Article

Uniqueness of Closed Equilibrium Hypersurfaces for Anisotropic Surface Energy and Application to a Capillary Problem

Miyuki Koiso

Institute of Mathematics for Industry, Kyushu University, Motooka Nishi-ku, Fukuoka 819-0395, Japan; koiso@math.kyushu-u.ac.jp

check for updates

Abstract: We study a variational problem for hypersurfaces in the Euclidean space with an anisotropic surface energy. An anisotropic surface energy is the integral of an energy density that depends on the surface normal over the considered hypersurface, which was introduced to model the surface tension of a small crystal. The purpose of this paper is two-fold. First, we give uniqueness and nonuniqueness results for closed equilibria under weaker assumptions on the regularity of both considered hypersurfaces and the anisotropic surface energy density than previous works and apply the results to the anisotropic mean curvature flow. This part is an announcement of two forthcoming papers by the author. Second, we give a new uniqueness result for stable anisotropic capillary surfaces in a wedge in the three-dimensional Euclidean space.

Keywords: anisotropic mean curvature; anisotropic surface energy; Wulff shape; anisotropic mean curvature flow; crystalline variational problem; capillary problem

1. Introduction

We study equilibrium hypersurfaces in the $(n + 1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$ for anisotropic surface energy, which serve as a mathematical model of small crystals and small liquid crystals with anisotropy. Therefore, the objects of our study should be not only smooth hypersurfaces but also hypersurfaces with singular points like vertices and edges. However, such equilibrium hypersurfaces with both smooth curved parts and singular points have not yet been studied sufficiently well. One of the purposes of this paper is to give a new concept a piecewise-C^2 weak immersion and to study variational problems in this class of hypersurfaces. Piecewise-C^2 weak immersions are hypersurfaces with the weakest regularity in order to study this type of variational problem by using essentially classical differential geometry. The other purpose of this paper is to give a new uniqueness result for stable anisotropic capillary surfaces in a wedge in $\mathbb{R}^3$, which is proved by using the method developed to pursue the first purpose together with careful treatment of the boundary part by using several new formulas.

Let $\gamma : S^n \to \mathbb{R}_{>0}$ be a positive continuous function on the unit sphere $S^n = \{\nu \in \mathbb{R}^{n+1} ; \|\nu\| = 1\}$ in $\mathbb{R}^{n+1}$. Let X be a closed hypersurface in $\mathbb{R}^{n+1}$ for which the tangent hyperplane is well defined at almost every point. X will be represented as a mapping $X : M \to \mathbb{R}^{n+1}$ from an n-dimensional oriented connected compact C^∞ manifold M into $\mathbb{R}^{n+1}$. Let ν be the unit normal vector field along $X|_{M \setminus S[X]}$, where $S[X]$ is the set of singular points of X. The anisotropic energy $\mathcal{F}_\gamma(X)$ of X is defined as $\mathcal{F}_\gamma(X) := \int_{M \setminus S[X]} \gamma(\nu) \, dA$, where dA is the n-dimensional volume form of M induced by X. Such an energy was introduced by Gibbs (1839–1903) in order to model the shape of small crystals, and it is used as a mathematical model of anisotropic surface energy [1,2]. It is known that, for any positive number $V > 0$, among all closed hypersurfaces as above enclosing the same $(n + 1)$-dimensional

volume V, there exists a unique (up to translation in $\mathbb{R}^{n+1}$) minimizer $W_\gamma(V)$ of $\mathcal{F}_\gamma$ [3]. The minimizer $W_\gamma(V_0)$ for the specific value $V_0 := (n+1)^{-1} \int_{S^n} \gamma(\nu) \, dS^n$ is called the Wulff shape for γ, and we will denote it by W_γ. When $\gamma \equiv 1$, $\mathcal{F}_\gamma(X)$ is the usual n-dimensional volume of the hypersurface X and W_γ is the unit sphere S^n.

In this paper, we study variational problems for the anisotropic energy $\mathcal{F}_\gamma$. In general, the Wulff shape and equilibrium hypersurfaces of such an energy for volume-preserving variations are not smooth. In view of the origin of the energy functional mentioned above, it is natural to try to study the problem under the lowest possible regularities of the considered hypersurfaces and the energy density function. In order to pursue it, we define a new concept "piecewise-C^2 weakly immersed hypersurface" and study the variational problem of the energy $\mathcal{F}_\gamma$ for C^2 functions γ on S^n. These assumptions on the regularity are weaker than any previous works that studied variational problems of anisotropic surface energies in differential geometry (cf. References [4–9]). Under such a weak regularity assumption, we concentrate upon the problem of uniqueness for closed equilibrium hypersurfaces. In order to give precise statements of our results, we prepare a few words.

For any given convex set $\tilde{W}$ having the origin of $\mathbb{R}^{n+1}$ inside, there exists a Lipschitz continuous function $\gamma : S^n \to \mathbb{R}_{>0}$ such that the boundary $W := \partial \tilde{W}$ of $\tilde{W}$ coincides with the Wulff shape W_γ for γ. However, such γ is not unique. The "smallest" γ is called the convex integrand for W (or, simply, convex) (for another equivalent definition, see Section 2).

Each equilibrium hypersurface X of $\mathcal{F}_\gamma$ for variations that preserve the enclosed $(n+1)$-dimensional volume (we will call such a variation a volume-preserving variation) has constant anisotropic mean curvature. Here, the anisotropic mean curvature Λ of a piecewise-C^2 hypersurface X is defined at each regular point of X as (cf. Reference [6,10]) $\Lambda := (1/n)(-\mathrm{div}_M D\gamma + nH\gamma)$, where $D\gamma$ is the gradient of γ and H is the mean curvature of X. If $\gamma \equiv 1$, $\Lambda = H$ holds.

We call a piecewise-C^2 weakly immersed equilibrium hypersurface X a CAMC (constant anisotropic mean curvature) hypersurface (see Definition 3 for details). A CAMC hypersurface is said to be stable if the second variation of the energy $\mathcal{F}_\gamma$ for any volume-preserving variation is nonnegative.

One of the most basic questions is as follows:

Question 1. *Is any closed CAMC hypersurface the Wulff shape?*

The answer to this uniqueness problem is not affirmative in general [11–13]. However, it is expected that, if one of the following "good" conditions (I)–(III) is satisfied, the image of any closed CAMC hypersurface X coincides with the Wulff shape (up to translation and homothety).

(I)　　X is an embedding; that is, X is an injective mapping.
(II)　　X is stable.
(III)　$n = 2$ and the genus of M is 0; that is, M is homeomorphic to S^2.

If we assume that W_γ is a smooth, strictly convex hypersurface (that is, the support function γ of W_γ is strictly convex. See Definition 1.), any closed CAMC hypersurface X is also smooth and the above expectation was already proved. In fact, if X satisfies one of (I)–(III), it is a homothety of the Wulff shape (which was proved by the following papers: For (I), Reference [14] for $\gamma \equiv 1$ and Reference [5] for general γ; for (II), Reference [15] for $\gamma \equiv 1$ and Reference [8] for general γ; and for (III), Reference [16] for $\gamma \equiv 1$ and Reference [7] for general γ). However, the situation is not the same for more general γ and/or W_γ. Actually, if γ is not strictly convex, even if it is convex (See Definition 1) and of class C^∞, the Wulff shape can have singular points and its principal curvatures can be unbounded (Example 3). Also, we have the following striking nonuniqueness results.

Theorem 1 ([17]). *There exists a C^∞ function $\gamma : S^n \to \mathbb{R}_{>0}$ such that there exist closed embedded CAMC hypersurfaces in $\mathbb{R}^{n+1}$ for γ, each of which is not (any homothety or translation of) the Wulff shape W_γ.*

Theorem 2 ([17])**.** *There exists a C^∞ function $\gamma : S^2 \to \mathbb{R}_{>0}$ such that there exist closed embedded CAMC surfaces in $\mathbb{R}^3$ with genus zero for γ, each of which is not (any homothety or translation of) the Wulff shape W_γ.*

Theorems 1 and 2 are proved by giving suitable examples (Section 4). The same examples give self-similar shrinking solutions with genus 0 for anisotropic mean curvature flow, which are not (any homotheties or translations of) the Wulff shape (Section 6). In contrast with this, the round sphere is the only closed embedded self-similar shrinking solution of the mean curvature flow in $\mathbb{R}^3$ with genus zero [18].

