Abstract: Value at Risk (VaR) is used to illustrate the maximum potential loss under a given confidence level, and is just a single indicator to evaluate risk ignoring any information about income. The present paper will generalize one-dimensional VaR to two-dimensional VaR with income-risk double indicators. We first construct a double-VaR with $(\mu, \sigma^2)$ (or $(\mu, \text{VaR}^2)$) indicators, and deduce the joint confidence region of $(\mu, \sigma^2)$ (or $(\mu, \text{VaR}^2)$) by virtue of the two-dimensional likelihood ratio method. Finally, an example to cover the empirical analysis of two double-VaR models is stated.

Keywords: double-VaR; joint confidence region; $(\mu, \text{VaR}^2)$

MSC: 91B30

1. Introduction

In the early 1990s, the international economic and financial consultancy G30 published the report “derivatives practices and principles” based on the research on financial derivatives, and then proposed Value at Risk (VaR) model to measure the market risk. JP Morgan Bank then launched the VaR risk measurement and control model. Since VaR is accurate and comprehensive to the application of risk measurement and makes up the deficiency of Markowitz mean-variance model, it is generally welcomed by the international financial community, including regulatory authorities, and has become a standard to manage and control financial risk. Furthermore, VaR is widely used to measure credit risks and trading risks.

The biggest benefit of VaR is the ability to critically analyze risk through systematic analysis. The organization can control the front end and back end of the business by calculating VaR to understand the financial risks it faces and establish an independent risk management mechanism.

However, VaR also has its own limitations. Mausser and Rosen (1999) put forward the most obvious limitations of VaR: it does not provide absolute maximum loss value, which can only be expected in a certain confidence level. Another drawback of VaR is that when the calculation is based on historical data, the future situation of any event should be duplicated or fitted by historical data. However, the reality is that we cannot guarantee the future case of an event is just as old as. In addition, some researchers such as Artzner et al. (1999) criticized the VaR model because it does not meet sub-additivity.

The obvious limitation existing in VaR is that it is only used to illustrate the maximum possible loss for the given conditions and is only a single index to characterize the risk, which provides less information to users about other information such as income. In practice, what people usually care about is how much profit they can get while taking risks.
Based on the above considerations, we hope to follow the definition of VaR and Markowitz’s portfolio theory, and then construct a double-VaR, that is, we will extend one-dimensional single-risk monitoring indicator—VaR to two-dimensional revenue-risk monitoring indicators—VaR (or double-VaR for short).

There is abundant literature about VaR. Here, the main research literature involved in this paper is described briefly as follows.

Duffie and Pan (1997) made a detailed background description of VaR, characteristics, applications, and the entire VaR-system. Beder (1995) used eight different models to calculate VaR values and compare them. Some foreign scholars studied mainly the calculation of VaR, such as Jorion (1996); Linsmeier and Pearson (1996); Duffie and Pan (1997); Engle and Manganelli (1999). Based on different situations, they arrived at many calculation skills such as the variance-covariance matrix method, historical simulation, and Monte Carlo simulation method, and the related characteristics of VaR, etc.

After 1999, there are an increasing number of new VaR models in the financial, industrial, and other different applied areas. Potters and Bouchaud (1999) proposed how to use the normality of asset volatilities to calculate the VaR of nonlinear combination. Some other scholars studying the nature of VaR and other risk measurement methods, such as Artzner et al. (1999), proposed VaR does not meet sub-additivity; Wang (1999) studied the characteristics of dynamic risk measures; Mausser and Rosen (1999) proposed that if a small-probability event happened in the case of loss exceeding the VaR, then VaR models cannot measure the size of potential losses; Chen et al. (2014) studied future cash arbitrage with VaR-portfolio problems; Tang et al. (2018) investigated the no-arbitrage problem with VaR-like arguments; Cong and Zhao (2018; 2019) posed a non-cash risk measure and a generalized non-cash risk measure, respectively, which improved in some sense VaR under the distribution of any random variable that is uncertain. In addition, some scholars extended the classical Markowitz mean-variance model to the mean-VaR model; for instance, Pearson (2002) and Jorion (2007) used these models with constraints to manage the risk-profit for a fund company.

In addition, many scholars have proposed a series of improved calculation methods based on different markets and different assumptions. Berkowitz (1999) proposed a new method for evaluation of VaR. Taylor et al. (2000) proposed to use the t distribution to fit the income sequence; Hu (2012) based their research on the mixed Copula model to study the evaluation value of VaR; Ze-To (2013) used the Heath-Jarrow-Morton model to measure the value of VaR, and pointed out that the model can capture the non-normal income distribution well and can accurately provide the value of VaR; Li et al. (2017) confirmed that using the Bootstrap method to calculate VaR and CVaR can effectively improve the estimation accuracy.

