*Photonics* **2015**, *2*, 449-458; doi:10.3390/photonics2020449

OPEN ACCESS

*photonics*

**ISSN 2304-6732**

www.mdpi.com/journal/photonics

*Article*

# Dispersive Response of a Disordered Superconducting Quantum Metamaterial

**Dmitriy S. Shapiro [1,2,3,*], Pascal Macha [4,5], Alexey N. Rubtsov [1,3] and Alexey V. Ustinov [1,4,6]**

[1] Russian Quantum Center, Novaya St. 100, Skolkovo, Moscow region 143025, Russia; E-Mail: rubtsov@ct-qmc.org (A.N.R.); alexey.ustinov@kit.edu (A.V.U.)

[2] Kotel'nikov Institute of Radio Engineering and Electronics of the Russian Academie of Sciences, Mokhovaya 11/7, Moscow 125009, Russia

[3] Center of Fundamental and Applied Research, N.L. Dukhov All-Russia Institute of Science and Research, Sushchevskaya 22, Moscow 123055, Russia

[4] Physikalisches Institut, Karlsruhe Institute of Technology, Karlsruhe D-76128, Germany; E-Mail: pascal.macha@gmail.com

[5] ARC Centre for Engineered Quantum Systems, University of Queensland, Brisbane, Queensland 4072, Australia

[6] National University of Science and Technology MISIS, Leninsky prosp. 4, Moscow 119049, Russia

* Author to whom correspondence should be addressed; E-Mail: shapiro@cplire.ru; Tel.: +7-495-629-34-59

*Received: 8 April 2015 / Accepted: 23 April 2015 / Published: 27 April 2015*

---

**Abstract:** We consider a disordered quantum metamaterial formed by an array of superconducting flux qubits coupled to microwave photons in a cavity. We map the system on the Tavis-Cummings model accounting for the disorder in frequencies of the qubits. The complex transmittance is calculated with the parameters taken from state-of-the-art experiments. We demonstrate that photon phase shift measurements allow to distinguish individual resonances in the metamaterial with up to 100 qubits, in spite of the decoherence spectral width being remarkably larger than the effective coupling constant. Our simulations are in agreement with the results of the recently reported experiment.

**Keywords:** quantum metamaterial; superconducting qubits; cavity QED; simulation

---

## 1. Introduction

The novel type of quantum metamaterials [1–3] composed of arrays of superconducting two-level systems (qubits) offers a platform for quantum simulators and quantum memories [4,5]. Quantum metamaterials can be employed as test bench for studies of fundamental phenomena as ensemble quantum electrodynamics and spin resonance physics on macroscopic level [6,7]. The flux qubit [8,9], which can be viewed as a artificial atom [10], is a μm-sized superconducting loop with several Josephson junctions acting as nonlinear circuit elements. The strongly anharmonic potential of the flux qubit results in an effective two-level structure of the lowest pair of energy levels of the system, where the excitation frequencies fall into GHz range. Ground and excited levels of the qubit correspond to quantum superpositions of states with opposite directions of macroscopic persistent currents in the qubit loop. When coupled to a photon field in a superconducting cavity, the qubit becomes "dressed" with photons. The experimental studies of superconducting qubit-cavity quantum electrodynamics have shown vacuum Rabi oscillations [11,12], spin echo and Ramsey fringes [13,14], emission of single microwave photons [15], and Lamb shift [16].

While the artificially made ensembles of superconducting qubits have an unavoidable spread of excitation frequencies, in contrast to identical natural atoms, they are easily tunable by varying an external magnetic flux. This fact makes possible observation of the fundamental phenomena as dynamical Casimir effect by applying of a non-stationary external drive [17,18], under which GHz photons are created from cavity vacuum fluctuations. In case of the multi-qubit system, a cm-long cavity is used as a transmission line where the virtual photon exchange provides a long-range qubit-qubit interaction in a sub-wavelength metamaterial. It was shown by Tavis and Cummings [19] that $N$ identical two-level systems coupled to the single photon mode generate a collective enhancement of the coupling constant proportional to $\sqrt{N}$. In a qubit array with disordered values of excitation frequencies photons are also coupled to a collective superradiant mode and decoupled from other $N-1$ subradiant modes. Collective enhancement combined with a possibility of tuning the parameters of the metamaterial by external magnetic field opens new possibilities for quantum information technology. For example, it has been suggested, that by applying gradients of magnetic field along the array of qubits, the system can be operated as a quantum memory with information encoded in collective qubit states [20].