As for the uniqueness of stable closed CAMC hypersurfaces, we obtain the following result.

Theorem 3 ([19])**.** *Assume that $\gamma : S^n \to \mathbb{R}_{>0}$ is of class C^2 and convex. Then, the image of any closed stable piecewise-C^2 CAMC hypersurface for γ of which the rth anisotropic mean curvature for γ (see Section 3) is integrable for $r = 1, \ldots, n$ is (up to translation and homothety) a covering of the Wulff shape W_γ.*

Let us mention the preceding works relating to Theorem 3. As for planer curves, Morgan [20] proved that, if $\gamma : S^1 \to \mathbb{R}_{>0}$ is continuous and convex, then any closed equilibrium rectifiable curve for $\mathcal{F}_\gamma$ in $\mathbb{R}^2$ is (up to translation and homothety) a covering of the Wulff shape. About uniqueness of closed stable equilibria in $\mathbb{R}^3$, Palmer [9] proved the same result as Theorem 3 but under the assumptions that γ is of C^3 and that considered surfaces and the Wulff shape satisfy some extra assumptions.

A similar method to prove Theorem 3 together with careful treatment on the free boundary gives a uniqueness result for a capillary problem. For simplicity, we assume that $\gamma : S^2 \to \mathbb{R}_{>0}$ is a strictly convex function of class C^∞. Let Ω be a wedge-shaped domain bounded by two planes Π_1, Π_2 in $\mathbb{R}^3$, and let ω be a positive constant. Let M be a two-dimensional oriented connected compact C^∞ manifold with boundary $\partial M = \sigma_1 \cup \sigma_2$, where each σ_j is a topological circle. Consider any C^∞-immersion $X : (M, \sigma_1, \sigma_2) \to (\overline{\Omega}, \Pi_1, \Pi_2)$ of which the restriction $X|_{\partial M}$ to ∂M is an embedding. Set $C_j = X(\sigma_j)$, and let $D_j \subset \Pi_j$ be the domain bounded by C_j ($j = 1, 2$). We define the wetting energy $\mathcal{W}(X)$ of X as follows:

$$\mathcal{W}(X) = \omega(\mathcal{H}^2(D_1) + \mathcal{H}^2(D_2)),$$

where $\mathcal{H}^2(D_i)$ is the area of D_i. Then, we define the total energy $E(X)$ of X by

$$E(X) = \mathcal{F}_\gamma(X) + \mathcal{W}(X).$$

Note that $X(M) \cup D_1 \cup D_2$ is a piecewise smooth surface without boundary. We denote by $V(X)$ the oriented volume enclosed by $X(M) \cup D_1 \cup D_2$. We call a critical point of E for volume-preserving variations an anisotropic capillary surface (or, simply, a capillary surface). A capillary surface is said to be stable if the second variation of E is nonnegative for all volume-preserving variations of X. In the following uniqueness theorem, we identify M with $X(M)$.

Theorem 4. *Let Ω be a wedge in $\mathbb{R}^3$ bounded by two planes Π_1 and Π_2, and let $M \subset \overline{\Omega}$ be a compact oriented immersed surface that is disjoint from the edge $\Pi_1 \cap \Pi_2$ of Ω, having embedded boundary $\partial M \subset \Pi_1 \cup \Pi_2$ and satisfying $\partial M \cap \Pi_j = \partial D_j$ for a nonempty bounded domain D_j in Π_j. If M is a stable anisotropic capillary surface in $\overline{\Omega}$ and both D_1 and D_2 are convex, then M is (up to translation and homothety) part of the Wulff shape W_γ. Conversely, if M is part of W_γ (up to translation and homothety), then it is stable.*

As for previous works which are closely related to Theorem 4, we have the following. We studied the existence and uniqueness of stable anisotropic capillary surfaces between two parallel planes Π_1 and Π_2 in $\mathbb{R}^3$ [21–23]. Moreover, in Reference [24], we proved the uniqueness result similar to Theorem 4 for isotropic capillary hypersurfaces in a wedge in $\mathbb{R}^{n+1}$; here, isotropic means that $\gamma \equiv 1$.

This article is organized as follows. In Section 2, we give the definition and a representation of the Wulff shape and some fundamental concepts relating to the Wulff shape. Also some typical examples

are given. In Section 3, we give the Euler–Lagrange equations for our variational problem and the definitions of anisotropic curvatures. In Section 4, we give outlines of the proofs of Theorems 1 and 2. In Section 5, an outline of the proof of Theorem 3 is given. In Section 6, we mention an application to anisotropic mean curvature flow. In Section 7, we give the proof of Theorem 4.

2. Preliminaries

Let $\gamma : S^n \to \mathbb{R}_{>0}$ be a positive continuous function. In this paper, we call the boundary W_γ of the convex set $\tilde{W}[\gamma] := \cap_{\nu \in S^n} \{ X \in \mathbb{R}^{n+1} \; ; \; \langle X, \nu \rangle \leq \gamma(\nu) \}$ the Wulff shape for γ, where $\langle \, , \, \rangle$ means the standard inner product in $\mathbb{R}^{n+1}$. In other literatures, $\tilde{W}[\gamma]$ is often called the Wulff shape.

From now on, any parallel translation of the Wulff shape W_γ will be also called the Wulff shape, and it will be denoted also by W_γ, if it does not cause any confusion.

Let us define two terminologies which represent the convexity of the energy density function γ.

Definition 1. *(i) (cf. References [3,25]) A continuous map $\gamma : S^n \to \mathbb{R}_{>0}$ is called a convex integrand (or, simply, convex) if its homogeneous extension $\overline{\gamma} : \mathbb{R}^{n+1} \to \mathbb{R}_{\geq 0}$ defined by*

$$\overline{\gamma}(rX) := r\gamma(X), \; \forall X \in S^n, \; \forall r \geq 0, \tag{1}$$

is a convex function.

(ii) If $\gamma : S^n \to \mathbb{R}_{>0}$ is of class C^2, we say that γ is strictly convex if the $n \times n$ matrix $D^2\gamma + \gamma \cdot I_n$ is positive definite at any point in S^n; here, $D^2\gamma$ is the Hessian of γ on S^n and I_n is the identity matrix of size n.

The Wulff shape W_γ is not smooth, in general. It is smooth and strictly convex (that is, each principal curvature of W_γ with respect to the inward-pointing normal is positive at each point of W_γ) if and only if γ is of class C^2 and strictly convex. On the other hand, if γ is of class C^2, γ is convex if and only if $D^2\gamma + \gamma \cdot I_n$ is positive semi-definite. For such γ, the Wuff shape can have singular points as Example 3 shows.

Assume that $\gamma : S^n \to \mathbb{R}_{>0}$ is of class C^2. The Cahn–Hoffman map $\xi_\gamma : S^n \to \mathbb{R}^{n+1}$ for γ is defined as $\xi_\gamma(\nu) = D\gamma|_\nu + \gamma(\nu)\nu$, $(\nu \in S^n)$. Here, the tangent space $T_\nu(S^n)$ of S^n at ν is naturally identified with a hyperplane in $\mathbb{R}^{n+1}$. If γ is convex, the image $\xi_\gamma(S^n)$ coincides with W_γ; that is, ξ_γ gives a representation of W_γ.

Here, we give four typical examples. We denote a point in S^n by $\nu = (\nu_1, \ldots, \nu_{n+1})$.

