Dear Editors,

For the published paper entitled “A new cumulative fatigue damage rule based on dynamic residual S-N curve and material memory concept” (Metals (2018; 8(6): 456)), Kris Hectors and Prof. Wim De Waele give a comment about the incorrect use of the formula of fatigue life prediction. Thanks for the suggestion. The following is our reply for the comment.

According to the comment “Equation (34) implicitly assumes that the number of cycles of each load block can be scaled with the same scalar up to failure, corresponding to \( \sum D_i = 1 \) and \( \sum n_i = N_{pre} \). This is however only true if the damage accumulation is described as a linear function of cycle ratios.”

This description is not true in some special cases. Considering a repeated two-level load block (with two blocks to failure), as shown in Figure 1.

![Figure 1. Repeated two-level load block (two blocks).](image-url)
It can also be viewed as a four-level fatigue loading, the specimen is fatigued by \( \sigma_1 \) for \( n_1 \) cycles, \( \sigma_2 \) for \( n_2 \) cycles, \( \sigma_3 \) for \( n_3 \) cycles, and \( \sigma_4 \) for \( n_4 \) cycles to failure, where \( \sigma_1 = \sigma_1', \sigma_2 = \sigma_2', \sigma_3 = \sigma_1', \sigma_4 = \sigma_2' \), \( n_1 = n_1' \), \( n_2 = n_2' \), \( n_3 = n_1' \), \( n_4 = n_2' \).

As is known, for a linear damage accumulation model, the damage variable \( D_i \) should be a linear function of cycle ratio \( n_i / N_{fi} \), that is:

\[
D_i \propto \frac{n_i}{N_{fi}}
\]

(1)

In Equation (1), for Miner rule, \( D_i = \frac{n_i}{N_{fi}} \). For a nonlinear damage model, \( D_i \) is defined as a nonlinear function of the cycle ratio, such as Marco-Starkey’s model \( (D_i = (n_i / N_{fi})^{a_i}) \), Chaboche’s continuum damage mechanics model (the formula can be found in Ref [47]).

(1) **Kwofie’s model**

For Kwofie’s model, \( D_i \) is described as:

\[
D_i = \frac{n_i}{N_{fi}} \times \frac{\ln(N_{f1})}{\ln(N_{f1})}
\]

(2)

where \( \ln(N_{f1}) / \ln(N_{f1}) \) is a constant without \( n_i / N_{fi} \). \( D_i \) is a linear function of cycle ratio \( n_i / N_{fi} \) and that it is a linear damage model.

Fatigue damage is accumulated by taking a summation of the segmental damage caused by each loading stress level.

(a) The first-level loading

\[
D_1 = \frac{n_1}{N_{f1}} = \frac{n_1'}{N_{f1}}
\]

(3)

(b) The second-level loading

\[
D_2 = \frac{n_2}{N_{f2}} \times \frac{\ln(N_{f2})}{\ln(N_{f1})} = \frac{n_2'}{N_{f2}} \times \frac{\ln(N_{f2})}{\ln(N_{f1})}
\]

(4)

(c) The third-level loading

\[
D_3 = \frac{n_3}{N_{f3}} \times \frac{\ln(N_{f3})}{\ln(N_{f1})} = \frac{n_3'}{N_{f3}} \times \frac{\ln(N_{f3})}{\ln(N_{f1})} = \frac{n_1'}{N_{f1}}
\]

(5)

(d) The fourth-level loading

\[
D_4 = \frac{n_4}{N_{f4}} \times \frac{\ln(N_{f4})}{\ln(N_{f1})} = \frac{n_4'}{N_{f4}} \times \frac{\ln(N_{f4})}{\ln(N_{f1})}
\]

(6)

The cumulative damage \( \sum D_i \) is expressed as:

\[
\sum D_i = D_1 + D_2 + D_3 + D_4
\]

\[
= \frac{n_1'}{N_{f1}} + \frac{n_2'}{N_{f2}} \times \frac{\ln(N_{f2})}{\ln(N_{f1})} + \frac{n_3'}{N_{f3}} \times \frac{\ln(N_{f3})}{\ln(N_{f1})} + \frac{n_4'}{N_{f4}} \times \frac{\ln(N_{f4})}{\ln(N_{f1})}
\]

\[
= 2 \times \frac{n_1'}{N_{f1}} + 2 \times \frac{n_2'}{N_{f2}} \times \frac{\ln(N_{f2})}{\ln(N_{f1})}
\]