The present work is motivated by recent experiments by Macha *et al.* [1] where the collective states of an array of superconducting flux qubits were probed by microwave photons transmitted through the resonator, in the regime of relatively strong decoherence and disorder in qubit excitation frequencies. The experiment has shown that resonant photon phase shift effect is quite prominent and stable against the decoherence. We perform simulations of the phase shift as a function of external flux that modifies the qubit excitation energies, in the setting similar to Ref. [1]. In our approach, we perform a diagonalization of the Hamiltonian in a single excitation basis. We analyze the phase shift portrait for different number of qubits and various values of disorder in qubit excitation frequencies. We show that photon phase shift measurements are capable to resolve individual resonances for up to $N = 100$ qubits. Our simulations closely reproduce the experimental observations.

## 2. Model

### *2.1. Hamiltonian of Disordered Metamaterial*

Considering the interaction of $N$ qubits with a photon in a resonator (see Figure 1), we start from the Tavis-Cummings Hamiltonian, which includes qubit-qubit interaction term

$$H = \omega_r a^+ a + \sum_{i=1}^{N} \epsilon_i \sigma_i^+ \sigma_i^- + \sum_{i=1}^{N} g_i(\sigma_i^+ a + a^+ \sigma_i^-) + \sum_{i=1}^{N-1} g_{qq,i} \sigma_{i+1}^+ \sigma_i^- \quad (1)$$

Bosonic operators $a^+, a$ in Equation (1) correspond to the photon mode $\omega_r$ in the cavity and the single $i$-th qubit is described in terms of uppering and lowering Pauli operators $\sigma_i^+ = |e\rangle\langle g|$, $\sigma_i^- = |g\rangle\langle e|$ acting on its ground and excited states. The first and the second terms in Equation (1) correspond to photon and qubit excitations of frequencies $\omega_r$ and $\epsilon_i$, the third one is qubit-cavity coupling written in rotating wave approximation. The last term is an effective direct nearest neighbor qubit-qubit interaction.