Example 1. *Define a function $\gamma : S^n \to \mathbb{R}_{>0}$ as $\gamma = 1$. Then, γ is convex and $\xi_\gamma(\nu) = \nu$ ($\forall \nu \in S^n$). Hence, $W_\gamma = \xi_\gamma(S^n) = S^n$.*

Example 2. *Define a function $\gamma : S^n \to \mathbb{R}_{>0}$ as $\gamma(\nu) = \sum_{i=1}^{n+1} |\nu_i|$. Then, $\gamma \in C^0$ and convex and the Wulff shape W_γ is the cube $\{ x = (x_1, \ldots, x_{n+1}) \in \mathbf{R}^{n+1} \; ; \; \max\{|x_1|, \ldots, |x_{n+1}|\} = 1 \}$.*

Since γ is differentiable only at points except points in $S[\gamma] := \{ \nu \in S^n \; ; \; \nu_i = \pm 1, \exists i \in \{1, \ldots, n+1\} \}$, the Cahn–Hoffman map ξ_γ is defined on $S^n \setminus S[\gamma]$ and the image $\xi(S^n \setminus S[\gamma])$ of $\xi : S^n \setminus S[\gamma] \to \mathbb{R}^{n+1}$ is the set of all vertices of the cube W_γ.

Example 3 ([19])**.** *We give a simple example, which shows that, if the energy density function γ is not strictly convex, even if it is convex, the Wulff shape can have singular points and its principal curvatures can be unbounded. Set $n = 1$. For $m \in \mathbb{N}$, define*

$$\gamma_m(\nu) := (\nu_1^{2m} + \nu_2^{2m})^{1/(2m)}. \tag{2}$$

Then, γ_m is of class C^∞ and convex. Hence, the Wulff shape and the image $\xi_m(S^1)$ of the Cahn–Hoffman map ξ_m for γ_m coincide, and they are shown in Figure 1. ξ_m is represented as

$$\xi_m(\cos\theta, \sin\theta) = (\cos^{2m}\theta + \sin^{2m}\theta)^{(1/(2m))-1}(\cos^{2m-1}\theta, \sin^{2m-1}\theta).$$

Moreover, we have

$$
\begin{aligned}
A_m & := D^2\gamma_m + \gamma_m \cdot 1 \\
& = (2m-1)\cos^{2m-2}\theta\sin^{2m-2}\theta(\cos^{2m}\theta + \sin^{2m}\theta)^{(1/(2m))-2}.
\end{aligned}
$$

Hence, the following can be stated.
  (i) If $m \geq 2$, $A_m = 0$ on $S[\gamma] := \{(\cos\theta, \sin\theta)\,;\, \theta = (1/2)\ell\pi, (\ell \in \mathbb{Z})\}$.
  (ii) A_m is positive definite on $S^1 \setminus S[\gamma]$ and positive semi-definite on S^1.
The curvature κ_m of ξ_m with respect to the outward-pointing normal ν is represented as

$$\kappa_m(\theta) = \frac{-1}{2m-1}\cos^{-2m+2}\theta\sin^{-2m+2}\theta(\cos^{2m}\theta + \sin^{2m}\theta)^{2-\frac{1}{2m}}.$$

Hence, near each point in $S[\gamma]$, κ_m is unbounded and the following holds.

$$\lim_{\theta\to\frac{\ell}{2}\pi}\kappa_m(\theta) = -\infty, \quad \ell \in \mathbb{Z}, \; m \geq 2.$$
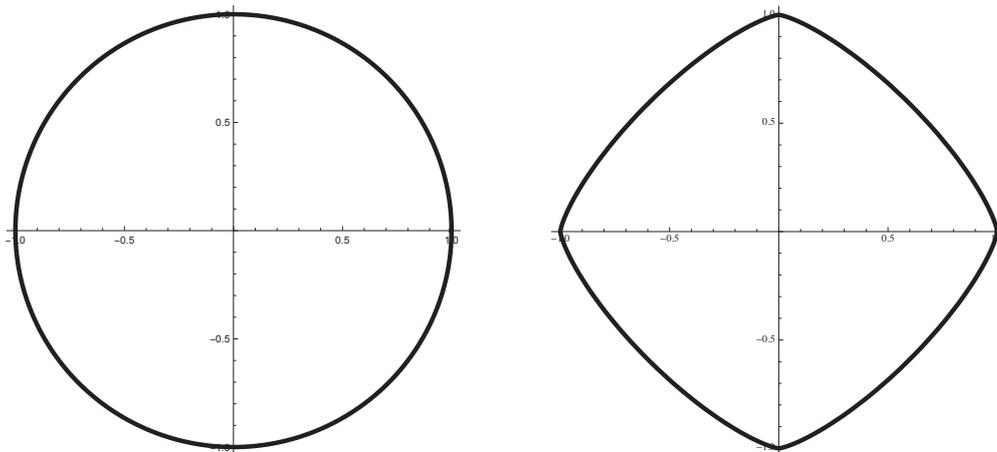

Figure 1. On the left, W_{γ_m} for $m = 1$ and, on the right, W_{γ_m} for $m = 2$ for γ_m defined by Equation (2).

Example 4. *We give a nonconvex example of γ. Set $n = 1$. Define γ as $\gamma(\nu) = 4\nu_1^3 - 3\nu_1 + 2$. Then, γ is of class C^∞ and it is not convex. The whole of the closed curve with self-intersection in Figure 2 is the image $\xi_\gamma(S^1)$ of the Cahn–Hoffman map ξ_γ, while the closed convex solid curve that is a proper subset of $\xi_\gamma(S^1)$ is the Wulff shape W_γ. Denote by κ_γ the curvature of ξ_γ at its regular points with respect to the normal ν. By computation, we obtain*

$$\xi_\gamma(\cos\theta, \sin\theta) = (8\cos^2\theta\sin^2\theta + 4\cos^2\theta + 2\cos\theta - 3, -8\cos^3\theta\sin\theta + 2\sin\theta),$$
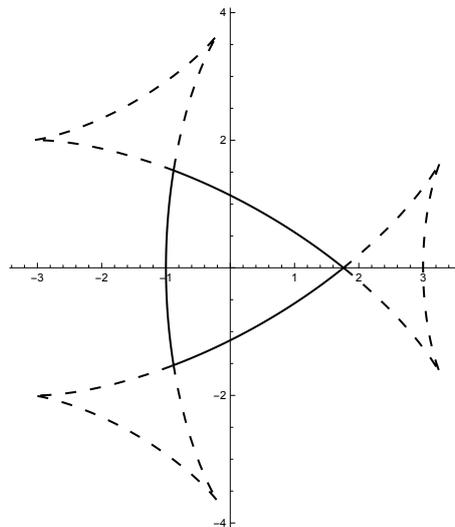
$$d\xi_\gamma = 2(-16\cos^3\theta + 12\cos\theta + 1)(-\sin\theta, \cos\theta) =: a(\theta)(-\sin\theta, \cos\theta),$$

$$\kappa_\gamma(\theta) = \frac{-1}{2|-16\cos^3\theta + 12\cos\theta + 1|} = \frac{-1}{|a(\theta)|}.$$

Hence, for any $\rho \in a^{-1}(0)$,

$$\lim_{\theta(\notin a^{-1}(0))\to\rho}\kappa_\gamma(\theta) = -\infty.$$

$a^{-1}(0) \subset S^1$ *is the set of exactly six points, and it is the set of the singular points of ξ_γ (Figure 2).*

Figure 2. The image of the Cahn–Hoffman map ξ_γ for γ in Example 4: The six vertices are the image of the singular points of ξ_γ, and the closed convex solid curve is the Wulff shape W_γ.

3. Euler–Lagrange Equations and Anisotropic Curvatures

From now on, we assume that $\gamma : S^n \to \mathbb{R}_{>0}$ is of class C^2. Let $M = \cup_{i=1}^{k} M_i$ be an n-dimensional oriented compact connected C^∞ manifold, where each M_i is an n-dimensional connected compact submanifold of M with piecewise-C^∞ boundary and $M_i \cap M_j = \partial M_i \cap \partial M_j$ $(i, j \in \{1, \ldots, k\}, i \neq j)$. We call a map $X : M \to \mathbb{R}^{n+1}$ a piecewise-C^2 weak immersion (or a piecewise-C^2 weakly immersed hypersurface) if X satisfies the following conditions (A1), (A2), and (A3) for $i = 1, \ldots, k$ [19].

(A1) X is continuous, and each $X_i := X|_{M_i} : M_i \to \mathbb{R}^{n+1}$ is of class C^2.

