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Article

A Simple First-Principles Homogenization Theory for Chiral Metamaterials

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Abstract: We discuss a simple first-principles homogenization theory for describing, in the long-wavelength limit, the effective bianisotropic response of a periodic metamaterial composite without intrinsic chiral and magnetic inclusions. In the case where the dielectric contrast is low, we obtain a full analytical description which can be considered the extension of Landau-Lifshitz-Looyenga effective-medium formulation in the context of periodic metamaterials.

Keywords: metamaterials; physics and modeling of metamaterial; all-dielectric metamaterials; chirality

1. Introduction

Tailoring the desired electromagnetic response of a composite structure is one of the main challenges of modern photonics and metamaterial science is the natural platform to achieve this goal. Metamaterials are composite materials artificially manufactured by repeating individual subwavelength elements (known as meta-atoms and meta-molecules) designed to mimic, at a mesoscopic scale, the electromagnetic response of atoms and molecules. Bearing in mind that the recent technology gives us the possibilities of achieving and combining subwavelength inclusions with various shapes, one can achieve an effective electromagnetic response at will. Exploiting the fact that a metamaterial is characterized by a subwavelength inhomogeneity scale, one generally assumes that its electromagnetic response coincides with that of a homogeneous medium and suitable phenomenological material parameters (such as effective permittivity and/or permeability) can be introduced for describing the effective medium response. The aim of a homogenization theory is to predict such effective electromagnetic parameters from the knowledge of the underlying composite structure. Using different approximation schemes, several researchers have developed suitable effective medium approach [1–10]. On the other hand, even if the inhomogeneity scale is much smaller than radiation wavelength, the description of the electromagnetic propagation in a metamaterial generally can not neglect spatial dispersion, which is a physical effect stemming from matter electromagnetic non-local response, and more phenomenological parameters are correspondingly in order. It is worth noting that the designing of spatial dispersion is a fundamental ingredient in numerous photonics devices. First-order spatial dispersion (described by terms proportional to first-order spatial derivatives of electric field in the constitutive relations) is equivalent to an artificial chiral response or, in other words, a reciprocal bianisotropic one [11]. Second-order spatial dispersion contributions (described by terms proportional to second-order spatial derivatives of electric field in the constitutive relations) can be partially interpreted as corrections to magnetic permeability so that spatial dispersion can, in this case, support a phenomenon known as optical or artificial magnetism [3].

In this paper, we present a simple first-principles homogenization theory for periodic metamaterials. Following the theory developed in Reference [12], we discuss a multiscale approach describing the electromagnetic (chiral) bianisotropic response, in the long wavelength regime, of a dielectric periodic medium whose underlying constituents are achiral and non-magnetic. Based on Fourier formalism, we suggest a numerical scheme for evaluating the effective dielectric and chiral tensors. In the case where the dielectric contrast is low, we develop a simple full analytical theory which can be considered the extension of Landau-Lifshitz-Looyenga (LLL) effective-medium approach in the context of periodic metamaterials. The LLL approach was independently developed both by Landau-Lifshitz [13] and by Looyenga [14] for evaluating the electrostatic effective dielectric permittivity of an isotropic mixture. In addition, in a specific example, by using the extended LLL approach, we deduce the analytical expressions of the dielectric and chiral tensor components.

The paper is organized as follows. In Section 2, we discuss the non-local effective medium theory of Ciattoni *et al.* [12]. In Section 3, by considering the low contrast approximation, we develop the extended version of LLL approach for periodic metamaterial. In Section 4, we draw our conclusions.

2. Effective Medium Theory

Let us consider propagation of a monochromatic electromagnetic field through an unbounded metal-dielectric composite whose underlying non-magnetic and achiral inclusions are patterned on a lattice. The electric $\mathbf{E}$ and magnetic $\mathbf{H}$ field amplitudes satisfy Maxwell's equations

$$\begin{aligned} \nabla \times \mathbf{E} &= i\omega\mu_0\mathbf{H} \\ \nabla \times \mathbf{H} &= -i\omega\varepsilon_0\varepsilon_r\mathbf{E} \end{aligned} \quad (1)$$

where time dependence $e^{-i\omega t}$ has been assumed (ε_0, μ_0 are the vacuum permittivity and permeability constants, respectively). The present paper deals with photonic crystals in the homogenized regime (periodic metamaterials). Indeed, we regard the dielectric permittivity as a periodic complex function, namely

