory of linear differential equations. One of the first studies in this domain is the book written by Thome [1]. Then, in [2], the asymptotics were constructed for a particular case of this problem, namely, for the case where the main symbol $H_0(p)$ of the operator $\hat{H}$, $H_0(p)=H(0,p)$ has simple roots.

Then, this particular case for systems of linear differential equations was considered, for example in such classical books as [3–5] and many other studies [6,7]. Paper [8] provides a review of the results obtained for the case of a system of ordinary linear differential equations.

In these studies, the asymptotic expansions of the solutions of some ordinary differential equations were constructed; they were obtained in the form of products of the corresponding exponentials and divergent power series, namely:

$$u = \sum_{i=1}^{n} e^{\alpha_i/r} r^{\sigma_i} \sum_{k=0}^{\infty} a_i^k r^k, \tag{2}$$

where $\alpha_i, i = 1, ..., n$ are the roots of the polynomial $H_0(p)$; σ_{j} and a_i^k are some complex numbers.

However, the problem of interpreting the obtained divergent series was left open; in other words, there is no method for summation of these divergent series. Such asymptotics were later called the WKB (Wentzel–Kramers–Brillouin) asymptotics [9,10]. This name appeared when solving some problems of quantum mechanics, where the asymptotic expansions of such a type were obtained.

At the end of the 1980s, an apparatus convenient for the summation of such series was obtained. It is based on the Laplace–Borel transform and the concept of the resurgent function introduced for the first time by Ecalle [11].

The apparatus for interpreting and constructing the asymptotic expansions of the form (3), based on the Laplace–Borel transform, is called the resurgent analysis.

The main idea of the resurgent analysis is that the formal Borel transforms $\widetilde{u}_1(p)$, $\widetilde{u}_2(p) \ldots$ are the power series of a dual variable p, converging in a neighborhood of the points $p = \lambda_{\mathrm{j}}$. The inverse Borel transform gives a regular way of summing series (2). However, it is necessary to prove the infinite extensibility of functions $\widetilde{u}_j(p)$, i.e., their extensibility along any path on the Riemann surface $\widetilde{u}_j(p)$ that does not pass through some discrete set that depends on the function (an exact definition of the concept of infinite extensibility is given below). As a rule, the proof of this fact represented a great difficulty in applying the resurgent analysis to constructing asymptotics of solutions to differential equations. For the equations with degeneracies, the proof of the infinite extensibility was obtained by Shatalov and Korovina [12,13]. It is shown in these works that all solutions to ordinary differential equations with arbitrary holomorphic coefficients and a resurgent right-hand part are resurgent functions. This result is generalized to the case of the linear partial differential equations with additional constraint conditions imposed on the differentiation operator symbol. This result makes it possible to apply the resurgent analysis methods to construct asymptotics of solutions to both ordinary and partial linear differential equations with holomorphic coefficients. For example, in [14], the resurgent analysis methods were applied to the theory of differential equations on manifolds with singularities. Namely, the problem of constructing the asymptotics of solutions to the Laplace equation on the manifold with a beak-like singularity was solved.

Owing to this result, in [12,15], uniform asymptotics of solutions were constructed for the case where the roots of the main symbol $H_0(p) = H(0, p)$ are of the first order. More detailed information on constructing uniform asymptotics can be found, for example, in [16].

However, the methods used to construct asymptotics of solutions in the case where the main symbol has simple roots have proven to be inapplicable to the case of multiple roots, except for the case of the second-order equations. In [17], the problem of constructing the asymptotic solutions for a second-order equation with arbitrary holomorphic coefficients was solved in the general case. In addition, the problem of constructing the asymptotics of solutions for the Laplace equation on a manifold with a beak-like singularity was solved in [17].

To solve the problem of multiple roots, in recent years, a method of re-quantization was created [18]. This method is used in the case where the integro-differential equation in the dual space cannot be solved by the method of successive approximations and is reduced, in turn, to an equation with beak-type degeneracies. In this case, the infinite extensibility theorem is proven (it was already proven for ordinary differential equations), then, the Laplace–Borel transform is applied one more time, and asymptotics are constructed for the resulting equation; they make it possible to find the asymptotics of the original equation. Now, using this method, a number of problems for equations with degeneracies in the case of multiple roots have been solved [19].

The essence of the re-quantization method is that the Laplace–Borel transform is performed twice. That is, having shifted the root p_j to the zero point by using the substitution:

$$u(r) = e^{\frac{p_j}{r}} u_j(r),$$

the Laplace–Borel transform is then carried out, which gives us an integro-differential equation for the function $\widetilde{u}_j(p)$, which will have a singularity at zero. To find the asymptotics of this function at zero, the Laplace–Borel transform is performed for the second time. Therefore, we obtain an integral equation with respect to the function $\widehat{u}_j(q)$, which is the Laplace–Borel transform of the function $\widetilde{u}_j(p)$. For the resulting equation, using the method of successive approximations, we construct the asymptotics of its solution in the neighborhood of the roots of the main symbol. Having found the asymptotics of the function $\widehat{u}_j(q)$ and then performing the inverse Laplace–Borel transform, we construct the asymptotics of the function $\widetilde{u}_j(p)$ in the neighborhood of zero. This asymptotics can be the WKB-asymptotics or have a more general form and contain exponents with fractional powers; so, the necessity arises to calculate the inverse Laplace–Borel transform of the functions of the form $e^{\frac{\alpha}{p^{n/k}}} g(p)$, where $n, k \in N$, $\alpha \in R$. This problem is solved in this paper. In addition, an example of the application of the re-quantization method for constructing the asymptotics for the fourth-order equations is considered.