(A2)   The restriction $X|_{M_i^o}$ of X to the interior M_i^o of M_i is a C^2-immersion.

(A3)   The unit normal vector field $\nu_i : M_i^o \to S^n$ along $X_i|_{M_i^o}$ can be extended to a C^1-mapping $\nu_i : M_i \to S^n$. Here, the orientation of ν_i is determined so that, if $(u^1, \ldots, u^n)$ is a local coordinate system in M_i, then $\{\nu_i, \partial/\partial u^1, \ldots, \partial/\partial u^n\}$ gives the canonical orientation in $\mathbb{R}^{n+1}$.

The Cahn–Hoffman field $\tilde{\xi}_i$ along X_i for γ is defined as $\tilde{\xi}_i := \xi_\gamma \circ \nu_i : M_i \to \mathbb{R}^{n+1}$. Note that the Cahn–Hoffman map $\xi_\gamma : S^n \to \mathbb{R}^{n+1}$ is a front ([19]). Hence the "tangent space" $T_q\xi_\gamma(S^n)$ of the image $\xi_\gamma(S^n)$ at each point $q \in \xi_\gamma(S^n)$ can be defined. Since the unit normal $\nu_i(p)$ of X_i at $p \in M_i$ coincides with the unit normal of ξ_γ at the point $\nu_i(p)$, we can identify $T_p M_i$ with $T_{\tilde{\xi}_i(p)}\xi_\gamma(S^n)$.

The linear map $S_p^\gamma : T_p M_i \to T_p M_i$ given by the $n \times n$ matrix $S^\gamma := -d\tilde{\xi}_i$ is called the anisotropic shape operatior of X_i. Various anisotropic curvatures of X are defined as follows.

Definition 2 (anisotropic curvatures; cf. References [5,10]). *(i) The eigenvalues of S^γ are called the anisotropic principal curvatures of X. We denote them by $k_1^\gamma, \ldots, k_n^\gamma$.*

(ii) Let σ_r^γ be the elementary symmetric functions of $k_1^\gamma, \ldots, k_n^\gamma$:

$$\sigma_r^\gamma := \sum_{1 \le l_1 < \cdots < l_r \le n} k_{l_1}^\gamma \ldots k_{l_r}^\gamma, \quad r = 1, \ldots, n. \tag{3}$$

Set $\sigma_0^\gamma := 1$. $H_r^\gamma := \sigma_r^\gamma / {}_nC_r$ is called the rth anisotropic mean curvature of X, where ${}_nC_r = \dfrac{n!}{k!(n-k)!}$.

(iii) H_1^γ is called the anisotropic mean curvature of X, and we often denote it by Λ; that is, $\Lambda = \dfrac{1}{n}\sum_{i=1}^{n} k_i^\gamma$.

The anisotropic curvatures of the most fundamental hypersurfaces W_γ and ξ_γ are simply stated as follows.

Remark 1 ([6,19])**.** *For the Cahn–Hoffman map $\xi_\gamma : S^n \to \mathbb{R}^{n+1}$, $\xi_\gamma(\nu) = D\gamma|_\nu + \gamma(\nu)\nu$, ($\nu \in S^n$), it is shown that the unit normal vector field ν_{ξ_γ} is given by ξ_γ^{-1}. Hence, the anisotropic shape operator of ξ_γ is $S^\gamma = -d(\xi_\gamma \circ \nu_{\xi_\gamma}) = -d(\mathrm{id}_{S^n}) = -I_n$. Therefore, the anisotropic principal curvatures of ξ_γ are -1, and hence, each rth anisotropic mean curvature of ξ_γ is $(-1)^r$. Particularly, the anisotropic mean curvature of ξ_γ for the normal ν and that of W_γ for the outward-pointing unit normal is -1 at any regular point.*

S^γ is not symmetric in general. However, we have the following good properties of the anisotropic curvatures.

Remark 2. *(i) If $d\xi_\gamma = D^2\gamma + \gamma \cdot I_n$ is positive definite at a point $\nu(p)$ ($p \in M_i^o$), then all of the anisotropic principal curvatures of X at p are real [26].*
(ii) k_i^γ is not a real value in general. However, each H_r^γ is always a real valued function on M_i^o [19].

We have the following first variation formula for the anisotropic surface energy $\mathcal{F}_\gamma$.

Proposition 1 ([19])**.** *Assume that the map $X : M_0 \to \mathbb{R}^{n+1}$ satisfies (A1), (A2), and (A3) above with $X_i = X$, $M_i = M_0$, and $\nu_i = \nu$. Let $X_\epsilon : M_0 \to \mathbb{R}^{n+1}$ ($\epsilon \in J := (-\epsilon_0, \epsilon_0)$), be a variation of X; that is, $\epsilon_0 > 0$ and $X_0 = X$. Assume for simplicity that X_ϵ is of class C^∞ in ϵ. We also assume that, for each $\epsilon \in J$, the anisotropic mean curvature Λ_ϵ of X_ϵ ($\epsilon \neq 0$) is bounded on M_0^o. Set*

$$\delta X := \frac{\partial X_\epsilon}{\partial \epsilon}\bigg|_{\epsilon=0}, \quad \psi := \langle \delta X, \nu \rangle.$$

Then, the first variation of the anisotropic energy $\mathcal{F}_\gamma$ is given as follows.

$$\frac{d\mathcal{F}_\gamma(X_\epsilon)}{d\epsilon}\bigg|_{\epsilon=0} = -\int_{M_0} n\Lambda\psi \, dA - \oint_{\partial M_0} \langle \delta X, R(p(\tilde{\xi})) \rangle \, ds, \tag{4}$$

where ds is the $(n-1)$-dimensional volume form of ∂M_0 induced by X, N is the outward-pointing unit conormal along ∂M_0, R is the $\pi/2$-rotation on the (N, ν)-plane, p is the projection from $\mathbb{R}^{n+1}$ to the (N, ν)-plane, and the first integral in the right hand side of Equation (4) which is an improper integral converges.

On the other hand, the first variation of the $(n+1)$-dimensional volume enclosed by X_ϵ is

$$\delta V = \int_{M_0} \psi \, dA \tag{5}$$

(cf. Reference [27]). This with Equation (4) gives the following Euler–Lagrange equations.

Proposition 2 (Euler–Lagrange equations, Koiso [19]. For $n = 2$, see Palmer [9])**.** *A piecewise-C^2 weak immersion $X : M = \cup_{i=1}^k M_i \to \mathbb{R}^{n+1}$ is a critical point of the anisotropic energy $\mathcal{F}_\gamma$ for volume-preserving variations if and only if the following conditions (i) and (ii) hold.*
(i) The anisotropic mean curvature of X is constant on $M \setminus S[X]$.
(ii) $\tilde{\xi}_i(\zeta) - \tilde{\xi}_j(\zeta) \in T_\zeta M_i \cap T_\zeta M_j = T_\zeta(\partial M_i \cap \partial M_j)$ holds at any $\zeta \in \partial M_i \cap \partial M_j$, where a tangent space of a submanifold of $\mathbb{R}^{n+1}$ is naturally identified with a linear subspace of $\mathbb{R}^{n+1}$.

In view of Proposition 2, we will use the following terminology.