$$\varepsilon_r(\mathbf{r}) = \varepsilon_r\left(\mathbf{r} + \mathbf{\Lambda}\right) \tag{2}$$

where $\mathbf{\Lambda}$ is any arbitrary lattice vector. Here, the main assumption is that dielectric permittivity is characterized by a spatial subwavelength modulation and hence it is natural to introduce the parameter $\eta = d/\lambda$ where d is the largest of the lattice basis vector lengths. Exploiting the condition $\eta \ll 1$, we can develop an asymptotic analysis of electromagnetic propagation. Since electromagnetic propagation is characterized by two very different scales, any field $\mathbf{A}$ ($\mathbf{A} = \mathbf{E}, \mathbf{H}$) separately depends on the slow and fast coordinates ($\mathbf{r}$, $\mathbf{R} = \mathbf{r}/\eta$, respectively) and $A(\mathbf{r}, \mathbf{R})$ can be decomposed as $\mathbf{A}(\mathbf{r}, \mathbf{R}) = \overline{\mathbf{A}}(\mathbf{r}) + \tilde{\mathbf{A}}(\mathbf{r}, \mathbf{R})$ where the overline denotes the spatial average over the metamaterial unit cell, namely

$$\overline{\mathbf{A}}(\mathbf{r}) = \frac{1}{V}\int_C d^3R\mathbf{A}(\mathbf{r}, \mathbf{R}) \tag{3}$$

(C is the unit cell and V is its volume scaled by η^3), and the tilde denotes the rapidly varying zero mean residual, *i.e.*, $\tilde{\mathbf{A}} = \mathbf{A} - \overline{\mathbf{A}}$. In our approach, the relative dielectric permittivity only depends on the fast coordinates ($\varepsilon(\mathbf{R}) = \varepsilon_r(\eta\mathbf{R})$) and it can be decomposed as $\varepsilon(\mathbf{R}) = \overline{\varepsilon} + \tilde{\varepsilon}(\mathbf{R})$. Representing each field $\mathbf{A} = \overline{\mathbf{A}} + \tilde{\mathbf{A}}$ as a Taylor expansion up to the first order in η, we get

$$\overline{\mathbf{A}} = \overline{\mathbf{A}}_0(\mathbf{r}) + \overline{\mathbf{A}}_1(\mathbf{r})\eta, \quad \tilde{\mathbf{A}} = \tilde{\mathbf{A}}_0(\mathbf{r}, \mathbf{R}) + \tilde{\mathbf{A}}_1(\mathbf{r}, \mathbf{R})\eta \tag{4}$$

After substituting Equation (4) into Equation (1) and noting that $\nabla \rightarrow \nabla + \frac{1}{\eta}\nabla_{\mathbf{R}}$ it is possible, in each equation, to separately balance the averaged contributions and zero mean residuals. As a result, for the averaged equations (after multiplying for η^n and summing over $n = 0, 1$), we obtain

$$\begin{aligned} \nabla \times \overline{\mathbf{E}} &= i\omega\mu_0\overline{\mathbf{H}} \\ \nabla \times \overline{\mathbf{H}} &= -i\omega\overline{\mathbf{D}} \end{aligned} \tag{5}$$

where

$$\overline{\mathbf{D}} = \varepsilon_0\left[\overline{\varepsilon}\overline{\mathbf{E}} + \overline{\varepsilon(\tilde{\mathbf{E}}_0 + \tilde{\mathbf{E}}_1\eta)}\right] \tag{6}$$

On the other hand, for the zero mean residual equations, we have

$$\begin{aligned} \nabla_{\mathbf{R}} \times \tilde{\mathbf{E}}_0 &= \mathbf{0} \\ \nabla_{\mathbf{R}} \times \tilde{\mathbf{H}}_0 &= \mathbf{0} \\ \nabla_{\mathbf{R}} \times \tilde{\mathbf{E}}_{n+1} &= -\nabla \times \tilde{\mathbf{E}}_n + i\omega\mu_0\tilde{\mathbf{H}}_n \\ \nabla_{\mathbf{R}} \times \tilde{\mathbf{H}}_{n+1} &= -\nabla \times \tilde{\mathbf{H}}_n - i\omega\varepsilon_0\left[\tilde{\varepsilon}\overline{\mathbf{E}}_n + \varepsilon\tilde{\mathbf{E}}_n - \overline{\varepsilon\tilde{\mathbf{E}}_n}\right] \end{aligned} \tag{7}$$