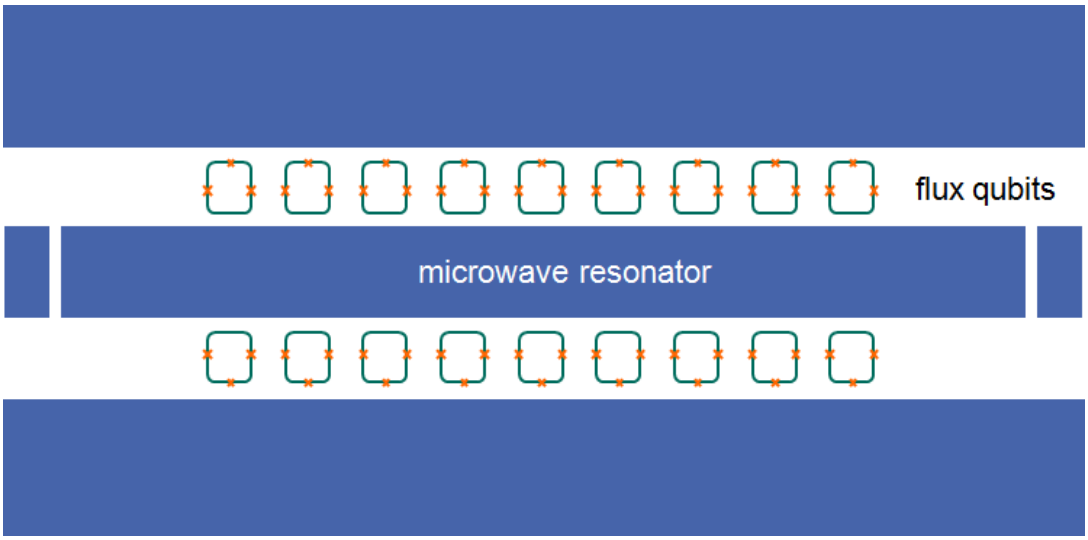


**Figure 1.** Sketch of quantum metamaterial formed by an array of flux qubits embedded into a microwave resonator.

The qubit excitation frequency $\epsilon$ follows from microscopic Josephson and charging energies, $E_J$ and $E_C$, which imply $E_J > E_C$ in the operating regime of flux qubit. When the external magnetic flux threading the qubit loop is equal to half of the magnetic flux quantum $\Phi = \Phi_0/2$, with $\Phi_0 = h/(2e)$, the total energy of the qubit has two symmetric minima related to the opposite circulations of persistent currents $|\downarrow\rangle$ and $|\uparrow\rangle$ where the tunneling rate $\Delta$ between two wells depends on $E_J$ and $E_C$. The condition that $\Delta$ is higher than dephasing rate $\Gamma_\varphi$ allows for quantum superpositions of $|\downarrow\rangle$ and $|\uparrow\rangle$ resulting in two non-degenerate states $|g\rangle, |e\rangle$ being the basis of Equation (1). By detuning the magnetic flux from $\Phi = \Phi_0/2$ one shifts the two minima by the energy

$$\varepsilon_i(\Phi) = \frac{2I_{p,i}}{\hbar}\left(\Phi - \frac{\Phi_0}{2}\right) \quad (2)$$

where $I_{p,i}$ is the $i$-th qubit nominal current [8]. The excitation frequency of $i$-th qubit is given by the Floquet relation

$$\epsilon_i(\Phi) = \sqrt{\Delta_i^2 + \varepsilon_i(\Phi)^2} \tag{3}$$

The qubit-cavity coupling constants in Equation (1) are renormalized bare constants $g_i = \frac{\Delta_i}{\epsilon_i(\Phi)} g_i^{bare}$ written in the rotated basis $|g\rangle, |e\rangle$. Everywhere below we work with the effective constants. Under uniform flux biasing conditions, the spread in $\epsilon_i$ depends mainly on qubit excitation gaps $\Delta_i$, rather than $I_{p,i}$, due to exponential dependence of $\Delta_i \propto \sqrt{E_J/E_C}\exp(-\alpha\sqrt{E_J/E_C})$ on the $E_J/E_C$ ratio which fluctuates from one flux qubit to another. The system under consideration [1] is strongly disordered because the spread of excitation gaps $\Delta_i$ is estimated as high as $\sigma_\Delta \leq 20\%$. The system has an intermediate collective coupling strength value $\Omega$, which is smaller than decoherence rate $\Gamma_\varphi$ but larger than relaxation rate $\kappa$ of the cavity

$$\Gamma_\varphi > \Omega > \kappa, \quad \Omega = \sqrt{\sum_{i=1\ldots N} g_i^2}$$

In the experiment by Macha *et al.* [1], the external magnetic flux $\Phi$ tunes qubit excitation frequencies in resonance with the cavity mode providing the shift of the phase $\varphi(\Phi)$ of a weak external probe signal at cavity mode frequency $\omega_r/2\pi = 7.78$ GHz. From the measured $\varphi(\Phi)$ it was found that number of qubits in an ensemble, which form a collective mode, corresponds to $N = 8$. Relevant parameters of the studied metamaterial are the following: qubit-cavity coupling strength $g_i/2\pi = (1.2 \pm 0.1)$ MHz, dephasing $\Gamma_\varphi/2\pi = 55$ MHz, persistent current $I_p = (74 \pm 1)$ nA and average value of the excitation gap $\Delta/2\pi = 5.6$ GHz. The qubit-qubit interaction, estimated as $g_{qq,i}/2\pi < 1$ MHz, has negligible effect on $\varphi(\Phi)$. The interaction only slightly shifts energies of collective states and does not contribute directly to the collective qubit-cavity coupling strength $\Omega$, which is the most relevant for the phase shift.