(7)
Using the Equation (34), the predicted fatigue life is

$$N_{\text{pre}} = \sum n_i \sum D_i = \frac{n_1 + n_2 + n_3 + n_4}{\sum D_i} = \frac{2n_1 + 2n_2}{\frac{n_1}{N_f} + 2n_2 \times \frac{\ln(N_f)}{N_f}}$$

Equation (8) can be scaled with a scalar C (C = 2).

(2) A modified Kwofie’s model

Similar to Equation (2), we change the damage variable as:

$$D_i = \frac{n_i}{N_f} = \frac{n_i'}{N_f}$$

Compared to Equation (2), \(\ln(N_f)\) becomes \(\ln(N_{f(i-1)})\), and \(\ln(N_f)/\ln(N_{f(i-1)})\) is a constant without of \(n_i/N_f\). \(D_i\) is still a linear function of cycle ratio \(n_i/N_f\).

(a) The first-level loading

$$D_1 = \frac{n_1}{N_f} = \frac{n_1'}{N_f}$$

(b) The second-level loading

$$D_2 = \frac{n_2}{N_f} = \frac{n_2'}{N_f}$$

(c) The third-level loading

$$D_3 = \frac{n_3}{N_f} = \frac{n_3'}{N_f}$$

(d) The fourth-level loading

$$D_4 = \frac{n_4}{N_f} = \frac{n_4'}{N_f}$$

The cumulative damage \(\sum D_i\) is expressed as:

$$\sum D_i = D_1 + D_2 + D_3 + D_4 = \frac{n_1'}{N_f} + \frac{n_2'}{N_f} \times \frac{\ln(N_f)}{N_f} + \frac{n_3'}{N_f} \times \frac{\ln(N_f)}{N_f} + \frac{n_4'}{N_f} \times \frac{\ln(N_f)}{N_f}$$

Using the Equation (34), the predicted fatigue life is:

$$N_{\text{pre}} = \sum n_i \sum D_i = \frac{n_1 + n_2 + n_3 + n_4}{\sum D_i} = \frac{2n_1 + 2n_2}{\frac{n_1}{N_f} + 2n_2 \times \frac{\ln(N_f)}{N_f}}$$
Equation (15) cannot be scaled with a scalar because \( \ln(N_{f1}) / \ln(N_{f2}) \neq 1 \). As a result, for the linear damage model, it is uncertain whether the number of cycles of each load block can be scaled. This is attributed to \( \ln(N_{f(i-1)}) \) in Equation (9). Although \( \ln(N_{f(i-1)}) \) is a constant irrelevant with cycle ratio, it is changed with previous fatigue life.

### Proposed model

For the proposed model, \( D_i \) is described as:

\[
D_i = \frac{n_i}{N_{f_i}} \times \prod_{j=1}^{i-1} \left( \frac{N_{f_j}}{n_j} \right)^{\frac{1}{\ln \alpha_i} - 1}
\]

(16)

where \( \alpha_k = \left( e^{-n_k/N_{f_k}} - e^{-1} \right) / (1 - e^{-1}) \).

In Equation (16), the loading effect coefficient \( \prod_{j=1}^{i-1} \left( \frac{N_{f_j}}{n_j} \right)^{\frac{1}{\ln \alpha_i} - 1} \) (using the symbol \( \lambda_i \) instead) is associated with fatigue lives \( N_{f1}, N_{f2}, \ldots, N_{f4} \) and previous cycle ratios \( n_{i-1}/N_{f(i-1)} \) but without \( n_i/N_{f_i} \). If these previous cycle ratios are determined, then \( \lambda_i \) becomes a determined value as a constant. The damage variable \( D_i \) will be a linear function of cycle ratio \( n_i/N_{f_i} \), in reality it is a linear damage model.