where $n = 0, 1$. Therefore the slowly varying electric and magnetic field amplitudes satisfy the macroscopic Maxwell Equations (5) with the slowly varying displacement vector $\overline{\mathbf{D}}$ of Equation (6) which has two contributions, the former due to the spatial average of the dielectric profile and latter due to the dielectric modulation. The latter contribution is obtained by summing the spatial average of the rapidly varying fields $\tilde{\mathbf{E}}_0$, $\tilde{\mathbf{E}}_1$ multiplied by the dielectric permittivity. In order to obtain an effective medium description of the metamaterial composite, the rapidly varying fields $\tilde{\mathbf{E}}_0$, $\tilde{\mathbf{E}}_1$ have to be related to the slowly varying ones. After applying the operator $\nabla_{\mathbf{R}}\cdot$ to both the third and fourth of Equation (7) we obtain

$$\begin{aligned} \nabla \cdot \left(\nabla_{\mathbf{R}} \times \tilde{\mathbf{E}}_n\right) &= -i\omega\mu_0 \nabla_{\mathbf{R}} \cdot \tilde{\mathbf{H}}_n \\ \nabla \cdot \left(\nabla_{\mathbf{R}} \times \tilde{\mathbf{H}}_n\right) &= i\omega\varepsilon_0 \nabla_{\mathbf{R}} \cdot \left[(\varepsilon - \overline{\varepsilon})\overline{\mathbf{E}}_n + \varepsilon\tilde{\mathbf{E}}_n\right] \end{aligned} \tag{8}$$

where we have used the identity $\nabla_{\mathbf{R}} \cdot (\nabla \times \mathbf{A}) = -\nabla \cdot (\nabla_{\mathbf{R}} \times \mathbf{A})$. Setting $n = 0$ and using the first and the second of Equation (7), Equation (8) become

$$\begin{aligned} \nabla_{\mathbf{R}} \cdot \tilde{\mathbf{H}}_0 &= 0 \\ \nabla_{\mathbf{R}} \cdot \left(\varepsilon\tilde{\mathbf{E}}_0\right) &= -\left(\nabla_{\mathbf{R}}\varepsilon\right) \cdot \overline{\mathbf{E}}_0 \end{aligned} \tag{9}$$

Relabelling $n \to n+1$ in Equation (8) and substituting the expressions for $\nabla_{\mathbf{R}} \times \tilde{\mathbf{E}}_{n+1}$ and $\nabla_{\mathbf{R}} \times \tilde{\mathbf{H}}_{n+1}$ from the third and fourth of Equation (7) we obtain

$$\begin{aligned} \nabla_{\mathbf{R}} \cdot \tilde{\mathbf{H}}_{n+1} &= -\nabla_{\mathbf{R}} \cdot \tilde{\mathbf{H}}_n \\ \nabla_{\mathbf{R}} \cdot \left(\varepsilon\tilde{\mathbf{E}}_{n+1}\right) &= -\nabla \cdot \left[(\varepsilon - \overline{\varepsilon})\overline{\mathbf{E}}_n + \varepsilon\tilde{\mathbf{E}}_n - \overline{\varepsilon\tilde{\mathbf{E}}_n}\right] - \left(\nabla_{\mathbf{R}}\varepsilon\right) \cdot \overline{\mathbf{E}}_{n+1} \end{aligned} \tag{10}$$

for $n = 0, 1$. Equations (7) together with Equations (9) and (10) can be used to evaluate the rapidly varying fields of order $n+1$ once those of order n are known and these fields are linearly dependent on the slowly-varying fields. As a result, from Equations (7), (9) and (10), we obtain

$$\begin{aligned} \tilde{\mathbf{E}}_0 &= \hat{\mathbf{e}}_i \left(\partial_i f_j\right) \overline{E}_{0j} \\ \tilde{\mathbf{E}}_1 &= \hat{\mathbf{e}}_i \left[\left(\partial_i f_j\right) \overline{E}_{1j} + \left(\delta_{ir}\tilde{f}_j + \partial_i W_{rj}\right) \frac{\partial \overline{E}_{0j}}{\partial x_r}\right] \end{aligned} \tag{11}$$