### *2.2. Exact Diagonalization Approach*

The qubit dephasing in the system under consideration leads to smearing of the photon density of states which becomes not informative. However, the experimentally measured phase locking effect is quite prominent [1]. In our work here, we focus on the influence of relatively large diagonal disorder in the qubit excitation frequencies and their number $N$ on the phase shift $\varphi(\Phi)$ of the transmitted photons. We calculate the phase shift from a complex phase of the photon Green function $\varphi(\Phi) = \arg \mathcal{D}_\omega$, where $\omega$ is the probe frequency and $\Phi$ is the external flux, counted from the symmetry point value $\Phi_0/2$, which shifts all excitation energies in the qubit ensemble.

Considering approximately resonant regime where the excitation frequencies of qubits and resonator mode are close to each other

$$|\epsilon_i - \omega_r| < \omega_r \tag{4}$$

we find a spectrum of the system in a regime of single excitation by means of exact diagonalization of the Hamiltonian. This regime is realized under experimental conditions due to small microwave probe

power and low temperature of the system. Our solution corresponds to the fully quantum regime where all the variables $\sigma_i^-, a$ are quantum fields.

Below we introduce the basis of $N+1$ states of approximately equal energies which are related to single excitation either in the photon cavity or in one of $N$ qubits. Namely, the state where a single photon is excited and all qubits are in the ground state reads

$$|\mathbf{1}\rangle = \underbrace{|1;0,\ldots,0\rangle}_{N+1} \tag{5}$$

The excitation in 1-st, $i$-th or $N$-th qubits without the photon are given by

$$|\mathbf{2}\rangle = |0;\ 1,0,\ldots,0\rangle$$

$$|\mathbf{i+1}\rangle = |0;\ 0,\ldots,1,\ldots,0\rangle$$

and

$$|\mathbf{N+1}\rangle = |0;\ 0,\ldots,0,1\rangle$$

The Hamiltonian Equation (1) in a matrix form $\mathbf{H}_{i,j} = \langle \mathbf{i}|H|\mathbf{j}\rangle$ in terms of this single excitation basis $|\mathbf{i}\rangle$ reads

$$\mathbf{H}_{i,j} = \begin{pmatrix} \omega_r & g_1 & g_2 & g_3 & g_4 & g_5 & \ldots \\ g_1 & \epsilon_1 & g_{qq,1} & 0 & 0 & 0 & \ldots \\ g_2 & g_{qq,1} & \epsilon_2 & g_{qq,2} & 0 & 0 & \ldots \\ g_3 & 0 & g_{qq,2} & \epsilon_3 & g_{qq,3} & 0 & \ldots \\ g_4 & 0 & 0 & g_{qq,3} & \epsilon_4 & g_{qq,4} & \ldots \\ g_5 & 0 & 0 & 0 & g_{qq,4} & \epsilon_5 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The photon Green function $\mathcal{D}_\omega$ is given by the first diagonal $\{1,1\}$ element of the resolvent matrix $\mathbf{G}$

$$\mathbf{G}_{i,j} = [\omega\delta_{i,j} - \mathbf{H}_{i,j} - \mathrm{diag}(i\kappa, i\Gamma_\varphi, ...)]^{-1}$$

This is the matrix of retarded Green functions, where the last term $\mathrm{diag}(i\kappa, i\Gamma_\varphi, ...)$ introduced in order to take into account the damping in the cavity $\kappa/2\pi = 715$ kHz and dephasing of qubits. We note that the above expression for the photon Green function $\mathcal{D}_\omega$ and $\mathbf{G}_{i,j}$ allowing us to calculate the phase shift coincides with the generic equations obtained earlier by Volkov and Fistul [3].