Definition 3 ([19])**.** *A piecewise-C^2 weak immersion $X : M = \cup_{i=1}^k M_i \to \mathbb{R}^{n+1}$ is called a hypersurface with constant anisotropic mean curvature (CAMC) if both conditions (i) and (ii) in Proposition 2 hold.*

4. Outline of the Proofs of Theorems 1 and 2

Theorems 1 and 2 are proven by giving explicit examples [17]. Here, we give an example for $n = 2$. Consider the C^∞ function $\gamma : S^2 \to \mathbb{R}_{>0}$ defined by

$$\gamma(\nu_1, \nu_2, \nu_3) = (\nu_1^2 + \nu_2^2)^3 + 15(\nu_1^2 + \nu_2^2)^2 \nu_3^2 + 15(\nu_1^2 + \nu_2^2)\nu_3^4 + \nu_3^6, \quad (\nu_1, \nu_2, \nu_3) \in S^2. \tag{6}$$

By computation, the corresponding Cahn–Hoffman map $\xi_\gamma : S^2 \to \mathbb{R}^3$ is given as follows.

$$\begin{aligned}
\xi_\gamma(\nu) = \frac{1}{4} \big(& (\cos\theta)(\cos^6\theta - 9\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta + 25\sin^6\theta)(\cos\rho), \\
& (\cos\theta)(\cos^6\theta - 9\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta + 25\sin^6\theta)(\sin\rho), \\
& (\sin\theta)(25\cos^6\theta + 15\cos^4\theta\sin^2\theta - 9\cos^2\theta\sin^4\theta + \sin^6\theta) \big),
\end{aligned} \tag{7}$$

where $\nu = (\cos\theta\cos\rho, \cos\theta\sin\rho, \sin\theta) \in S^2$. The image $\xi_\gamma(S^2)$ of ξ_γ is shown in Figure 3, left. It is a surface of revolution, and we show by the right figure in Figure 3 its section in the (x_1, x_3)-plane. In Figure 4, we show three closed embedded surfaces of revolution with genus 0, each of which is a subset of $\xi_\gamma(S^2)$. Since the anisotropic mean curvature of ξ_γ is -1 for the unit normal ν (Remark 1), all of these three surfaces have constant anisotropic mean curvature -1. The surface in Figure 3a is the Wulff shape W_γ, and the surfaces in Figure 3b,c are closed piecewise-C^∞ CAMC surfaces for γ, which are not the Wulff shape W_γ (up to homothety and translation). These two surfaces give the proof of Theorems 1 and 2 for $n = 2$.
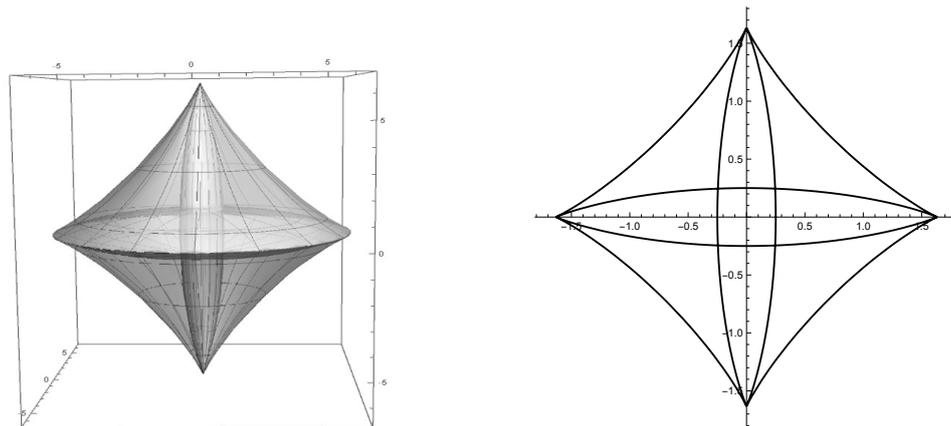
Figure 3. Left: The image $\xi_\gamma(S^2)$ of the Cahn–Hoffman map $\xi_\gamma : S^2 \to \mathbb{R}^3$ for $\gamma : S^2 \to \mathbb{R}_{>0}$ defined by Equation (6). **Right**: The section of $\xi_\gamma(S^2)$ in the (x_1, x_3)-plane.

(a) (b) (c)
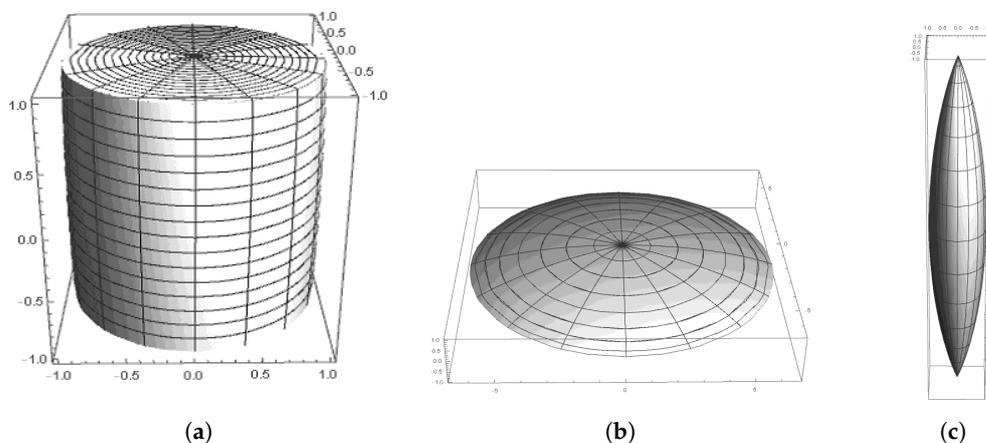
Figure 4. Some of the closed surfaces which are subsets of $\xi_\gamma(S^2)$ for γ defined by Equation (6) (Figure 3). The anisotropic mean curvature for the outward-pointing normal is -1: (**a**) Wulff shape W_γ; (**b**) a constant anisotropic mean curvature (CAMC) surface for γ; and (**c**) a CAMC surface for γ.

The section $\gamma_1 : S^1 \to \mathbb{R}_{>0}$ of γ and that of ξ_γ in the (x_1, x_3)-plane (Figure 3, right) give an example to prove Theorem 1 for $n = 1$. Suitable rotations of γ_1 give a proof of Theorem 1 for $n > 2$ similar to the above discussion for $n = 2$.

5. Outline of the Proof of Theorem 3

In this section, first, we give two useful integral formulas that are generalizations of the Steiner formula and the Minkowski formula (see Reference [26] for smooth case). They are proven in Reference [19] and are used to prove Theorem 3.

"Anisotropic parallel hypersurface" is a generalization of parallel hypersurface and is defined as follows.

Definition 4 (Anisotropic parallel hypersurface, cf. Reference [10])**.** *Let X be a piecewise-C^2 weak immersion. For any real number t, we call the map $X_t := X + t\tilde{\xi} : M \to \mathbb{R}^{n+1}$ the anisotropic parallel deformation of X of height t. If X_t is a piecewise-C^2 weak immersion, then we call it the anisotropic parallel hypersurface of X of height t.*

The anisotropic energy $\mathcal{F}_\gamma(X_t)$ of the anisotropic parallel hypersurface $X_t := X + t\tilde{\xi}$ is a polynomial of t with degree at the most n as follows.

Theorem 5 (Steiner-type formula [19])**.** *Assume that $\gamma : S^n \to \mathbb{R}_{>0}$ is of class C^2. Let $X : M = \cup_{i=1}^k M_i \to \mathbb{R}^{n+1}$ be a piecewise-C^2 weak immersion. Consider anisotropic parallel hypersurfaces $X_t = X + t\tilde{\xi} : M \setminus S[X] \to \mathbb{R}^{n+1}$, where $S[X]$ is the set of singular points of X. Then, the following integral formula holds.*

$$\mathcal{F}_\gamma(X_t) \quad = \quad \int_M \gamma(\nu) \sum_{r=0}^n (-1)^r t^r ({}_n C_r) H_r^\gamma \, dA,$$

where $\int_M$ means $\sum_{i=1}^k \int_{M_i}$.

The isotropic version of Theorem 5 is known as the Weyl's tube formula [28]. The isotropic 2-dimensional version is the well-known Steiner's formula.

Next, we give a generalization of the Minkowski formula.

Theorem 6 (Minkowski-type formula [19])**.** *Assume that $\gamma : S^n \to \mathbb{R}_{>0}$ is of class C^2. Assume also that $X : M = \cup_{i=1}^k M_i \to \mathbb{R}^{n+1}$ is a closed piecewise-C^2 weak immersion and that X satisfies the following condition.*

$$\tilde{\xi}_i(\zeta) = \tilde{\xi}_j(\zeta), \quad \forall \zeta \in \partial M_i \cap \partial M_j,$$

where $\tilde{\xi}_i := \xi_\gamma \circ \nu_i : M_i \to \mathbb{R}^{n+1}$ is the Cahn–Hoffman field along $X|_{M_i}$ and the tangent space of a submanifold of $\mathbb{R}^{n+1}$ is naturally identified with a linear subspace of $\mathbb{R}^{n+1}$. Then, we have the following integral formulas.
(i)

$$\int_M (\gamma(\nu) + \Lambda\langle X, \nu \rangle) \, dA = 0.$$

(ii) Assume that the rth anisotropic mean curvature of X for γ is integrable on M for $r = 1, \ldots, n$. Then,

$$\int_M (\gamma(\nu) H_r^\gamma + \langle X, \nu \rangle H_{r+1}^\gamma) \, dA = 0, \quad r = 0, \ldots, n-1,$$

holds, where H_r^γ is the rth anisotropic mean curvature of X.