where the sum is hereafter understood over repeated indices, $\hat{\mathbf{e}}_i$ is the unit vector along the i-th direction, ∂_i is the partial derivative along $X_i = \hat{\mathbf{e}}_i \cdot \mathbf{R}$, $\overline{E}_{0j} = \hat{\mathbf{e}}_j \cdot \overline{\mathbf{E}}_0$. In Equation (11), we have introduced the potential vector $\mathbf{f}$ ($f_j = \mathbf{f} \cdot \hat{\mathbf{e}}_j$ and $\tilde{f}_j$ is the zero mean residual of f_j) and the functions W_{rj} satisfying the equations

$$\begin{aligned} &\nabla_{\mathbf{R}} \cdot (\varepsilon \nabla_{\mathbf{R}} f_j) = -\partial_j \varepsilon \\ &\nabla_{\mathbf{R}} \cdot (\varepsilon \nabla_{\mathbf{R}} W_{rj}) = -\partial_r \left(\varepsilon \tilde{f}_j\right) - \left(Q_{rj} - \overline{Q_{rj}}\right) \end{aligned} \tag{12}$$

respectively, where $Q_{rj} = \epsilon(\delta_{rj} + \partial_r f_j)$ (δ_{rj} is the Kronecker's delta). Next, inserting Equation (11) into Equation (6), using the identity $\overline{\varepsilon \partial_i W_{rj}} = \overline{\varepsilon \left(\partial_r \tilde{f}_i\right) \tilde{f}_j - Q_{rj} \tilde{f}_i}$ [12], adding suitable higher order term for restoring the electric fields $\overline{E}_i$, the effective constitutive relations can be written as

$$\begin{aligned} \overline{D}_i &= \varepsilon_0 \left(\varepsilon_{ij}^{(eff)} \overline{E}_j + \alpha_{ijr}^{(eff)} \frac{\partial \overline{E}_j}{\partial x_r}\right) \\ \overline{B}_i &= \mu_0 \overline{H}_i \end{aligned} \tag{13}$$

where

$$\begin{aligned} \varepsilon_{ij}^{(eff)} &= \frac{1}{2}\overline{(Q_{ij} + Q_{ji})} \\ \alpha_{ijr}^{(eff)} &= \eta \overline{\left(Q_{ri}\tilde{f}_j - Q_{rj}\tilde{f}_i\right)} \end{aligned} \tag{14}$$

Note that Equation (13) describe media showing weakly spatial nonlocal dielectric response which stems from spatial dispersion as reported by Landau and Lifshitz [13]. On the other hand, the constitutive relations can be transformed to a symmetric form [12], *i.e.*,

$$\begin{aligned} \overline{\mathbf{D}'} &= \varepsilon_0 \varepsilon'^{(eff)} \overline{\mathbf{E}'} - \frac{i}{c} \kappa^T \overline{\mathbf{H}'} \\ \overline{\mathbf{B}'} &= \frac{i}{c} \kappa \overline{\mathbf{E}'} + \mu_0 \overline{\mathbf{H}'} \end{aligned} \tag{15}$$

where

$$\begin{aligned} \varepsilon'^{(eff)}_{ij} &= \varepsilon^{(eff)}_{ij} + \kappa^{(eff)}_{ir} \kappa^{(eff)}_{rj} \\ \kappa^{(eff)}_{ij} &= \eta k_0 \left[\epsilon_{imj} \overline{\varepsilon f_m} + \left(\epsilon_{imn} \delta_{jq} + \frac{1}{2} \epsilon_{mqn} \delta_{ij} \right) \overline{\varepsilon f_m \partial_q f_n} \right] \end{aligned} \tag{16}$$

$\kappa^{(eff)}_{ij}$ is the effective chiral medium tensor and it is provided by the first order spatial dispersion. We stress that Equation (16), to the best of our knowledge, are the simplest effective dielectric and chiral tensor expressions obtained by means of a first-principles homogenization approach. In fact the two tensors appearing in Equation (16) turn out to only depend on the functions f_i which can be obtained by solving the first of Equation (12) displaying a simple magnetostatic-like structure.