Using the substitution $x = \frac{1}{r}$, Problem (1) is reduced to the equation with a second-order beak-type degeneracy at zero, which can be written in the form:

$$H\left(r, -r^2 \frac{d}{dr}\right) u = 0, \tag{3}$$

where:

$$H\left(r, -r^2 \frac{d}{dr}\right) = \left(-r^2 \frac{d}{dr}\right)^n + \sum_{i=0}^{n-1} a_i(r) \left(-r^2 \frac{d}{dr}\right)^i.$$

The main symbol $H_0(p)$ of the operator $\hat{H}$ is $H_0(p) = H(0, p)$.

The question arises about the asymptotics in the neighborhood of a multiple root.

Let us consider the case where the main symbol of the differential operator has one root. Without loss of generality, we assume that this root is at zero.

It follows from this assumption that in this case, $a_i(x) = \sum_{j=1}^{\infty} \frac{b_i^j}{x^j}$. Let us write Equation (1) in the form:

$$\begin{aligned} &\left(-r^2 \tfrac{d}{dr}\right)^n u + a_0 r^{m_k} \left(-r^2 \tfrac{d}{dr}\right)^k u + a_1 r^{m_k+1} \left(-r^2 \tfrac{d}{dr}\right)^{k-1} u + a_2 r^{m_k+2} \left(-r^2 \tfrac{d}{dr}\right)^{k-2} u + \ldots + a_{k+1} r^{m_k+k} u + \\ &+ \ldots + \sum_{j=1}^{h} r^j \sum_{i=h_j}^{n-1} a_j^i \left(-r^2 \tfrac{d}{dr}\right)^i u + r^{h+1} \sum_{i=0}^{n-1} a^i(r) \left(-r^2 \tfrac{d}{dr}\right)^i u = 0. \end{aligned} \tag{4}$$

Here, a_i, a_j^i are the corresponding constants; $a^i(r)$ are holomorphic functions; $h_j + j > m_k + k$. Let us call $h = m_k + k$ the singularity index; in other words, provided that $j + i > h$, the terms of the form $a_j^i r^j \left(r^2 \frac{d}{dr}\right)^i$ are the lowest terms.

2. Definition of the Laplace–Borel Transform and Its Properties

Let us give definitions to some concepts of the resurgent analysis, which we will need below.

Let us denote by $S_{R,\varepsilon}$ the sector $S_{R,\varepsilon} = \{r|-\varepsilon < \arg r < \varepsilon, |r| < R\}$. Let us say that a function f that is analytical on $S_{R,\varepsilon}$ has no more than k-exponential growth if there are such non-negative constants C and α that the inequality $|f| < Ce^{a\frac{1}{|r|^k}}$ holds in the sector $S_{R,\varepsilon}$.

We denote by $E_k(S_{R,\varepsilon})$ the space of functions holomorphic in a region $S_{R,\varepsilon}$ that grow k-exponentially at zero.

The main technical tool for constructing the theory of resurgent functions is the Laplace–Borel transform. The definition and properties of this transform in the class of hyperfunctions of k-exponential growth for $k = 1$ are presented in monograph [16].

Let us denote by $E\left(\widetilde{\Omega}_{R,\varepsilon}\right)$ the space of holomorphic functions of exponential growth in the domain $\widetilde{\Omega}_{R,\varepsilon} = \left\{p|-\frac{\pi}{2} - \varepsilon < \arg p < \frac{\pi}{2} + \varepsilon, |p| > R\right\}$ and denote by $E(C)$ the space of the entirety of the functions of exponential growth.

The Laplace–Borel k-transform of the function $f(r) \in E_k(S_{R,\varepsilon})$ is the mapping $B_k : E_k(S_{R,\varepsilon}) \to \mathrm{E}(\widetilde{\Omega}_{R,\varepsilon})/\mathrm{E(C)}$ such that:

$$B_k f = \int_0^{r_0} e^{-p/r^k} f(r)\, \frac{dr}{r^{k+1}}.$$

The inverse Laplace–Borel k-transform is defined by the formula:

$$B_k{}^{-1}\widetilde{f} = \frac{k}{2\pi i}\int_{\widetilde{\gamma}} e^{p/r^k}\widetilde{f}(p)\, dp.$$

The contour $\widetilde{\gamma}$ is shown in Figure 1 in [12]. It should be noted that for the Laplace–Borel k-transform, the following formulas are valid:

$$\begin{gathered} B_k \quad \circ\left(-\frac{1}{k}r^{k+1}\frac{\partial}{\partial r}\right)f(r) = pB_k f, \\ \frac{\partial}{\partial p} \circ B_k f = -B_k\left(\frac{1}{r^k}f(r)\right). \end{gathered}$$

3. The Main Result

In this section, we find the Laplace–Borel transform of the function $e^{-\frac{\alpha}{p^{k/n}}}g(p)$.