(a) The first-level loading

\[
D_1 = \frac{n_1}{N_{f1}} = \frac{n_1'}{N_{f1}}
\]

(17)

(b) The second-level loading

\[
D_2 = \frac{n_2}{N_{f2}} \times \lambda_2 = \frac{n_2'}{N_{f2}} \times \lambda_2
\]

(18)

(c) The third-level loading

\[
D_3 = \frac{n_3}{N_{f3}} \times \lambda_3 = \frac{n_3'}{N_{f1}} \times \lambda_3
\]

(19)

(d) The fourth-level loading

\[
D_4 = \frac{n_4}{N_{f4}} \times \lambda_4 = \frac{n_4'}{N_{f2}} \times \lambda_4
\]

(20)

The cumulative damage \( \sum D_i \) is expressed as:

\[
\sum D_i = D_1 + D_2 + D_3 + D_4
\]

\[
= \frac{n_1'}{N_{f1}} + \frac{n_2'}{N_{f2}} \times \lambda_2 + \frac{n_3'}{N_{f1}} \times \lambda_3 + \frac{n_4'}{N_{f2}} \times \lambda_4
\]

(21)

Using the Equation (34), the predicted fatigue life is

\[
N_{pre} = \sum D_i = \frac{n_1 + n_2 + n_3 + n_4}{\sum D_i} = \frac{2n_1' + 2n_2' + 2n_3' + 2n_4'}{(1 + \lambda_3) \times \frac{n_1}{N_{f1}} + (1 + \lambda_4) \times \frac{n_2}{N_{f2}}}
\]

(22)

Equation (22) cannot be scaled with a scalar because \( 1 + \lambda_3 \neq 2 \) and \( \lambda_2 + \lambda_4 \neq 2 \). This is attributed to \( \lambda_i \) in Equation (16). \( \lambda_i \) is a constant without of the current cycle radio \( n_i/N_{f_i} \), it is changed and determined by previous cycle ratios from \( n_1/N_{f1} \) to \( n_{i-1}/N_{f(i-1)} \). It
should be pointed out that the loading effect coefficient $\lambda_i$ reveals a significant correlation with previous fatigue loading histories. This finding is an essential distinction with the Miner rule ($\lambda_i = 1$).

(4) The reason for using Equation (34) to predict fatigue life

In the paper, the prediction of fatigue life contain two steps:

(a) The first step:

The cumulative fatigue damage $\sum_{i=1}^{k} D_i$ is first calculated using different fatigue damage models. It should be emphasized that $\sum_{i=1}^{k} D_i$ denotes the predicted cumulative damage and its calculated value may not be equal to unity for different models. For the experimental value of cumulative damage, it is always defined as unity without using damage models, i.e., $\sum_{i=1}^{k} D_i = 1$.

(b) The second step:

According to the calculated values of cumulative damage obtained by different models, the fatigue life is thus calculated by Equation (34).

For the experimental result, the cumulative fatigue damage is unity. For different models, if the predicted cumulative damage is unity (i.e., $\sum_{i=1}^{k} D_i \to 1$), then the predicted fatigue life should be close to the experimental result (i.e., $N_{pre} \to N_{exp}$) and the corresponding damage model becomes more effective.

For the comment in Table 1, $\sum D_i$ denotes the predicted cumulative damage by different models but not the real value of experimental result ($\sum D_i = 1$). In theory, the predicted results cannot be equal to the experimental ones. In row (a), the specimen is fatigue by $\sigma_1$ for 13,749 cycles and $\sigma_2$ for 51,304 cycles to failure. This is correct because it denotes the real experimental results. However, in row (b) and row (c), the number of loading cycles of $n_1$ and $n_2$ are defined as the hypothetical values. The specimen is supposed first fatigue by $\sigma_1$ for 18,454 (or 17,497) cycles and then $\sigma_2$ for 68,860 (or 65,290) cycles. This situation is not correct (the specimen may be failure when $n_2 = 50,000$ or other value, and $n_2$ can only be determined by the experiments). In addition, if $n_1$ and $n_2$ are changed, the predicted damage ($D_1$ and $D_2$) should also be changed by the cumulative damage models as well as the corresponding predicted fatigue lives. It is meaningless for supposing a particular value of the loading cycles.