Next, we discuss a semi-analytical method for evaluating the effective electromagnetic parameters. Since the considered composite medium is periodic, we can expand the dielectric permittivity and the potential vector $\mathbf{f}$ in a Fourier series

$$\begin{aligned} \varepsilon &= \sum_{\mathbf{G}} \varepsilon_{\mathbf{G}} e^{i\mathbf{G}\cdot\mathbf{R}} \\ \mathbf{f} &= \sum_{\mathbf{G}} \mathbf{f}_{\mathbf{G}} e^{i\mathbf{G}\cdot\mathbf{R}} \end{aligned} \tag{17}$$

where $\mathbf{G} = \eta \mathrm{g}$ and g runs over all the reciprocal lattice vectors. Inserting the Fourier series of Equation (17) into the first of Equation (12), we obtain

$$\sum_{\mathbf{G},\mathbf{G}'} \varepsilon_{\mathbf{G}'} (\mathbf{G}' + \mathbf{G}) \cdot \mathbf{G} (\mathbf{f}_{\mathbf{G}} \cdot \hat{\mathbf{e}}_j) e^{i(\mathbf{G}'+\mathbf{G})\cdot\mathbf{R}} = i \sum_{\mathbf{G}} \varepsilon_{\mathbf{G}} (\mathbf{G} \cdot \hat{\mathbf{e}}_j) e^{i\mathbf{G}\cdot\mathbf{R}} \tag{18}$$

By introducing the vector $\mathbf{G}'' = \mathbf{G}' + \mathbf{G}$, Equation (18) can be written as

$$\sum_{\mathbf{G}''} \left[\sum_{\mathbf{G}} \varepsilon_{\mathbf{G}''-\mathbf{G}} \mathbf{G}'' \cdot \mathbf{G} (\mathbf{f}_{\mathbf{G}} \cdot \hat{\mathbf{e}}_j) - i \varepsilon_{\mathbf{G}''} (\mathbf{G}'' \cdot \hat{\mathbf{e}}_j) \right] e^{i\mathbf{G}''\cdot\mathbf{R}} = 0 \tag{19}$$

This equation is satisfied when all the Fourier coefficients of $e^{i\mathbf{G}''\cdot\mathbf{R}}$ vanish. As a consequence,

$$\sum_{\mathbf{G}} \varepsilon_{\mathbf{G}''-\mathbf{G}} \mathbf{G}'' \cdot \mathbf{G} (\mathbf{f}_{\mathbf{G}} \cdot \hat{\mathbf{e}}_j) = i \varepsilon_{\mathbf{G}''} (\mathbf{G}'' \cdot \hat{\mathbf{e}}_j) \tag{20}$$

for all $\mathbf{G}''$. Equation (20) is an infinite set of linear algebraic equations for the unknown coefficients $\mathbf{f}_{\mathbf{G}} \cdot \hat{\mathbf{e}}_j$ and it can be solved in principle. On the other hand, by truncating the Fourier series at a suitable order term, one can get numerically a fairly good solution.

3. Extended Landau-Lifshitz-Looyenga Effective-Medium Approach

In this section, as a case admitting full analytical description, we consider the situation where the dielectric contrast is low. More precisely, we show that the first of Equation (12) admits an analytical solution if the zero mean residual of underlying dielectric permittivity is much smaller than its mean value, *i.e.*, if

$$\varepsilon = \overline{\varepsilon} + \tau\Delta\varepsilon \tag{21}$$

where $\tau \ll 1$ and $\tilde{\varepsilon} = \tau\Delta\varepsilon$. In this approximation, we expand the potential field $\mathbf{f}$ in a perturbation series in the small parameter τ up to the second order, namely

$$\mathbf{f} = \mathbf{f}^{(0)} + \tau\mathbf{f}^{(1)} + \tau^2\mathbf{f}^{(2)} \tag{22}$$

Substituting Equations (21,22) into the first of Equation (12) and extracting equations for each order in τ, we get

$$\begin{aligned} \nabla^2_{\mathbf{R}} f_j^{(0)} &= 0 \\ \overline{\varepsilon}\nabla^2_{\mathbf{R}} f_j^{(1)} &= -\partial_j\delta\varepsilon - \nabla_{\mathbf{R}}\cdot(\Delta\varepsilon\nabla_{\mathbf{R}} f_j^{(0)}) \\ \overline{\varepsilon}\nabla^2_{\mathbf{R}} f_j^{(2)} &= -\nabla_{\mathbf{R}}\cdot(\Delta\varepsilon\nabla_{\mathbf{R}} f_j^{(1)}) \end{aligned} \tag{23}$$