Theorem 1. *The asymptotics of the function $B^{-1}e^{-\frac{\alpha}{p^{k/n}}}g(p)$ in the neighborhood of zero has the form:*

$$\sum_{j=1}^{n+k} \exp\left(\sum_{i=1}^{k} \frac{\alpha_i^j}{r^{\frac{i}{n+k}}}\right) r^{\frac{\sigma_j}{n+k}} \sum_l c_l^j r^{\frac{l}{n+k}}. \tag{5}$$

Here, α_{n-1}^j, $j = 1, \ldots, k+n$ are the roots of polynomial $p^{n+k} + \left(\frac{n+k}{k}\right)^{n+k}(-1)^{n+1}\left(\frac{\alpha k}{n}\right)^n$; coefficients α_i^j at $i < n-1$ and c_i^j are some constants.

Proof of Theorem 1. Let us consider the equation:

$$\begin{aligned} &\left(-r^2\frac{d}{dr}\right)^{n+k}u + c_{n-1}r\left(-r^2\frac{d}{dr}\right)^{k+n-1}u + c_{n-2}r^2\left(-r^2\frac{d}{dr}\right)^{k+n-2} + \\ &+ \ldots + c_1 r^{n-1}\left(-r^2\frac{d}{dr}\right)^{k+1}u + cr^n u + c_0 r^n\left(-r^2\frac{d}{dr}\right)^k u = 0, \end{aligned} \tag{6}$$

where $c_i, i = 0, \ldots, n-1$ and c are some constants. Let us perform the Borel transform:

$$p^{n+k}\widetilde{u} + c_0(-1)^n\int_1^p \ldots \int_1^{p_2} p_1^k\widetilde{u}(p_1)dp_1\ldots dp_n + c_1(-1)^{n-1}\int_1^p \ldots \int_1^{p_2} {p_1}^{k+1}\widetilde{u}(p_1)dp_1\ldots dp_{n-1} + \ldots$$
$$+c_2(-1)^{n-2}\int_1^p\int_1^{p_2} {p_1}^{k+2}\widetilde{u}(p_1)dp_1dp_2 + \ldots +$$
$$+c_{n-1}(-1)\int_1^p {p_1}^{k+n-1}\widetilde{u}(p_1)dp_1 + c(-1)^n\int_1^p \ldots \int_1^{p_2} \widetilde{u}(p_1)dp_1\ldots dp_n = f(p).$$

We denote by $f(p)$ an arbitrary holomorphic function. Let us differentiate it n times:

$$\left(\frac{d}{dp}\right)^n p^{n+k}u + c_0(-1)^n p^k u + c_1(-1)^{n-1}\left(\frac{d}{dp}\right)p^{k+1}u + c_2(-1)^{n-2}\left(\frac{d}{dp}\right)^2 p^{k+2}u + \ldots +$$
$$+c_{n-1}(-1)\left(\frac{d}{dp}\right)^{n-1} p^{k+n-1}u + (-1)^n cu = f(p). \tag{7}$$

Let us show that it is possible to choose constants $c,\ c_i,\ i = 0, \ldots, n-1$ so that the function $\widetilde{u}(p) = p^\sigma e^{-\frac{\alpha}{p^{k/n}}}$ is a solution to Equation (7). Substituting this function into Equation (7), we get the following equation for the constants $c,\ c_i,\ i = 0, \ldots, n-1$:

$$\left(\alpha_0^n p^{\sigma+k} + \alpha_1^n p^{\sigma+k-\frac{k}{n}} + \alpha_2^n p^{\sigma+k-\frac{2k}{n}} + \alpha_3^n p^{\sigma+k-\frac{3k}{n}} + \ldots + \alpha_n^n p^\sigma\right) +$$
$$+c_{n-1}\left(\alpha_0^{n-1}p^{\sigma+k} + \alpha_1^{n-1}p^{\sigma+k-\frac{k}{n}} + \alpha_2^{n-1}p^{\sigma+k-\frac{2k}{n}} + \alpha_3^{n-1}p^{\sigma+k-\frac{3k}{n}} + \ldots + \alpha_{n-1}^{n-1}p^{\sigma+k-\frac{(n-1)k}{n}}\right) +$$
$$+c_{n-2}\left(\alpha_0^{n-2}p^{\sigma+k} + \alpha_1^{n-2}p^{\sigma+k-\frac{k}{n}} + \ldots + \alpha_{n-2}^{n-2}p^{\sigma+k-\frac{(n-2)k}{n}}\right) +$$
$$+ \ldots + c_1\left(\alpha_0^1 p^{\sigma+k} + \alpha_1^1 p^{\sigma+k-\frac{k}{n}}\right) + c_0 p^{\sigma+k} + c(-1)^n p^\sigma = 0.$$