Meanwhile, many foreign scholars have also attached great importance to the empirical applications of VaR. Jackson et al. (1997) studied how to apply VaR into bank reserves. Berkowitz and O’Brien (2002) proposed how to evaluate the transaction risk of commercial banks through the prediction accuracy of the VaR model. Basak and Shapiro (2001) analyzed the optimal dynamic portfolio risks with VaR model.

In the present paper, we will extend one-dimensional single-risk monitoring indicator—VaR—to two-dimensional benefit-risk monitoring indicators—double-VaR.

Firstly, a reasonable definition of two-dimensional VaR (double-VaR) is given. For a better understanding of double-VaR we choose mean $\mu$ and variance $\sigma$ as the parameters to build the first double-VaR model. Thus, for a given confidence level $\alpha$, one can not only know the scope of asset risks but also can know their income range. Such indicators are better able to make trade-offs to the risk-return of assets.

Secondly, to solve the first model—double-VaR with respect to $(\mu, \sigma^2)$—we extend the one-dimensional likelihood ratio method to two-dimensional likelihood ratio method, and derive the joint confidence region containing the unknown parameters. Then, we can solve a specific joint confidence region with ideal point method and area minimization method as well as to compare the results of these two methods.
Finally, according to the accuracy theory of VaR we study the double-VaR based on \((\mu, \text{VaR}^2)\) so that for a given confidence interval we cannot only know the biggest value of possible asset loss but we can also get the gain range. In this situation a better trade-off of the asset is possible.

The organization of the paper is as follows. Section 2 introduces some necessary conditions and terminologies. Section 3 is devoted to constructing double-VaR models with \((\mu, \sigma^2)\) and \((\mu, \text{VaR}^2)\). Section 4 confirms that the double-VaR models are effective via some examples.

2. Preliminaries

2.1. VaR

2.1.1. Definitions and Basic Descriptions

**Definition 1.** The basic meaning of VaR is the maximum potential loss of risk assets under normal market conditions, for a given confidence level \(\alpha\) and holding period \(t\). One can describe it as follows

\[
P(\Delta p > \text{VaR}) = \alpha
\]

where \(\Delta p\) is the loss of risk asset \(W\) within the holding period \(t\) and VaR is the value at risk under the confidence level \(\alpha\).

**Remark 1.** Under a normal market environment and a given confidence level \(\alpha\), let the probability distribution density function of a risky asset value be \(f(w)\), the initial value of a risky asset be \(w_0\), the lowest value of a risky asset under a confidence level \(\alpha\) be \(w^*\) and the yield on holding period \(t\) be \(r\), then we have

\[
\text{VaR} = E[w] - w^*
\]

where \(w = w_0(1 + r)\) and \(w^*\) can be achieved with the following two formulas

\[
\alpha = \int_{w^*}^{\infty} f(w)dw \quad \text{or} \quad 1 - \alpha = \int_{-\infty}^{w^*} f(w)dw
\]

In particular, when the distribution of the risk asset yield \(r\) is a normal distribution, that is, \(r \sim N(\mu, \sigma^2)\), then one can get VaR of the risk asset by the following steps:

Let

\[
1 - \alpha = \int_{-\infty}^{-\xi^*} \phi(\epsilon) d\epsilon
\]

where \(\phi\) is the probability density function of a standard normal distribution. According to (2) and (3), we can get

\[
\int_{-\infty}^{-\xi^*} \phi(\epsilon) d\epsilon = \int_{-\infty}^{w^*} f(w)dw = \int_{-\infty}^{r^*} g(r)dr
\]

where \(g(r)\) is the normal probability density function of the risk asset with yield \(r\). Since one has

\[
\int_{-\infty}^{-\xi^*} \phi(\epsilon) d\epsilon = \int_{-\infty}^{w^*} \frac{1}{\sqrt{2\pi}} e^{-\frac{\epsilon^2}{2}} d\epsilon
\]

and

\[
\int_{-\infty}^{r^*} g(r)dr = \int_{-\infty}^{r^*} \frac{1}{\sqrt{2\pi}} e^{-\frac{(r-\mu)^2}{2\sigma^2}} dr = \int_{-\infty}^{t^*} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt
\]

where \(t = \frac{r-\mu}{\sigma}\) and \(t \sim N(0, 1)\). We arrive here by (4) and (5)

\[
r^* = \mu - \sigma \xi^*
\]
Substitute (7) into (1), we then have

\[ \text{VaR} = E[w] - w^* = w_0 + w_0 E[r] - (w_0 + w_0 r^*) = w_0 \sigma \xi \]

that is, there holds

\[ \text{VaR} = w_0 \sigma \xi \] \quad (8)

More generally, if one considers the time factor for VaR, then the Formula (8) can be written as

\[ \text{VaR} = w_0 \sigma \xi \sqrt{\Delta t} \]

In other words, when the yield on a risk asset follows a normal distribution, VaR measure of the risk is equivalent to the variance measure.