In order to solve such differential equations, we consider the Fourier series for $\Delta\varepsilon$ and for the fields $f_j^{(m)}$ $(m = 0, 1, 2)$ which are given by, respectively,

$$\begin{aligned} \Delta\varepsilon &= \sum_{\mathbf{G}\neq\mathbf{0}} \Delta\varepsilon_{\mathbf{G}} e^{i\mathbf{G}\cdot\mathbf{R}} \\ \mathbf{f}^{(m)} &= \sum_{\mathbf{G}} \mathbf{f}_{\mathbf{G}}^{(m)} e^{i\mathbf{G}\cdot\mathbf{R}} \end{aligned} \tag{24}$$

Using the Fourier series of Equation (24), we obtain an explicit solution of the set of Equation (23) which reads

$$f_j = i\tau \sum_{\mathbf{G}\neq\mathbf{0}} \left(\frac{\Delta\varepsilon_{\mathbf{G}}}{\overline{\varepsilon}} \frac{G_j}{|\mathbf{G}|^2} - \tau \sum_{\mathbf{G}'\neq\mathbf{0}} \frac{\Delta\varepsilon_{\mathbf{G}'}\Delta\varepsilon_{\mathbf{G}-\mathbf{G}'}}{\overline{\varepsilon}^2} \frac{(\mathbf{G}\cdot\mathbf{G}')G'_j}{|\mathbf{G}|^2|\mathbf{G}'|^2} \right) e^{i\mathbf{G}\cdot\mathbf{R}} \tag{25}$$

Substituting Equation (25) into Equation (16), the analytical expression for the electromagnetic dielectric and chiral tensors are given by, respectively,

$$\begin{aligned} \varepsilon'_{ij} &= \overline{\varepsilon}\delta_{ij} - \frac{\tau^2}{\overline{\varepsilon}} \sum_{\mathbf{G}\neq\mathbf{0}} \frac{\Delta\varepsilon_{-\mathbf{G}}G_i}{|\mathbf{G}|^2} \left[\Delta\varepsilon_{\mathbf{G}}G_j - \tau \sum_{\mathbf{G}'\neq\mathbf{0}} \frac{\Delta\varepsilon_{\mathbf{K}}\Delta\varepsilon_{\mathbf{G}-\mathbf{G}'}}{\overline{\varepsilon}} \frac{(\mathbf{G}\cdot\mathbf{G}')G'_j}{|\mathbf{G}'|^2} \right] \\ \kappa_{ij} &= i\eta k_0 \frac{\tau^3}{\overline{\varepsilon}^2} \sum_{\mathbf{G}\neq\mathbf{0},\mathbf{G}'\neq\mathbf{0}} \frac{\Delta\varepsilon_{-\mathbf{G}-\mathbf{G}'}\Delta\varepsilon_{\mathbf{G}}\Delta\varepsilon_{\mathbf{G}'}}{|\mathbf{G}|^2|\mathbf{G}'|^2} \times \\ & \left[(\mathbf{G}\cdot\mathbf{G}')\epsilon_{imj}G'_m + \left(1 + 2\frac{\mathbf{G}\cdot\mathbf{G}'}{|\mathbf{G}|^2}\right)\left(\epsilon_{imn}\delta_{jq} + \frac{1}{2}\epsilon_{mqn}\delta_{ij}\right) G_q G_m G'_n \right] \end{aligned} \tag{26}$$

where we have neglected the fourth and higher order terms in τ. In the zero order approximation ($\tau = 0$), the effective permittivity tensor is the average of the "microscopic" one ($\varepsilon'_{ij} \simeq \overline{\varepsilon}\delta_{ij}$), whereas the chiral tensor vanishes. It is interesting to evaluate the dielectric permittivity for an isotropic medium. In this case, the dielectric permittitity is diagonal with identical elements and one can easily prove the relations $\sum_{\mathbf{G}\neq\mathbf{0}}(G_x^2/|\mathbf{G}|^2)\delta\varepsilon_{-\mathbf{G}}\delta\varepsilon_{\mathbf{G}} = \sum_{\mathbf{G}\neq\mathbf{0}}(G_y^2/|\mathbf{G}|^2)\delta\varepsilon_{-\mathbf{G}}\delta\varepsilon_{\mathbf{G}} = \sum_{\mathbf{G}\neq\mathbf{0}}(G_z^2/|\mathbf{G}|^2)\delta\varepsilon_{-\mathbf{G}}\delta\varepsilon_{\mathbf{G}}$. Using these relations and the identities $\sum_{\mathbf{G}\neq\mathbf{0}}(G_x^2+G_y^2+G_z^2)/|\mathbf{G}|^2)\delta\varepsilon_{-\mathbf{G}}\delta\varepsilon_{\mathbf{G}} = \sum_{\mathbf{G}\neq\mathbf{0}}\delta\varepsilon_{-\mathbf{G}}\delta\varepsilon_{\mathbf{G}} = \overline{(\delta\varepsilon^2)}$, after neglecting the third order contribution in τ in the first of Equation (26), the effective dielectric tensor becomes