Here, $\alpha_i^j,\ i = 1, \ldots, n,\ j = 1, \ldots, n$ are the corresponding constants. It should be noted that $\alpha_n^n = \left(\frac{\alpha k}{n}\right)^n$. It follows from the last equation that:

$$\begin{aligned}
&c(-1)^n = -\alpha_n^n,\\
&\alpha_{n-1}^{n-1}c_{n-1} = -\alpha_{n-1}^n,\\
&\alpha_{n-1}^{n-2}c_{n-2} + \alpha_{n-2}^{n-2}c_{n-1} = -\alpha_{n-2}^n,\\
&\ldots\\
&\alpha_0^{n-1}c_{n-1} + \alpha_0^{n-2}c_{n-2} + \ldots + \alpha_0^1 c_1 + c_0 = -\alpha_0^n.
\end{aligned}$$

It is obvious that this system has a unique solution. Therefore, we can choose the constants $c,\ c_i,\ i = 0, \ldots, n-1$ so that the function $\widetilde{u}(p) = p^\sigma e^{-\frac{\alpha}{p^{k/n}}}$ is a solution to Equation (7). The solution $u(x)$ of Equation (6) is the inverse Laplace–Borel transform of this function.

Let us return to Equation (6) and find its solution. Dividing this equation by r^n, we can write it in the form:

$$\left(-r^{2-\frac{n}{n+k}}\frac{d}{dr}\right)^{n+k}u + {c'}_{n-1}r^{\frac{k}{n+k}}\left(-r^{2-\frac{n}{n+k}}\frac{d}{dr}\right)^{k+n-1}u + {c'}_{n-2}r^{2\frac{k}{n+k}}\left(-r^{2-\frac{n}{n+k}}\frac{d}{dr}\right)^{k+n-2} + \ldots +$$
$$+{c'}_1 r^{(n-1)\frac{k}{n+k}}\left(-r^{2-\frac{n}{n+k}}\frac{d}{dr}\right)^{k+1}u + \tag{8}$$
$$+cu + {c'}_0 r^{\frac{nk}{n+k}}\left(-r^{2-\frac{n}{n+k}}\frac{d}{dr}\right)^k u + \sum_{i=1}^{k-1}\alpha_i r^{\frac{(n+i)k}{n+k}}\left(-r^{2-\frac{n}{n+k}}\frac{d}{dr}\right)^{k-1-i}u = 0.$$

Here, $\alpha_i,\ {c'}_j, i = 1, \ldots, k-1,\ j = 0, \ldots, n-1$ are the corresponding constants. Performing the substitution $c_2^i,\ i = 1, 2, 3$ in (8), we get the following equation:

$$q = 0,\ q = -c_1 \tag{9}$$

Since the main symbol of this equation is:

$$H_0(p) = p^{n+k} + (-1)^{n+1}\left(\frac{n+k}{k}\right)^{n+k} c = p^{n+k} + (-1)^{n+1}\left(\frac{n+k}{k}\right)^{n+k}\left(\frac{\alpha k}{n}\right)^{n}$$

and all roots of the polynomial $H_0(p)$ are of the first order, then the asymptotics of the solution of Equation (9) at zero has the form:

$$u(x) = \sum_{j=1}^{n+k} \exp\left(\sum_{i=1}^{k} \frac{\alpha_i^j}{x^i}\right) x^{\sigma_j} \sum_l c_l^j x^l,$$

where $\alpha_k^j,\ j = 1, \ldots, n+k$ are the roots of the polynomial $H_0(p)$. Since:

$$\sum_{j=1}^{n+k} e^{\sum_{i=1}^{k} \frac{\alpha_i^j}{x^j}} x^{\sigma_j} \sum_l c_l^j x^l = \sum_{j=1}^{n+k} \exp\left(\sum_{i=1}^{k} \frac{\alpha_i^j}{r^{\frac{i}{n+k}}}\right) r^{\frac{\sigma_j}{n+k}} \sum_l c_l^j r^{\frac{l}{n+k}},$$

we finally obtain that the asymptotics of the function $B_1^{-1} p^{\sigma} e^{-\frac{\alpha}{p^{k/n}}}$ in the neighborhood of zero has the form:

$$B_1^{-1} p^{\sigma} e^{\frac{1}{p^{\frac{k}{n}}}} \approx \sum_{j=1}^{n+k} \exp\left(\sum_{i=1}^{k} \frac{\alpha_i^j}{r^{\frac{i}{n+k}}}\right) r^{\frac{\sigma_j}{n+k}} \sum_l c_l^j r^{\frac{l}{n+k}}.$$

□

4. Example 1

Let us consider the application of the re-quantization method in practice. For this purpose, let us consider the following ordinary differential equation:

$$\left(r^2 \frac{d}{dr}\right)^2 u - \frac{1}{4} r u - \frac{1}{2} r \left(r^2 \frac{d}{dr}\right) u = 0.$$

Using the Laplace–Borel transform, we get the equation:

$$p^2 \widetilde{u}(p) + \frac{1}{4} \int \widetilde{u}(p) dp - \frac{1}{2} \int p \widetilde{u}(p) dp = f(p).$$

Here, $f(p)$ is an arbitrary holomorphic function. Differentiating the obtained equation with respect to p, we get:

$$p^2 \frac{d}{dp} \widetilde{u}(p) + \frac{3}{2} p \widetilde{u}(p) + \frac{1}{4} \widetilde{u}(p) = \frac{d}{dp} f(p)$$

Solving this equation by the re-quantization method, we obtain the asymptotics for the solution of the last equation:

$$\widetilde{u}(p) \approx C p^{-\frac{3}{2}} e^{\frac{1}{4p}}.$$

Let us calculate the inverse transform $B^{-1} e^{-\frac{1}{p^{4p}}} p^{-\frac{3}{2}}$.