Moreover, for Corten’s model as stated by Equation (29), the prediction formula of fatigue life can be found in the paper entitled “A Practical Method for Determining the Corten-Dolan Exponent and Its Application to Fatigue Life Prediction” by Shun-Peng Zhu, Hong-Zhong Huang, Yu Liu, Li-Ping He and Qiang Liao published in Int. J. Turbo Jet-Engines (2012;29: 79–87), on page 81, in Equation (5), that is:

$$\sum_{i=1}^{k} n_i \left( \frac{\sigma_i}{\sigma_1} \right)^d$$

(23)

where $\sum_n = n_1 + n_2 + \ldots + n_k = \sum_{i=1}^{k} n_i$, $N_g$ is the predicted fatigue life.

The predicted fatigue life can be written as:

$$N_{pre} = N_g = \frac{\sum_{i=1}^{k} n_i \left( \frac{\sigma_i}{\sigma_1} \right)^d}{\sum_{i=1}^{k} \frac{n_i}{\sigma_1}}$$

(24)
This formula is the same as the Equation (34) in our paper.

As stated previously, the use of the Equation (34) for fatigue life prediction should be reasonable.

### Table 1. Predicted lifetime calculated by Peng et al. for their non-linear damage model and two block loading sequences (a), with the corresponding damage (b) and the corrected lifetime prediction (c).

<table>
<thead>
<tr>
<th></th>
<th>n&lt;sub&gt;1&lt;/sub&gt;/n&lt;sub&gt;2&lt;/sub&gt;</th>
<th>n&lt;sub&gt;1&lt;/sub&gt;</th>
<th>n&lt;sub&gt;2&lt;/sub&gt;</th>
<th>∑n&lt;sub&gt;1&lt;/sub&gt;</th>
<th>α&lt;sub&gt;1&lt;/sub&gt;</th>
<th>∑D&lt;sub&gt;i&lt;/sub&gt;</th>
<th>N&lt;sub&gt;pre&lt;/sub&gt;</th>
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<tbody>
<tr>
<td><strong>High–low loading sequence σ&lt;sub&gt;1&lt;/sub&gt; = 485 MPa – σ&lt;sub&gt;2&lt;/sub&gt; = 400 MPa</strong></td>
<td></td>
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</tr>
<tr>
<td>(a) N&lt;sub&gt;pre&lt;/sub&gt; Peng et al.</td>
<td>0.268</td>
<td>13,749</td>
<td>51,304</td>
<td>65,053</td>
<td>0.6501</td>
<td>0.7450</td>
<td>87,314</td>
</tr>
<tr>
<td>(b) Corresponding damage</td>
<td>0.268</td>
<td>18,454</td>
<td>68,860</td>
<td>87,314</td>
<td>0.5491</td>
<td>1.0687</td>
<td></td>
</tr>
<tr>
<td>(c) Correct N&lt;sub&gt;pre&lt;/sub&gt;</td>
<td>0.268</td>
<td>17,497</td>
<td>65,290</td>
<td>82,787</td>
<td>0.5689</td>
<td>1.0000</td>
<td>82,786</td>
</tr>
<tr>
<td><strong>Low–high loading sequence σ&lt;sub&gt;1&lt;/sub&gt; = 400 MPa – σ&lt;sub&gt;2&lt;/sub&gt; = 485 MPa</strong></td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>(a) N&lt;sub&gt;pre&lt;/sub&gt; Peng et al.</td>
<td>2.341</td>
<td>10,9310</td>
<td>46,693</td>
<td>156,003</td>
<td>0.1653</td>
<td>1.1264</td>
<td>138,502</td>
</tr>
<tr>
<td>(b) Corresponding damage</td>
<td>2.341</td>
<td>97,047</td>
<td>41,455</td>
<td>138,502</td>
<td>0.2309</td>
<td>1.0221</td>
<td></td>
</tr>
<tr>
<td>(c) Correct N&lt;sub&gt;pre&lt;/sub&gt;</td>
<td>2.341</td>
<td>94,482</td>
<td>40,359</td>
<td>134,841</td>
<td>0.2453</td>
<td>1.0000</td>
<td>134,847</td>
</tr>
</tbody>
</table>

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