Now, let us give an outline of the proof of Theorem 3. We first prove that the Cahn–Hoffman field $\tilde{\xi}$ along a closed CAMC hypersurface $X : M = \cup_{i=1}^k M_i \to \mathbb{R}^{n+1}$ can be defined as a continuous map on the whole of M. Then, we consider the anisotropic parallel hypersurfaces $X_t := X + t\tilde{\xi}$ ($t \in \mathbb{R}$, $|t| << 1$) of X. By taking homotheties of X_t if necessary, we have a volume-preserving variation $Y_t =$

$\mu(t)X_t = \mu(t)(X + t\tilde{\xi})$ ($\mu(t) > 0$, $\mu(0) = 1$) of X. Using Theorems 5 and 6, by long computation, we prove that the following holds: $\left.\dfrac{d^2\mathcal{F}_\gamma(Y_t)}{dt^2}\right|_{t=0} = \dfrac{-1}{n}\displaystyle\int_M \gamma(\nu)\sum_{1\le i<j\le n}(k_i^\gamma - k_j^\gamma)^2\,dA$. Since γ is convex, all k_i^γ are real values on $M \setminus S[X]$ (Remark 2). Hence, if X has constant anisotropic mean curvature Λ and is stable, then $k_1^\gamma = \cdots = k_n^\gamma = \Lambda/n \ne 0$ must hold on $M \setminus S[X]$. Therefore, from Corollary 1 in Reference [10], $X(M \setminus S[X]) \subset rW_\gamma$ holds for some $r > 0$. Because M is closed and W_γ has anisotropic mean curvature -1, this inclusion implies that the following holds: $X(M) = (1/|\Lambda|)W_\gamma$.

6. Application to Anisotropic Mean Curvature Flow

Let $\gamma : S^n \to \mathbb{R}_{>0}$ be of class C^2 with Cahn–Hoffman map ξ_γ. Let $X_t : M \to \mathbb{R}^{n+1}$ be a one-parameter family of embedded piecewise-C^2 hypersurfaces with anisotropic mean curvature Λ_t. Assume that the Cahn–Hoffman field $\tilde{\xi}_t$ along X_t is defined as a continuous map on M. If X_t satisfies $\partial X_t/\partial t = \Lambda_t \tilde{\xi}_t$, it is called an anisotropic mean curvature flow, which diminishes the anisotropic energy if $\Lambda_t \not\equiv 0$ [17]. The example given in Section 4 (see Figure 4b,c) gives self-similar shrinking solutions with genus 0, which are not (any homotheties or translations of) the Wulff shape.

7. Proof of Theorem 4

Most of the work in this section can be generalized to hypersurfaces in $\mathbb{R}^{n+1}$. We will discuss it in another paper [29].

Let Π_1 and Π_2 be two planes in $\mathbb{R}^3$ containing the x_1-axis and making angles α and $-\alpha$ ($0 < \alpha < \pi/2$) with the horizontal plane $\{x_3 = 0\}$, respectively. Let $\Omega \subset \{x_2 > 0\}$ be the wedge-shaped domain bounded by Π_1 and Π_2 (Figure 5). We denote by $\overline{\Omega}$ the closure of Ω. The x_1-axis is called the edge of the wedge Ω. Denote by $\tilde{N}_j$ the unit normal to Π_j which points outward from Ω.
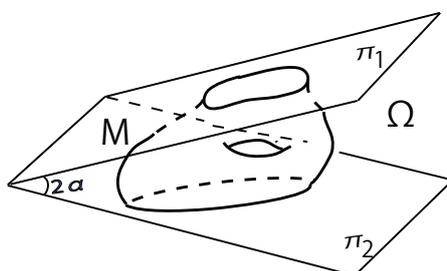

Figure 5. The wedge Ω and an admissible surface M.

Let $\gamma : S^2 \to \mathbb{R}_{>0}$ be a strictly convex function of class C^∞. As in Section 1, let $X : (M, \sigma_1, \sigma_2) \to (\overline{\Omega}, \Pi_1, \Pi_2)$ be a C^∞-immersion of which the restriction $X|_{\partial M}$ is an embedding onto two simple closed curves $C_1 \subset \Pi_1$ and $C_2 \subset \Pi_2$. Denote by $D_j \subset \Pi_j$ the (nonempty) domain bounded by C_j. Then, by a similar way to the way to derive the Euler–Lagrange equations in Reference [22], we can prove the following Euler–Lagrange equations for our capillary problem.

Lemma 1. *X is a capillary surface if and only if both of the following conditions (i) and (ii) hold.*
(i) The anisotropic mean curvature Λ of X is constant on M.
(ii) $\langle \tilde{\xi}, \tilde{N}_j \rangle = \omega$ on σ_j ($j = 1, 2$), where $\tilde{\xi}$ is the Cahn–Hoffman field along X.

In view of the property of condition (ii) in Lemma 1, it is useful to consider the anisotropic energy for curves in Π_j. First, define planes P_j ($j = 1, 2$) by

$$P_j := \{x \in \mathbb{R}^3 \; ; \; \langle x, \tilde{N}_j \rangle = \omega\}.$$

Then, set the following:

$$\hat{W}_j := W_\gamma \cap P_j, \quad \hat{O}_j := \omega\tilde{N}_j.$$

Assume that $\omega > 0$ is sufficiently small so that $\hat{W}_j$ includes at least two distinct points. Then, $\hat{W}_j$ is a strictly convex closed C^∞ curve in the plane P_j. We regard the point $\hat{O}_j$ as the origin of P_j. Denote by $\hat{\gamma}_j : S^1 \to \mathbb{R}_{>0}$ the support function of $\hat{W}_j$. Then, $\hat{W}_j$ is the Wulff shape for $\hat{\gamma}_j$. For later use, we denote by $\hat{\xi}_j$ the Cahn–Hoffman map for $\hat{\gamma}_j$.

Now, let $\chi : S^1 \to \Pi_j$ be a C^∞ regular embedded curve with outward unit normal $\rho = \rho_j$. Define the anisotropic energy of χ by

$$\hat{\mathcal{F}}_j(\chi) := \int_{S^1} \hat{\gamma}_j(\rho) \, ds, \tag{8}$$

where ds is the line element of χ.

From now on, we assume that X is a capillary surface. Set the following:

$$\chi_j := X|_{\sigma_j}.$$

Denote by ρ the outward-pointing unit normal to χ_j in the plane Π_j. X has the following property, which we call the balancing formula that is a generalization of the balancing formula for the isotropic case [24].