Since in this case $n = k = 1$ and $\alpha = -\frac{1}{4}$, then the roots $\alpha_1^j,\ j = 1,2$ are the roots of the polynomial $p^2 - 1$; in other words, $\alpha_1^1 = 1$ and $\alpha_1^2 = -1$. Therefore, we get:

$$B^{-1}e^{-\frac{\frac{\alpha}{k}}{p^{\frac{n}{k}}}}g(p) = \sum_{j=1}^{n+k}\exp\left(\sum_{i=1}^{k}\frac{\alpha_i^j}{r^{\frac{i}{n+k}}}\right)r^{\frac{\sigma_j}{n+k}}\sum_l c_l^j r^{\frac{l}{n+k}},$$
$$B^{-1}e^{-\frac{1}{p^{4p}}}p^{-\frac{3}{2}} = e^{\frac{1}{\sqrt{r}}}g_1(r) + e^{-\frac{1}{\sqrt{r}}}g_2(r).$$

Here, $g_i(r) = r^{\sigma_i}\sum_j c_j^i r^{\frac{j}{2}},\ i = 1,2$, where $\sigma_j,\ c_j^i$ are some constants. Substituting them into the initial equation, we can find $g_i(r) = C_i$ and, thus, the asymptotics of the equation solution will be as follows:

$$u(r) \approx C_1 e^{\frac{1}{\sqrt{r}}} + C_2 e^{-\frac{1}{\sqrt{r}}}.$$

5. Example 2

As an example of application of Formula (5), let us construct the asymptotics of the solution to the fourth-order equation in the case where the main symbol is homogeneous, i.e., the multiplicity of its root is four.

First, let us consider the case where the singularity index is three:

$$\begin{aligned}&\left(-r^2\tfrac{d}{dr}\right)^4 u + c_1 r\left(-r^2\tfrac{d}{dr}\right)^2 u + c_2 r^2\left(-r^2\tfrac{d}{dr}\right)u + c_3 r^3 u + c_4 r^2\left(-r^2\tfrac{d}{dr}\right)^2 u + \\ &+ c_5 r\left(-r^2\tfrac{d}{dr}\right)^3 u + r^3 H\left(r,\left(-r^2\tfrac{d}{dr}\right)\right)u = 0.\end{aligned} \tag{10}$$

Here, c_i are some constants, $i = 1,\ldots,5$; the condition $c_1 \neq 0$ is satisfied.

$$H\left(r,\left(-r^2\frac{d}{dr}\right)\right) = a_1\left(-r^2\frac{d}{dr}\right) + a_2\left(-r^2\frac{d}{dr}\right)^2 + a_3\left(-r^2\frac{d}{dr}\right)^3 + r\sum_{i=0}^{3} b_i(r)\left(-r^2\frac{d}{dr}\right)^i. \tag{11}$$

Here, a_i are some constants, $i = 1,2,3$; $b_i(r)$ are holomorphic functions. Performing the Laplace–Borel transform of Equation (10), we get the equation:

$$\begin{aligned}&p^4\widetilde{u}(p) + c_1(-1)\int p^2\widetilde{u}(p)dp + c_2(-1)^2\int_1^p\int_1^{p_2} p_1\widetilde{u}(p_1)dp_1dp_2 + c_4(-1)^2\int_1^p\int_1^{p_2} {p_1}^2\widetilde{u}(p_1)dp_1dp_2 \\ &+ c_5(-1)\int p^3\widetilde{u}(p)dp + +(-1)^3\int_1^p\int_1^{p_3}\int_1^{p_2} BH\left(r,\left(r^2\tfrac{d}{dr}\right)\right)u(r)dp_1dp_2dp_3 = f(p).\end{aligned} \tag{12}$$

Here, $f(p)$ is an arbitrary holomorphic function. Differentiating this equation twice, we get:

$$\begin{aligned}&\left(-p^2\tfrac{d}{dp}\right)^2\widetilde{u}(p) + c_1\left(-p^2\tfrac{d}{dp}\right)\widetilde{u}(p) + c_2^1 p\widetilde{u}(p) + c_3^1 p^2\widetilde{u}(p) + \\ &+ c_4^1 p\left(-p^2\tfrac{d}{dp}\right)\widetilde{u}(p) - \int BH\left(r,\left(r^2\tfrac{d}{dr}\right)\right)u(r)dp = \left(\tfrac{d}{dp}\right)^2 f(p).\end{aligned} \tag{13}$$