## 3. Results

### *3.1. Strong Disorder and Large $N$*

In our numerical studies, we model the system with large amount of qubits and in the presence of disorder between them. Namely, we fix metamaterial parameters and plot the phase shift increasing qubit number $N$, but keeping spread of $\Delta_i$ constant, see Figure 2, and vice versa, see Figure 3. The reported results for the $\varphi(\Phi)$ are obtained at the experimental values $\omega/2\pi = \omega_r/2\pi = 7.78$ GHz, $I_p = 74$ nA and the following parameters: decoherence rate $\Gamma_\varphi/2\pi = 33$ MHz, average qubit excitation gaps $\Delta/2\pi = 5.9$ GHz with normal distribution and dispersion $\sigma_\Delta = 3.6\%$, effective qubit-cavity coupling $g/2\pi = 1.1$ MHz with small dispersion $\sigma_g = 1\%$ and $N = 7$ qubits in the ensemble. These parameters give best fit in the experimental regime at small $N$ discussed in the next Section 3.2.

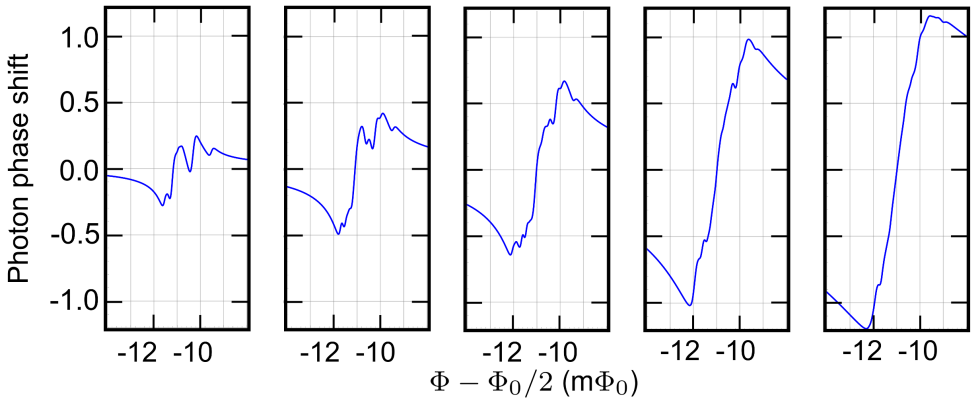


**Figure 2.** Numerical results for the phase shift $\varphi(\Phi)$ at qubit energy gap disorder $\sigma_\Delta = 3.6\%$ of $N = 20, 50, 100, 250, 500$ qubits.