2.1.2. Properties of VaR

VaR has the following properties:

1. Transformation invariance
   For any \( c \in \mathbb{R} \) and positive \( x \), there holds \( \text{VaR}(x + c) = \text{VaR}(x) + c \).
2. Positive homogeneity
   For any \( c > 0 \), there holds \( \text{VaR}(cx) = c \text{VaR}(x) \).
3. Co-monotonic additivity
   For any \( x, y \) is co-monotonic, there holds \( \text{VaR}(x + y) = \text{VaR}(x) + \text{VaR}(y) \).
4. First-order stochastic dominance
   For \( x, y \), if the first order of \( x \) is better than that of \( y \), there holds \( \text{VaR}(x_1) \leq \text{VaR}(x_2) \).
5. Discontinuity on the confidence level
   With respect to the confidence level \( 1 - \alpha \), VaR is not continuous.
6. Convexity is not satisfied
   This property means that the local minimizer of an optimizing problem with VaR as an objective function is not unique.
7. Sub-additivity is not satisfied
   This means that VaR for a portfolio is not less than the sum of VaR of all risky assets.

3. Double-VaR

3.1. Introduction of Double-VaR

We now chose the mean and variance as two proposed parameters-based on Markowitz’s portfolio theory to form a revenue-risk region \( D \). Then we can define VaR-like as follows

\[ P\{ (\mu, \sigma^2) \in D \} = 1 - \alpha \]

**Remark 2.** It is not hard to speculate that the role of the revenue-risk region \( D \) in fact is similar to that of VaR. For convenience, we may call the boundary (or partial boundary) of \( D \) a double-VaR with respect to indexes \( \mu \) and \( \sigma^2 \). A detailed definition of double-VaR will be given later.

Its real economic significance is under the normal market environment and a given confidence level, an area in which the maximum possible loss (expressed by \( \text{var} \sigma^2 \)) and the benefits (expressed by mean \( \mu \)) of an asset within a certain time in the future falls.

3.2. Double-VaR Model with Respect to \((\mu, \sigma^2)\)

3.2.1. Two-Dimensional Likelihood Ratio Argument

Chen and Jiang (2017) proposed and studied a high-dimensional likelihood method for normal distribution. However, for the use of the two-dimensional likelihood method for the derivation of the
joint confidence domain, we have not found relevant literature. Thus, we innovate based on learning their ideas to solve the two-dimensional joint confidence region for double-VaR problem.

Assume that the total distribution of $\xi$ follows a distribution whose density function is $f(x; \theta)$, where the parameters $\theta = (\theta_1, \theta_2)$ are unknown. For the given sample observations, it is easy to get the likelihood function:

$$L(\theta_1, \theta_2) = L(x_1, x_2, \cdots, x_n; \theta_1, \theta_2) = \prod_{i=1}^{n} f(x_i; \theta).$$

If the maximum likelihood estimation of $(\theta_1, \theta_2)$ is $(\hat{\theta}_1, \hat{\theta}_2)$ that is, $L(\hat{\theta}_1, \hat{\theta}_2) = \sup_{\theta \in \Theta} L(\theta_1, \theta_2)$, $\Theta$ is the parameter space of $\theta$, the likelihood ratio can be defined as

$$R = R(x_1, x_2, \cdots, x_n; \theta_1, \theta_2) = \frac{L(\hat{\theta}_1, \hat{\theta}_2)}{L(\theta_1, \theta_2)}.$$

For a given confidence level, the joint confidence region $D$ of the parameter $\theta = (\theta_1, \theta_2)$ can be calculated by $R$.

When the total $\xi \sim N(\mu, \sigma^2)$, where $(x_1, x_2, \cdots, x_n)$ is a set of sample values and the confidence level is $1 - a$, the joint confidence region of $(\mu, \sigma^2)$ can be gained by the following two-dimensional likelihood ratio, where the unknown parameter is $\theta = (\mu, \sigma^2)$.