Here, c_2^i are the corresponding constants, $i = 1,2,3$. Let us apply the re-quantization method. The main symbol has two first-order roots: $q = 0,\ q = -c_1$. It is easy to show that the term $\int BH\left(r,\left(r^2\frac{d}{dr}\right)\right)udp$ gives the lowest terms of the asymptotics of the function $\widetilde{u}(p)$ in the neighborhood of zero. Performing the Laplace–Borel transform of Equation (13), we obtain the equation:

$$\begin{aligned}&q(q+c_1)\hat{u}(q) - \int(c_2^1 + c_4^1 q)\hat{u}(q)dq + c_3^1\int_1^q\int_1^{q_2}\hat{u}(q_1)dq_1dq_2 - \\ &-B\int BH\left(r,\left(r^2\tfrac{d}{dr}\right)\right)u(r)dp = B\left(\tfrac{d}{dp}\right)^2 f(p).\end{aligned}$$

Hence, we get from this equation:

$$\begin{aligned}\hat{u}(q) = \tfrac{1}{q(q+c_1)}\int(c_2^1 + c_4^1 q)\hat{u}(q)dq - \tfrac{c_3^1}{q(q+c_1)}\int_1^q\int_1^{q_2}\hat{u}(q_1)dq_1dq_2 - \\ -\tfrac{1}{q(q+c_1)}B\int BH\Big(r,\Big(r^2\tfrac{d}{dr}\Big)\Big)u(r)dp + \tfrac{1}{q(q+c_1)}B\Big(\tfrac{d}{dp}\Big)^2 f(p).\end{aligned} \tag{14}$$

It should be noted that we can choose the constant σ_1 so that making the substitution $u(r) = r^{\sigma_1}u_1(r)$ in Equation (10), we can get a coefficient c_2^1 such that the root of the polynomial $c_2^1 + c_4^1 q$ would be equal to $-c_1$. As was shown in [13], the equality:

$$B\left(\int_{q_0}^{p}\widetilde{u}(p)dp\right) = -\frac{1}{q}\int\int_{q_0}^{q}\hat{u}(q_1)dq_1dq \tag{15}$$

holds.

It is easy to show that $B\int\Big(BH\Big(r,\Big(r^2\frac{d}{dr}\Big)\Big)u(r)\Big)dp$ are the lowest terms of the asymptotics of the function $\hat{u}(q)$ in the vicinity of points $q = 0,\ q = c_1$. This can be done by applying the method of successive approximations to Equation (12) with Formulas (11) and (15) taken into account in the same way as was done in [18]. Therefore, the asymptotics of the solution to Equation (13) in the vicinity of zero has the form:

$$\widetilde{u}(p) \approx e^{-\frac{c_1}{p}}p^{\sigma_1}\sum_{i=0}^{\infty}a_i^1p^i + p^{\sigma_2}\sum_{i=0}^{\infty}a_ip^i.$$

Let us perform the inverse Laplace–Borel transform. Using Formula (5), we derive:

$$B^{-1}\left(e^{-\frac{c_1}{p}}p^{\sigma_1}\sum_{i=0}^{\infty}a_i^1p^i + p^{\sigma_2}\sum_{i=0}^{\infty}a_ip^i\right) = e^{\frac{\alpha_1}{\sqrt{r}}}r^{\lambda_1}\sum_{l=0}^{\infty}c_l^1r^{\frac{l}{2}} + e^{\frac{\alpha_2}{\sqrt{r}}}r^{\lambda_2}\sum_{l=0}^{\infty}c_l^2r^{\frac{l}{2}} + r^{\lambda_3}\sum_{l=0}^{\infty}c_l^3r.$$

Here, $\alpha_i,\ i = 1,2$ are the roots of the polynomial $p^2 + 4c_1$.

Let $c_1 = 0$, then Equation (10) takes the form:

$$\begin{aligned}\Big(-r^2\tfrac{d}{dr}\Big)^4u + c_2r^2\Big(-r^2\tfrac{d}{dr}\Big)u + c_3r^3u + c_4r^2\Big(-r^2\tfrac{d}{dr}\Big)^2u + \\ +c_5r\Big(-r^2\tfrac{d}{dr}\Big)^3u + r^3H\Big(r,\Big(-r^2\tfrac{d}{dr}\Big)\Big)u = 0.\end{aligned} \tag{16}$$

Performing the Borel transform in (16), we obtain:

$$\begin{aligned}\Big(-p^{\frac{3}{2}}\tfrac{d}{dp}\Big)^3\widetilde{u}(p) + c_2\Big(-p^{\frac{3}{2}}\tfrac{d}{dp}\Big)\widetilde{u}(p) + c_3^1p^{\frac{1}{2}}\widetilde{u}(p) + c_4^1p\Big(-p^{\frac{3}{2}}\tfrac{d}{dp}\Big)\widetilde{u}(p) + \\ +c_5^1p^{\frac{1}{2}}\Big(-p^{\frac{3}{2}}\tfrac{d}{dp}\Big)^2\widetilde{u}(p) + BH\Big(r,\Big(-r^2\tfrac{d}{dr}\Big)\Big)u(r) = f(p).\end{aligned} \tag{17}$$

Let us apply the re-quantization method. Performing the Borel transform one more time, we get:

$$\Big(q^2 + c_2\Big)q\hat{u}(q) + \int(c_3^1 + c_5^1q^2)\hat{u}(q)dq + c_4\int\int_{q_0}^{q}q\hat{u}(q_1)dq_1 + B\left[BH\Big(r,\Big(-r^2\frac{d}{dr}\Big)\Big)u(r)\right] = Bf(p).$$