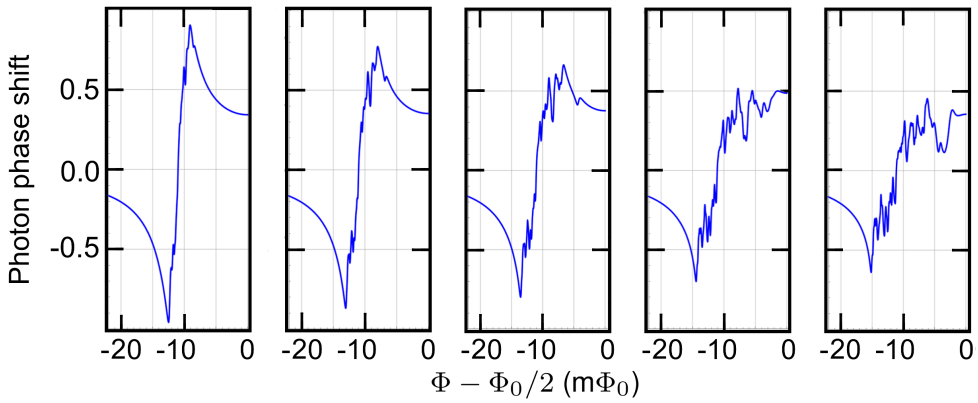


**Figure 3.** Numerical results for the phase shift $\varphi(\Phi)$ of $N = 250$ qubits using five different values of disorder $\sigma_\Delta = 5, 7.5, 10, 15, 25\%$. Parameters of the system: probe frequency $\omega/2\pi = \omega_r/2\pi = 7.78$ GHz and relaxation rate $\kappa/2\pi = 715$ kHz, average excitation gap $\Delta/2\pi = 5.9$ GHz of qubits with normal distribution and dispersion $\sigma_\Delta = 3.6\%$, average coupling constant $g/2\pi = 1.1$ MHz with $\sigma_g = 1\%$, decoherence $\Gamma_\varphi/2\pi = 33$ MHz, persistent current $I_p = 74$ nA.

Figure 2 shows plots of the phase shift dependence on magnetic flux for $N = 20, 50, 100, 250, 500$ qubits in the system and fixed spread $\sigma_\Delta = 3.6\%$ of the normal distribution of $\Delta_i$. The pronounced

irregular peaks in Figure 2 at finite number of qubits, e.g., at $N = 20$, reflects the disordered structure of the eigenmodes. Under the increasing of $N$ in our simulation Figure 2, fluctuations on curves are smearing gradually. One can see that for $N < 100$ the phase shift fluctuates and depends on a particular realization of the qubit frequencies, while for $N > 100$ it appears rather smooth and regular, meaning that the array of qubits can be treated as a continuous system at these parameters. In this limit the phase shift curve depends on the shape of the inhomogenous broadening of qubit frequencies in the ensemble, rather than the particular disorder realization. In Figure 3, we show results for different values of disorder with the following dispersions $\sigma_\Delta = 5, 7.5, 10, 15, 25\%$, keeping qubit number $N = 250$ fixed. One can see from the Figure 3 that, with increasing disorder, the phase shift pictures start to reveal fluctuations. For the realistic spread $\sigma_\Delta = 10\%$ the system remains far from the ensemble limit, even for this relatively large number of qubits. We demonstrate in Figure 3, that effects of finite number of qubits are significant and it requires several hundreds of qubits in a real superconducting metamaterials to form an effectively continuous system showing a regular response functions.

### *3.2. Experimental Regime, Small N*

Next, we compare experimental [1] (Figure 4, left panel) and numerical (Figure 4, right panel) results for the phase shift $\varphi(\Phi)$. The resonator frequency $\omega/2\pi = \omega_r/2\pi = 7.78$ GHz and persistent current $I_p = 74$ nA are fixed. In our numerical method, we assume a normal distribution of the qubit excitation gaps and effective coupling. We selected a parameter distribution with dispersions $\sigma_\Delta = 3.6\%$ and $\sigma_g = 1\%$ for a system containing $N = 7$ qubits with an average excitation gap $\Delta/2\pi = 5.9$ GHz, which closely resembles the experimental data. Subsequently, we fitted the decoherence rate and effective qubit coupling and found $\Gamma_\varphi/2\pi = 33$ MHz and $g/2\pi = 1.3$ MHz, respectively. While the assumed spread is less than the experimentally expected one ($\sigma_\Delta^{exp} < 20\%$) all found parameters are in correspondence with those quoted in [1], where $\Delta^{exp}/2\pi = 5.6$ GHz, $g^{exp}/2\pi = 1.2$ MHz, $\Gamma_\varphi^{exp}/2\pi = 55$ MHz and $N^{exp} = 8$. The values for qubit number and dephasing reported in Ref. [1] were found under the assumption of identical qubits. Here we show, that the experimental data can be reproduced fairly well under the assumption of randomly distributed qubit parameters. In the experiment it was observed, that an ensemble of $N^{exp} = 8$ qubits, resonantly interacting with the cavity mode and monitored over long time, spontaneously dissolved into two sub-ensembles of 4 qubits each, resulting into two jumps in Figure 4 (left panel). The reduction of the peaks amplitude in the central region from $\approx\pm0.15$ to $\approx\pm0.05$ in Figure 4 (left panel) follows from the interference between the tails of the resonant jumps at $\Phi - \Phi_0/2 \approx 10.5, 11.5\ \mathrm{m}\Phi_0$ related to each of two sub-ensembles. In our exact diagonalization procedure we do not find the formation of sub-ensembles but arrive at correspondence with the experimental curves if parameters of the metamaterial and disorder are close to the experimental values.