Lemma 2.

$$\hat{\mathcal{F}}_j(\chi_j) = -2\Lambda \mathcal{H}^2(D_j), \quad j = 1, 2. \tag{9}$$

Proof. Let u be a constant vector in $\mathbb{R}^3$. Consider the parallel translations:

$$X_t = X + tu.$$

Then, by the first variation formulas in Equations (4) and (5), we have

$$
\begin{aligned}
0 &= \left.\frac{d}{dt}\right|_{t=0} E(X_t) = \left.\frac{d}{dt}\right|_{t=0} (E(X_t) + 2\Lambda V(X_t)) \\
&= -2 \int_M \Lambda \langle u, v \rangle \, dA - \oint_{\partial M} \langle u, R(p(\tilde{\xi})) \rangle \, ds \\
&\quad + \omega \sum_{j=1}^{2} \oint_{\sigma_j} \langle p(u), \rho \rangle \, ds + 2\Lambda \int_M \langle u, v \rangle \, dA + 2\Lambda \sum_{j=1}^{2} \int_{D_j} \langle u, \tilde{N}_j \rangle \, dA \\
&= -\oint_{\partial M} \langle u, R(p(\tilde{\xi})) \rangle \, ds + \omega \sum_{j=1}^{2} \oint_{\sigma_j} \langle u, \rho \rangle \, ds + 2\Lambda \sum_{j=1}^{2} \int_{D_j} \langle u, \tilde{N}_j \rangle \, dA \\
&= -\oint_{\partial M} \langle u, R(p(\tilde{\xi})) \rangle \, ds + 2\Lambda \sum_{j=1}^{2} \int_{D_j} \langle u, \tilde{N}_j \rangle \, dA.
\end{aligned}
$$

Hence, by setting $u = (1, 0, 0), (0, 1, 0), (0, 0, 1)$, we have

$$-2\Lambda \sum_{j=1}^{2} \mathcal{H}^2(D_j) \tilde{N}_j = -\sum_{j=1}^{2} \oint_{\sigma_j} R(p(\tilde{\xi})) \, ds. \tag{10}$$

On σ_j, since $\langle \tilde{\xi}, \tilde{N}_j \rangle = \omega$ and $\langle \tilde{\xi}, \rho_j \rangle = \langle \hat{\xi}_j, \rho_j \rangle = \hat{\gamma}_j$ hold, we can write

$$\tilde{\xi} = \omega \tilde{N}_j + \hat{\gamma}_j \rho_j + \tau,$$

where τ is tangent to C_j. Then, we have

$$R(p(\tilde{\xi})) = R(\omega \tilde{N}_j + \hat{\gamma}_j \rho_j) = \omega \rho_j - \hat{\gamma}_j \tilde{N}_j. \tag{11}$$

Note that, by the divergence theorem, the following holds:

$$\oint_{\sigma_j} \rho_j \, ds = 0.$$

Hence, substituting Equation (11) into Equation (10), we obtain

$$-2\Lambda \sum_{j=1}^{2} \mathcal{H}^2(D_j)\tilde{N}_j = \sum_{j=1}^{2} \oint_{\sigma_j} \hat{\gamma}_j \tilde{N}_j \, ds = \sum_{j=1}^{2} \hat{\mathcal{F}}_j(\chi_j)\tilde{N}_j. \tag{12}$$

Because $\tilde{N}_1$, $\tilde{N}_2$ are linearly independent, Equation (12) implies Equation (9). □

Now, consider the anisotropic parallel surfaces $X_t := X + t\tilde{\xi}$ ($t \in \mathbb{R}$, $|t| << 1$) of X. Set $B := \left(0, \dfrac{\omega}{\sin \alpha}, 0\right)$. Then, the variation $Z_t := X_t + tb$ of X satisfies the boundary condition. Take homotheties

$$Y_t := \mu(t)Z_t = \mu(t)(X + t(\tilde{\xi} + b)), \ \mu(t) > 0, \ \mu(0) = 1,$$

of Z_t if necessary so that $Y_t : (M, \partial M) \to (\overline{\Omega}, \partial \Omega)$ is a volume-preserving variation of X (Figure 6).
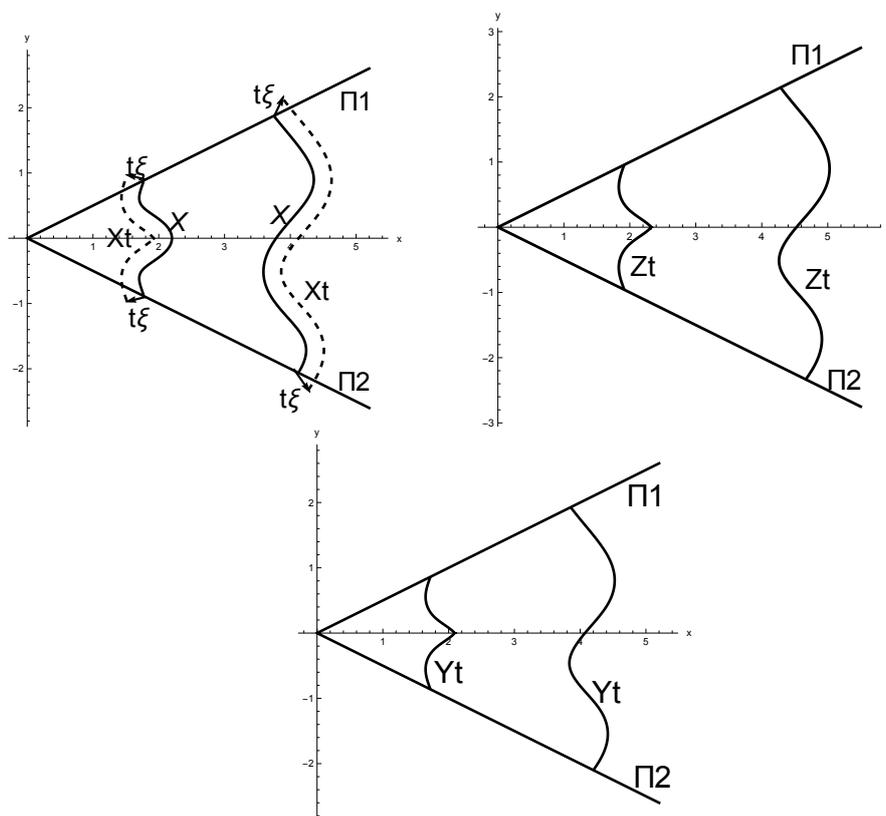
Figure 6. Construction of volume-preserving variation Y_t using anisotropic parallel surfaces X_t of X. **Upper left:** A capillary surface X and its anisotropic parallel surface X_t. **Upper right:** A parallel translation Z_t of X_t that satisfies the boundary condition. **Bottom:** A homothety Y_t of Z_t that satisfies the volume condition.

Denote by $e(t)$ the total energy $E(Y_t)$ of Y_t. Then, by a similar way to the proof of Theorem 3, we obtain

$$e''(0) = \frac{-1}{2} \int_M \gamma(\nu)(k_1^\gamma - k_2^\gamma)^2 \, dA - \omega \sum_{j=1}^{2} \left(2\Lambda^2 \mathcal{H}^2(D_j) + \int_{\sigma_j} \hat{\gamma}_j(\rho)\hat{\Lambda} \, ds\right), \tag{13}$$

where $\hat{\Lambda}$ is the anisotropic (mean) curvature of χ_j for $\hat{\gamma}_j$.

Note that, from Remark 2(i), k_j^{γ} is real. Since X has constant anisotropic mean curvature Λ, the first term of the right hand side of Equation (13) is nonnegative if and only if $k_1^{\gamma} = k_2^{\gamma} = \Lambda/2 \neq 0$. Hence, again by Corollary 1 in Reference [10], $X(M) \subset (1/|\Lambda|)W_\gamma$ holds.

Let us study the second term of the right hand side of Equation (13). Set the following:

$$B_j := 2\Lambda^2 \mathcal{H}^2(D_j) + \int_{\sigma_j} \hat{\gamma}_j(\rho)\hat{\Lambda} \, ds.$$

We will prove that $B_j \geq 0$ holds and that the equality holds if and only if $\chi_j(\sigma_j) = r\hat{W}_j$ for some $r > 0$. Using Equation (9), we obtain

$$B_j = \frac{(\hat{\mathcal{F}}_j(\chi_j))^2}{2\mathcal{H}^2(D_j)} + \int_{\sigma_j} \hat{\gamma}_j(\rho)\hat{\Lambda} \, ds. \tag{14}$$

Below, for simplicity, we identify an embedded closed curve in a plane with the domain bounded by this curve.