Here, the polynomial $\big(q^2 + c_2\big)$ has two roots $\beta_1 = i\sqrt{c_2},\ \beta_2 = -i\sqrt{c_2}$ and a root $\alpha_0 = 0$; then, the asymptotics of the solution to Equation (17) has the form:

$$\widetilde{u}(p) \approx e^{\frac{\beta_1}{p}}p^{\sigma_1}\sum_{i=0}^{\infty}a_i^1p^i + e^{\frac{\beta_2}{p}}p^{\sigma_2}\sum_{i=0}^{\infty}a_i^2p^i + p^{\sigma_3}\sum_{i=0}^{\infty}a_ip^i.$$

Using Formula (5), we obtain the asymptotics of the solution to Equation (16):

$$B^{-1}\left(e^{\frac{\beta_1}{p}}p^{\sigma_1}\sum_{i=0}^{\infty}a_i^1p^i+e^{\frac{\beta_2}{p}}p^{\sigma_2}\sum_{i=0}^{\infty}a_i^2p^i+p^{\sigma_2}\sum_{i=0}^{\infty}a_ip^i\right)=\sum_{i=1}^{4}e^{\frac{\alpha_i}{\sqrt{r}}}r^{\lambda_i}\sum_{l=0}^{\infty}c_l^ir^{\frac{l}{2}}+r^{\lambda_0}\sum_{l=0}^{\infty}c_l^3.$$

Here, $\alpha_1=2\sqrt{\beta_1}$, $\alpha_2=-2\sqrt{\beta_1}$, $\alpha_3=2\sqrt{\beta_2}$, $\alpha_4=-2\sqrt{\beta_2}$.
Let $c_1=c_2=0$, then:

$$\left(-r^2\frac{d}{dr}\right)^4u+c_3r^3u+c_4r^2\left(-r^2\frac{d}{dr}\right)^2u+c_5r\left(-r^2\frac{d}{dr}\right)^3u+r^3H\left(r,\left(-r^2\frac{d}{dr}\right)\right)u=0. \quad (18)$$

Dividing Equation (18) by r^3, we transform it as follows:

$$\left(-r^{\frac{5}{4}}\frac{d}{dr}\right)^4u+c_3u+c_4r^{\frac{2}{4}}\left(-r^{\frac{5}{4}}\frac{d}{dr}\right)^2u+c_5r^{\frac{1}{4}}\left(-r^{\frac{5}{4}}\frac{d}{dr}\right)^3u+H\left(r,\left(-r^2\frac{d}{dr}\right)\right)u=0.$$

The main symbol of this equation is q^4+c_3. It has simple roots. In this case, the application of the method of successive approximations is not required; the asymptotics of the solutions to Equation (18) for this case was constructed in [15]:

$$u(r)\approx\sum_{j=1}^{4}e^{\frac{\alpha_j}{r^{\frac{1}{4}}}}r^{\sigma_j}\sum_{i=0}^{\infty}a_i^jr^i.$$

Similar to the case where the singularity index is three, the asymptotics are constructed for the case where the singularity index is two. Namely, in the case where the singularity index is two, Equation (10) takes the form:

$$\begin{gathered}\left(-r^2\tfrac{d}{dr}\right)^4u+a_0r\left(-r^2\tfrac{d}{dr}\right)u+c_0r^2u+c_1r\left(-r^2\tfrac{d}{dr}\right)^2u+c_2r^2\left(-r^2\tfrac{d}{dr}\right)u+\\+c_3r^3u+c_4r^2\left(-r^2\tfrac{d}{dr}\right)^2u+c_5r\left(-r^2\tfrac{d}{dr}\right)^3u+r^3H\left(r,\left(-r^2\tfrac{d}{dr}\right)\right)u=0.\end{gathered}$$

Let $a_0\neq0$, then, performing the Laplace–Borel transform, as was done above, we obtain:

$$\begin{gathered}\left(-p^3\tfrac{d}{dp}\right)^2\widetilde{u}+a_0\left(-p^3\tfrac{d}{dp}\right)\widetilde{u}+c_0p^2\widetilde{u}+c_1p\left(-p^3\tfrac{d}{dp}\right)\widetilde{u}+c_2p^3\widetilde{u}+{p_1}^4\widetilde{u}+\\+p^2\left(p^3\tfrac{d}{dp}\right)\widetilde{u}+(-1)^3p^2\int_1^p BH\left(r,\left(r^2\tfrac{d}{dr}\right)\right)u(r)dp=f(p).\end{gathered} \quad (19)$$

The main symbol of the last operator in the left-hand part of Equation (19) has simple roots: $q=0$, $q=-a_0$. Taking Formula (5) into account, we get that the solution to Equation (18) can be represented in the form of the WKB-asymptotics.

Let $a_0=0$, then:

$$\begin{gathered}\left(-r^2\tfrac{d}{dr}\right)^4u+c_0r^2+c_1r\left(-r^2\tfrac{d}{dr}\right)^2u+c_2r^2\left(-r^2\tfrac{d}{dr}\right)u+c_3r^3u+c_4r^2\left(-r^2\tfrac{d}{dr}\right)^2u+\\+c_5r\left(-r^2\tfrac{d}{dr}\right)^3u+r^3H\left(r,\left(-r^2\tfrac{d}{dr}\right)\right)u=0.\end{gathered}$$

Dividing this equation by r^2, we reduce it to the equation with simple roots; the asymptotics of solutions to this equation are constructed as was done for Equation (18). The same algorithm can be used for the case where the singularity index is one.