$$\varepsilon'_{ij} = \delta_{ij}\left[\overline{\varepsilon} - \frac{\overline{(\tilde{\varepsilon}^2)}}{3\overline{\varepsilon}}\right] \tag{27}$$

on account of the isotropy of the electromagnetic response. As stated above, Equation (27) is accurate up to the second order in τ, so that one can write

$$\varepsilon'_{ij} = \delta_{ij}\overline{\varepsilon^{\frac{1}{3}}} \tag{28}$$

The expression of effective permittivity of the Equation (28) coincides with the Landau-Lifshitz-Looyenga (LLL) formula [13,14]. The LLL effective medium approach generally describes the dielectric response of an isotropic and homogeneous finely dispersed mixture, whereas, in this paper, we consider the homogenization of a photonic crystal (or a periodic metamaterial) with underlying low contrast dielectric modulation. As a consequence, the analytical expression of the dielectric and chiral tensors of Equation (26) can be considered the extended version of the LLL formula of the Equation (28) in the situation where the medium is periodic and its effective macroscopic electromagnetic response is bi-anisotropic.

In order to check the predictions of the extend version of LLL approach, we consider a one-dimensional sub-wavelength grating whose effective dielectric and chiral parameters can be evaluated without resorting the low contrast approximation. Specifically, as a theoretical benchmark, we assume the slab grating to be described by the underlying permittivity

$$\varepsilon_r = \overline{\varepsilon} + 2\tau\left[\cos\left(g_0 x_1\right) + \sin\left(2 g_0 x_1\right)\right] \tag{29}$$

where $x_1 = \hat{\mathbf{e}}_1 \cdot \mathbf{r}$, $g_0 = 2\pi/d$ and d is the grating period. In Reference [12], Ciattoni et al. have shown that, for a one-dimensional periodic medium (for which $\varepsilon_r(x_1) = \varepsilon_r(x_1+d)$), the first of Equation (12) can be solved analytically without additional assumptions. According to this approach the effective dielectric and chiral tensors resulting from a general periodic dielectric profile are given by

$$\begin{aligned} \varepsilon_{ij}^{(eff)} &= \overline{\varepsilon}\delta_{ij} + \left[\left(\overline{\varepsilon^{-1}}\right)^{-1} - \overline{\varepsilon}\right]\delta_{i1}\delta_{j1} \\ \kappa_{ij}^{(eff)} &= \eta\kappa_0\epsilon_{ij1} \end{aligned} \tag{30}$$

where

$$\kappa_0 = \left[\overline{\varepsilon^{-1}}\right]^{-1}\frac{2\pi}{\lambda^2}\int_0^\lambda dZ_1\int_0^\lambda dZ_2\frac{\varepsilon(Z_1)}{\varepsilon(Z_2)}\left[\left(\frac{Z_1-Z_2}{\lambda}\right) - \frac{1}{2}\text{sign}\left(\frac{Z_1-Z_2}{\lambda}\right)\right] \tag{31}$$

Note that the effective dielectric tensor in the first of Equation (30) coincides with the well-known result of the standard effective medium theory (EMT) of layered media [15]. In addition, it is worth noting that the expression of κ_0 in Equation (31) can be manipulated and recast in a different form (see [12]) which coincides with the expression reported in the Reference [16] where the one-dimensional homogenization theory up to the first order is considered and numerically checked through full-wave simulations.