Since the Wulff shape $\hat{W}_j$ is the minimizer of $\hat{\mathcal{F}}_j$ among closed curves enclosing the same area, it holds that

$$\frac{(\hat{\mathcal{F}}_j(\chi_j))^2}{2\mathcal{H}^2(D_j)} \geq \frac{(\hat{\mathcal{F}}_j(\hat{W}_j))^2}{2\mathcal{H}^2(\hat{W}_j)}, \tag{15}$$

where the equality holds if and only if $\chi_j(\sigma_j) = r\hat{W}_j$ for some $r > 0$. Denote by $\hat{\xi}_j$ the Cahn–Hoffman map for $\hat{\gamma}_j$ and by $ds_{\hat{W}_j}$ the line element of $\hat{\xi}_j : S^1 \to \mathbb{R}^2$. Then, by the definition of $\hat{\mathcal{F}}_j$ and since $\hat{\gamma}_j$ is the support function of $\hat{W}_j$, we have

$$\frac{(\hat{\mathcal{F}}_j(\hat{W}_j))^2}{2\mathcal{H}^2(\hat{W}_j)} = \frac{(\int_{S^1} \hat{\gamma}_j(\nu) \, ds_{\hat{W}_j})^2}{2\mathcal{H}^2(\hat{W}_j)} = \frac{(2\mathcal{H}^2(\hat{W}_j))^2}{2\mathcal{H}^2(\hat{W}_j)} = 2\mathcal{H}^2(\hat{W}_j). \tag{16}$$

Let us compute the second term of B_j. For simplicity, we denote by $\hat{\xi}_j$ also the Cahn–Hoffman field of χ_j. Then, by Equations (4) and (5), we have

$$\int_{\sigma_j} \hat{\gamma}_j(\rho)\hat{\Lambda} \, ds = -\frac{d\hat{\mathcal{F}}_j(\chi_j + t\hat{\xi}_j))}{dt}\bigg|_{t=0} = -\frac{d^2\mathcal{H}^2(\chi_j + t\hat{\xi}_j)}{dt^2}\bigg|_{t=0}. \tag{17}$$

Assume now that D_j is convex. Then, for $t \geq 0$, it holds that

$$\mathcal{H}^2(\chi_j + t\hat{\xi}_j) = \mathcal{H}^2(D_j + t\hat{W}_j) = \mathcal{H}^2(D_j) + 2tv(D_j, \hat{W}_j) + t^2\mathcal{H}^2(\hat{W}_j), \tag{18}$$

where $v(D_j, \hat{W}_j)$ is a real number depending on D_j and $\hat{W}_j$ ([30], Theorem 5.1.7). Equation (17) with Equation (18) gives

$$\int_{\sigma_j} \hat{\gamma}_j(\rho)\hat{\Lambda} \, ds = -2\mathcal{H}^2(\hat{W}_j). \tag{19}$$

From Equations (14)–(16) and (19), we obtain

$$B_j \geq 0, \tag{20}$$

where the equality holds if and only if $\chi_j(\sigma_j) = r\hat{W}_j$ for some $r > 0$.

If the capillary surface X is stable, then $e''(0) \geq 0$. Hence, by the above observations, $X(M) \subset (1/|\Lambda|)W_\gamma$ holds. Conversely, if $X(M)$ is part of a homothety of the Wulff shape W_γ, by a similar way to the proof of Theorem 4.1 in Reference [22], it is shown that X is stable, which proves Theorem 4.

References

1. Winterbottom, W.L. Equilibrium shape of a small particle in contact with a foreign substrate. *Acta Metall.* **1967**, *15*, 303–310. [CrossRef]
2. Wulff, G. Zur Frage der Geschwindigkeit des Wachsthums und der Auflösung der Krystallflächen. *Zeitschrift für Krystallographie und Mineralogie* **1901**, *34*, 449–530.
3. Taylor, J.E. Crystalline variational problems. *Bull. Am. Math. Soc.* **1978**, *84*, 568–588. [CrossRef]
4. Ando, N. Hartman-Wintner's theorem and its applications. *Calc. Var. Partial Differ. Equ.* **2012**, *43*, 389–402. [CrossRef]
5. He, Y.; Li, H.; Ma, H.; Ge, J. Compact embedded hypersurfaces with constant higher order anisotropic mean curvatures. *Indiana Univ. Math. J.* **2009**, *58*, 853–868. [CrossRef]
6. Koiso, M.; Palmer, B. Geometry and stability of surfaces with constant anisotropic mean curvature. *Indiana Univ. Math. J.* **2005**, *54*, 1817–1852. [CrossRef]
7. Koiso, M.; Palmer, B. Anisotropic umbilic points and Hopf's theorem for surfaces with constant anisotropic mean curvature. *Indiana Univ. Math. J.* **2010**, *59*, 79–90. [CrossRef]
8. Palmer, B. Stability of the Wulff shape. *Proc. Am. Math. Soc.* **1998**, *126*, 3661–3667. [CrossRef]
9. Palmer, B. Stable closed equilibria for anisotropic surface energies: Surfaces with edges. *J. Geom. Mech.* **2012**, *4*, 89–97. [CrossRef]
10. Reilly, R.C. The relative differential geometry of nonparametric hypersurfaces. *Duke Math. J.* **1976**, *43*, 705–721. [CrossRef]
11. Kapouleas, N. Complete constant mean curvature surfaces in euclidean three-space. *Ann. Math.* **1990**, *131*, 239–330. [CrossRef]
12. Kapouleas, N. Constant mean curvature surfaces constructed by fusing Wente tori. *Invent. Math.* **1995**, *119*, 443–518. [CrossRef]
13. Wente, H.C. Counterexample to a conjecture of H. Hopf. *Pac. J. Math.* **1986**, *121*, 193–243. [CrossRef]
14. Alexandrov, A.D. A characteristic property of spheres. *Annali di Matematica Pura ed Applicata* **1962**, *58*, 303–315. [CrossRef]
15. Barbosa, J.L.; do Carmo, M. Stability of hypersurfaces with constant mean curvature. *Math. Z.* **1984**, *185*, 339–353. [CrossRef]
16. Hopf, H. *Differential Geometry in the Large*; Springer: Berlin, Germany, 1989.
17. Jikumaru, Y.; Koiso, M. Non-uniqueness of closed embedded non-smooth hypersurfaces with constant anisotropic mean curvature. *arXiv* **2019**, arXiv:1903.03958.
18. Brendle, S. Embedded self-similar shrinkers of genus 0. *Ann. Math.* **2016**, *183*, 715–728. [CrossRef]
19. Koiso, M. Uniqueness of stable closed non-smooth hypersurfaces with constant anisotropic mean curvature. *arXiv* **2019**, arXiv:1903.03951.
20. Morgan, F. Planar Wulff shape is unique equilibrium. *Proc. Am. Math. Soc.* **2005**, *133*, 809–813. [CrossRef]
21. Koiso, M.; Palmer, B. Stability of anisotropic capillary surfaces between two parallel planes. *Calc. Var. Partial Differ. Equ.* **2006**, *25*, 275–298. [CrossRef]
22. Koiso, M.; Palmer, B. Anisotropic capillary surfaces with wetting energy. *Calc. Var. Partial Differ. Equ.* **2007**, *29*, 295–345. [CrossRef]
23. Koiso, M.; Palmer, B. Uniqueness theorems for stable anisotropic capillary surfaces. *SIAM J. Math. Anal.* **2007**, *39*, 721–741. [CrossRef]
24. Choe, J.; Koiso, M. Stable capillary hypersurfaces in a wedge. *Pac. J. Math.* **2016**, *280*, 1–15. [CrossRef]
25. Morgan, F. The cone over the Clifford torus in $\mathbf{R}^4$ is Φ-minimizing. *Math. Ann.* **1991**, *289*, 341–354. [CrossRef]
26. He, Y.; Li, H. Integral formula of Minkowski type and new characterization of the Wulff shape. *Acta Math. Sin.* **2008**, *24*, 697–704. [CrossRef]
27. Barbosa, J.L.; do Carmo, M.; Eschenburg, J. Stability of hypersurfaces of constant mean curvature in Riemannian manifolds. *Math. Z.* **1988**, *197*, 123–138. [CrossRef]

28. Weyl, H. On the volume of tubes. *Am. J. Math.* **1939**, *61*, 461–472. [CrossRef]
29. Koiso, M. Stable anisotropic capillary hypersurfaces in a wedge. In preparation.
30. Schneider, R. *Convex Bodies: The Brunn–Minkowski Theory*; Cambridge University Press: Cambridge, UK, 1993.