For the case where the singularity index is greater than or equal to four, dividing the equation by r^4, we can transform it to an equation with a conic degeneracy in its coefficients. In this case, the asymptotics of the solution are conormal.

In this case, dividing this equation by r^4, we obtain the equation of the form:

$$\left(-r\frac{d}{dr}\right)^4 u + c_1 r\left(-r^2\frac{d}{dr}\right)^3 u + c_2 r^2\left(-r^2\frac{d}{dr}\right)^2 u + c_3 r^3\left(-r^2\frac{d}{dr}\right) u + r^4 H\left(r, \left(-r^2\frac{d}{dr}\right)\right) u = 0.$$

Note that the main symbol of the differential operator has one root. Let us assume the opposite. Let the main symbol have two roots, i.e.,

$$H_0(p) = p^{i_1}(p-b)^{i_2},$$

then:

$$\left(-r^2\frac{d}{dr}\right)^{i_1}\left(\left(-r^2\frac{d}{dr}\right)-b\right)^{i_2} u + r\sum_{i=0}^{2} c_i^1\left(-r^2\frac{d}{dr}\right)^i u + r^2\sum_{i=0}^{1} c_i^2\left(-r^2\frac{d}{dr}\right)^i u + c_0^3 r^3 u +$$
$$+ r^3 H\left(r, \left(-r^2\frac{d}{dr}\right)\right) u = 0,$$

where $i_1 + i_2 = 4$. Performing the Laplace–Borel transform, we get the equation:

$$\begin{aligned} &p^{i_1}(p-b)^{i_2}\widetilde{u}(p) + \int \sum_{i=0}^{2} c_i^1 p^i \widetilde{u}(p)dp + \int_1^p\int_1^{p_1}\sum_{i=0}^{1} c_i^2 {p_1}^2\widetilde{u}(p_1)dp_1dp + \\ &+ c_0^3\int\int_1^p\int_1^{p_1}\widetilde{u}(p_2)dp_2dp_1dp + c_0^3\int\int_1^p\int_1^{p_1} BH\left(r, \left(-r^2\frac{d}{dr}\right)\right) udp_2dp_1dp = f(p). \end{aligned} \tag{20}$$

Let us construct the asymptotics of the solution to the transformed equation at the point $p = 0$. If the singularity index is greater than or equal to i_2, then the asymptotics of the solution will be conormal; if the singularity index is smaller than i_2, then we rewrite Equation (20) in the form:

$$\begin{aligned} &p^{i_1}\widetilde{u}(p) + \frac{1}{(p+b)^{i_2}}\int \sum_{i=0}^{2} c_i^1 p^i \widetilde{u}dp + \frac{1}{(p+b)^{i_2}}\int_1^p\int_1^{p_2}\sum_{i=0}^{1} c_i^2 p^2\widetilde{u}dp_1dp_2 + \\ &+ \frac{c_0^3}{(p+b)^{i_2}}\int\int_1^p\int_1^{p_1}\widetilde{u}(p_2)dp_2dp_1dp + \frac{c_0^3}{(p+b)^{i_2}}\int\int_1^p\int_1^{p_1} BH\left(r, \left(-r^2\frac{d}{dr}\right)\right) udp = \frac{f(p)}{(p+b)^{i_2}}. \end{aligned} \tag{21}$$

Since the functions $\frac{1}{(p+b)^{i_2}}$ do not have singularities at zero, the asymptotics of the solution to Equation (21) in the vicinity of the point $p = 0$ can be obtained by using the method of successive approximations, as was done for Equation (12). To find the asymptotics corresponding to the root $-b$, it is necessary to shift this root to zero by performing the substitution $u(r) = e^{-\frac{b}{r}}u_1(r)$ and then find the asymptotics in the neighborhood of zero, as was done above.

6. Conclusions

In this study, we have developed the re-quantization method. Using this method, we have obtained some important results for constructing uniform asymptotics for some types of equations with irregular singularities, for example, the asymptotics of the solutions to linear differential equations with holomorphic coefficients in the neighborhood of infinity. To solve this problem, it was necessary to solve the problem of constructing the asymptotics for the inverse Laplace–Borel transform of a particular type of functions that exponentially grow at zero.

This paper provides the solution of this problem. In addition, we have considered two examples of constructing uniform asymptotics at infinity for differential equations of the second and fourth order by using the re-quantization method. Developing this method further, we expect to solve, for example, the classical problem of constructing uniform asymptotics in the neighborhood of infinity for the solutions to linear differential equations of any type with holomorphic coefficients in the general case, including the systems of non-Fuchsian type. This method can also be applied to partial differential equations with separable variables to construct asymptotics of solutions to such equations as, for example, the Kadomtsev–Petviashvili equation that describes nonlinear waves in two-dimensional media with weak dispersion or to the biharmonic equation.

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