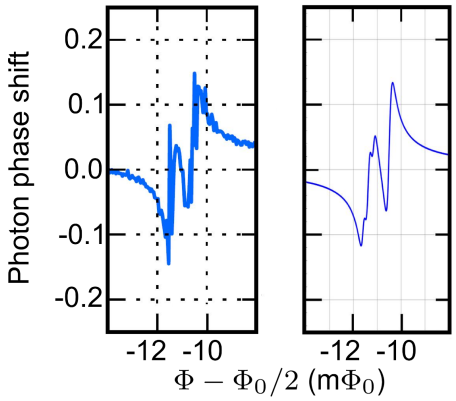


**Figure 4.** Experimental (**left panel**) and numerical (**right panel**) results for the photon phase shift $\varphi(\Phi)$ as a function of external flux bias $\Phi$, calculated for the system of $N = 7$ qubits.

## 4. Conclusions

We theoretically studied the model of a flux qubit array coupled to a cavity, with disorder in qubit excitation frequencies. The system under consideration contains a finite number of qubits and operates in the intermediate regime where disorder range and decoherence rate exceed the effective qubit-cavity coupling. We calculated the photon Green function through the exact diagonalization of the Hamiltonian in the single excitation regime, assuming low power of external microwave driving. We found that the resonant phase shift of a transmitted probe signal shows quantitative correspondence with the experimental data [1]. Variations in phase shift characteristics at different values of the disorder and number of qubits in the system were presented.

## Acknowledgments

This work was supported by the Ministry for Education and Science of Russian Federation under contract No. 11.G34.31.0062 and in the framework of Increase Competitiveness Program of the National University of Science and Technology MISIS under contract No. K2-2014-025.

## Author Contributions

A.N.R., A.V.U. and P.M. contributed to the conception of the work. D.S.S. and A.N.R. performed theoretical analysis and calculations. All the authors contributed to the interpretation of the results and to the preparing and revising of the manuscript.

## Conflicts of Interest

The authors declare no conflict of interest.