Considering the specific dielectric profile of Equation (29) and by using Equation (30) and Equation (31), we obtain the effective permittivity tensor components and the chiral parameter κ_0 predicted by the nonlocal effective medium theory (NEMT). Furthermore, by using Equation (26) for the profile of the dielectric constant of Equation (29), we evaluate the dielectric and chiral tensor in the LLL approach, which are, respectively,

$$\begin{aligned} \epsilon_{ij}^{(eff)} &= \bar{\varepsilon}\delta_{ij} - 4\frac{\tau^2}{\bar{\varepsilon}}\delta_{i1}\delta_{j1} \\ \kappa_{ij}^{(eff)} &= \eta\kappa_0\epsilon_{ij1} \end{aligned} \tag{32}$$

where

$$\kappa_0 = 3\frac{\tau^3}{\bar{\varepsilon}^2} \tag{33}$$

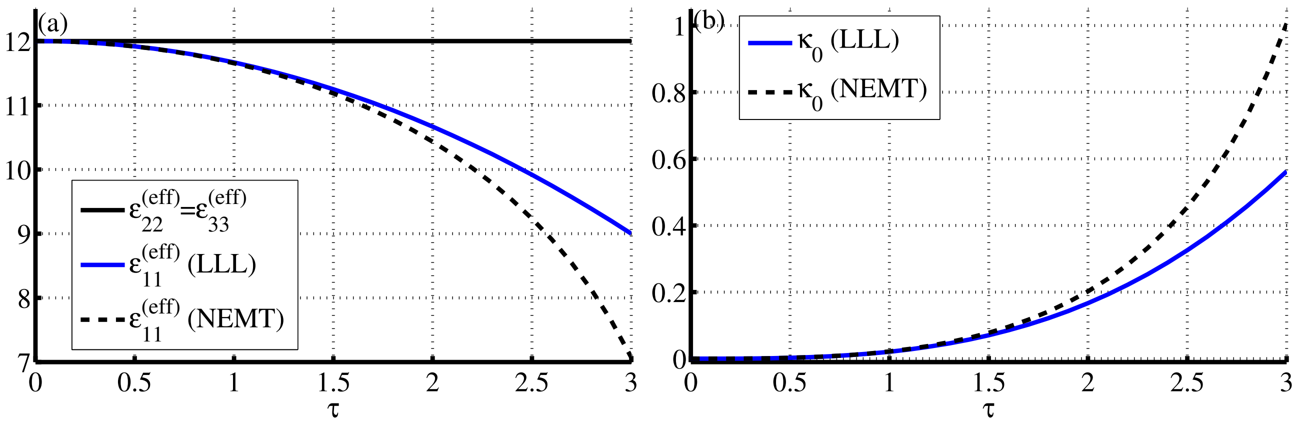

Figure 1. Comparison between effective parameters evaluated from the Landau-Lifshitz-Looyenga (LLL) approach (blue solid line) and from the non-local effective medium theory (NEMT). We have set $\bar{\varepsilon} = 12$, $\eta = 0.1$. (a) Effective dielectric tensor components. (b) Chiral parameter.

In Figure 1a, we plot the effective dielectric tensor component $\varepsilon_{11}^{(eff)}$ predicted by LLL approach (blue solid line) and by the non-local effective medium theory (NEMT) described in Reference [12] (dark dashed line). Note that the dielctric tensor components $\varepsilon_{22}^{(eff)}$, $\varepsilon_{33}^{(eff)}$ coincide with the average $\bar{\varepsilon}$ in both approaches (black solid line in Figure 1a). In Figure 1b, we compare the chiral parameter κ_0 evaluated from the LLL approach (blue solid line) and the NEMT. As expected, the LLL approach is in good agreement with the NEMT in the region where the grating depth is shallow (in this case $\tau < 2$); whereas, for higher values of τ, the LLL approach is not adequate to describe the effective electromagnetic response.

4. Conclusions

In conclusion, we have considered a simple first-principles homogenization theory for describing the effective electromagnetic response of a periodic metamaterial and we have performed the analysis up to the first order of the small ratio between material period and field wavelength. As a consequence, in addition to the effective permittivity tensor, our approach provides a simple way for evaluating the medium chirality tensor. In particular, in the specific situation where the dielectric spatial modulation is shallow, we have deduced an analytical expression for both tensors thus generalizing the Landau-Lifshitz-Looyenga effective-medium formulation to the context of anisotropic metamaterials. It is worth stressing that our approach can easily be generalized to encompass magnetic inclusions as well as non-reciprocal or nonlinear ones.

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Author Contributions

C.R. developed the Fourier space version of the homogenization technique and the LLL approach. A.C. developed the general homogenization technique. Both authors contributed to writing the paper.

Conflicts of Interest

The authors declare no conflict of interest.

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