## References


1. Macha, P.; Oelsner, G.; Reiner, J.-M.; Marthaler, M.; André, S.; Schön, G.; Hübner, U.; Meyer, H.-G.; Il'ichev, E.; Ustinov, A.V. Implementation of a quantum metamaterial using superconducting qubits. *Nat. Commun.* **2014**, *5*, 5146.
2. Rakhmanov, A.L.; Zagoskin, A.M.; Savel'ev, S.; Nori, F. Quantum metamaterials: Electromagnetic waves in a Josephson qubit line. *Phys. Rev. B* **2008**, *77*, 144507.
3. Volkov, P.; Fistul, M.V. Collective quantum coherent oscillations in a globally coupled array of qubits. *Phys. Rev. B* **2014**, *89*, 054507.
4. DiCarlo, L.; Chow, J.M.; Gambetta, J.M.; Bishop, L.S.; Johnson, B.R.; Schuster, D.I.; Majer, J.; Blais, A.; Frunzio, L.; Girvin, S.M.; *et al.* Demonstration of two-qubit algorithms with a superconducting quantum processor. *Nature* **2009**, *460*, 240.
5. Nation, P. D.; Johansson, J.R.; Blencowe, M.P.; Nori, F. Stimulating uncertainty: Amplifying the quantum vacuum with superconducting circuits. *Rev. Mod. Phys.* **2012**, *84*, 1.
6. Blais, A.; Huang, R.-S.; Wallraff, A.; Girvin, S.M.; Schoelkopf, R.J. Cavity quantum electrodynamics for superconducting electrical circuits: An architecture for quantum computation. *Phys. Rev. A* **2004**, *69*, 062320.
7. You, J.Q.; Nori, F. Atomic physics and quantum optics using superconducting circuits. *Nature* **2011**, *474*, 589.
8. Orlando, T.P.; Mooij, J.E.; Tian, L.; van der Wal, C.H.; Levitov, L.S.; Lloyd, S.; Mazo, J.J. Superconducting persistent-current qubit. *Phys. Rev. B* **1999**, *60*, 15398.
9. Mooij, J.E.; Orlando, T.P.; Levitov, L.; Tian, L.; van der Wal, C.H.; Lloyd, S. Josephson persistent-current qubit. *Science* **1999**, *285*, 1036.
10. Astafiev, O.; Zagoskin, A.M.; Abdumalikov, A.A., Jr.; Pashkin, Y.A.; Yamamoto, T.; Inomata, K.; Nakamura, Y.; Tsai, J.S. Resonance fluorescence of a single artificial atom. *Science* **2010**, *327*, 840.
11. Vijay, R.; Macklin, C.; Slichter, D.H.; Weber, S.J.; Murch, K.W.; Naik, R.; Korotkov, A.N.; Siddiqi, I. Stabilizing Rabi oscillations in a superconducting qubit using quantum feedback. *Nature* **2012**, *490*, 77.
12. Fink, J.M.; Bianchetti, R.; Baur, M.; Göppl, M.; Steffen, L.; Filipp, S.; Leek, P.J.; Blais, A.; Wallraff, A. Dressed Collective Qubit States and the Tavis-Cummings Model in Circuit QED. *Phys. Rev. Lett.* **2009**, *103*, 083601.
13. Bertet, P.; Chiorescu, I.; Burkard, G.; Semba, K.; Harmans, C.J.P.M.; DiVincenzo, D.P.; Mooij, J.E. Dephasing of a Superconducting Qubit Induced by Photon Noise. *Phys. Rev. Lett.* **2005**, *95*, 257002.
14. Wallraff, A.; Schuster, D.I.; Blais, A.; Frunzio, L.; Majer, J.; Devoret, M.H.; Girvin, S.M.; Schoelkopf, R.J. Approaching Unit Visibility of a Superconducting Qubit with Dispersive Readout. *Phys. Rev. Lett.* **2005**, *95*, 060501.
15. Romero, G.; García-Ripoll, J.J.; Solano, E. Microwave photon detector in circuit QED. *Phys. Rev. Lett.* **2009**, *102*, 173602.

16. Fragner, A.; Göppl, M.; Fink, J.M.; Baur, M.; Bianchetti, R.; Leek, P.J.; Blais, A.; Wallraff, A. Resolving Vacuum Fluctuations in an Electrical Circuit by Measuring the Lamb Shift. *Science* **2008**, *322*, 1357.
17. Lähteenmäki, P.; Paraoanua, G.S.; Hasselb, J.; Hakonena, P.J. Dynamical Casimir effect in a Josephson metamaterial. *Proc. Natl. Acad. Sci.* **2011**, *111*, 10485.
18. Wilson, C.M.; Johansson, G.; Pourkabirian, A.; Simoen, M.; Johansson, J.R.; Duty, T.; Nori F.; Delsing, P. Observation of the Dynamical Casimir Effect in a Superconducting Circuit. *Nature* **2011**, *479*, 376.
19. Tavis, M.; Cummings, F.W. Exact solution for an n-molecule-radiation-field hamiltonian. *Phys. Rev.* **1968**, *170*, 379.
20. Julsgaard, B.; Mølmer, K. Fundamental limitations in spin-ensemble quantum memories for cavity fields. *Phys. Rev. A* **2013**, *88*, 062324.

© 2015 by the authors; licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution license (http://creativecommons.org/licenses/by/4.0/).