In fact, we have known the maximum likelihood estimations of mean $\mu$ and variance $\sigma^2$ are respectively by $\bar{x} = \sum_{i=1}^{n} x_i$ and $\hat{\sigma}^2 = s^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2$. Then we get

$$L(\mu, \sigma^2) = (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left\{- \frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \bar{x})^2\right\},$$

$$L(\bar{x}, \hat{\sigma}^2) = (2\pi\hat{\sigma}^2)^{-\frac{n}{2}} \exp\left\{- \frac{1}{2\hat{\sigma}^2} \sum_{i=1}^{n} (x_i - \bar{x})^2\right\}.$$

Next we can get the likelihood ratio

$$R = \frac{L(\hat{\theta}_1, \hat{\theta}_2)}{L(\theta_1, \theta_2)} = \left(\frac{s^2}{\hat{\sigma}^2}\right)^{\frac{n}{2}} \exp\left\{- \frac{1}{2\hat{\sigma}^2} \sum_{i=1}^{n} (x_i - \bar{x})^2 + \frac{1}{2\hat{\sigma}^2} \sum_{i=1}^{n} (x_i - x)^2\right\}$$

(9)

Since

$$\sum_{i=1}^{n} (x_i - \mu)^2 = \sum_{i=1}^{n} x_i^2 + n\mu^2 - 2\mu \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i^2 + n\mu^2 - 2n\mu x = \sum_{i=1}^{n} x_i^2 - nx^2 + n(x - \mu)^2.$$

Denote by $\frac{\bar{x} - \mu}{\hat{\sigma}/\sqrt{n}}, \frac{\hat{\sigma}^2}{\sigma^2} \approx \bar{M}$ and $\frac{n\hat{\sigma}^2}{\sigma^2} \approx T$, thus we get the following

$$- \frac{1}{2\hat{\sigma}^2} \sum_{i=1}^{n} (x_i - \mu)^2 = - \frac{1}{2\hat{\sigma}^2} \sum_{i=1}^{n} (x_i^2 - nx^2) - \frac{n}{2\hat{\sigma}^2} (x - \mu)^2 = - \frac{1}{2\hat{\sigma}^2} \sum_{i=1}^{n} (x_i^2 - nx^2) - \frac{1}{2} M^2.$$
According to the meaning of likelihood ratio, let
\begin{align*}
R &= \left(\frac{T}{n}\right)^{\frac{n}{2}} \exp \left\{ -\frac{1}{2} \sum_{i=1}^{n} \left( x_i^2 - n \bar{x}^2 \right) - \frac{1}{2} T^2 + \frac{n}{2} \right\} \\
&= \left(\frac{T}{n}\right)^{\frac{n}{2}} \left( -\frac{1}{2} \sum_{i=1}^{n} (x_i^2 - n \bar{x}^2)T \right) \frac{n}{2} - \frac{1}{2} T^2 + \frac{n}{2} \\
&= \left(\frac{T}{n}\right)^{\frac{n}{2}} \exp \left\{ -\frac{1}{2} T - \frac{1}{2} M^2 + \frac{n}{2} \right\}
\end{align*}
\begin{align*}
\frac{\partial R}{\partial T} &= n^{-\frac{n}{2}} \left( \frac{n}{2} T^{\frac{n}{2}-1} - \frac{1}{2} T^\alpha \right) \exp \left\{ -\frac{1}{2} T - \frac{1}{2} M^2 + \frac{n}{2} \right\},
\end{align*}
and \( T \neq 0 \), if we let \( \frac{\partial R}{\partial T} = 0 \), then there holds \( T_0 = n \). Since
\begin{align*}
\frac{\partial R}{\partial M} &= \left(\frac{T}{n}\right)^{\frac{n}{2}} (-M) \exp \left\{ -\frac{1}{2} T - \frac{1}{2} M^2 + \frac{n}{2} \right\},
\end{align*}
and \( M \neq 0 \). Let \( \frac{\partial R}{\partial M} = 0 \), we arrive at \( M_0 = 0 \). For convenience, we denote by
\begin{align*}
A &= \frac{\partial^2 R}{\partial M^2} \bigg|_{(M_0, T_0)} = \left(\frac{T}{n}\right)^{\frac{n}{2}} \left( M^2 - 1 \right) \exp \left\{ -\frac{1}{2} T - \frac{1}{2} M^2 + \frac{n}{2} \right\} = -1,
B &= \frac{\partial^2 R}{\partial T \partial M} \bigg|_{(M_0, T_0)} = n^{-\frac{n}{2}} \left( \frac{n}{2} T^{\frac{n}{2}-1} - \frac{1}{2} T^\alpha \right) (-M) \exp \left\{ -\frac{1}{2} T - \frac{1}{2} M^2 + \frac{n}{2} \right\} = 0,
C &= \frac{\partial^2 R}{\partial T^2} \bigg|_{(M_0, T_0)} = n^{-\frac{n}{2}} \exp \left\{ -\frac{1}{2} T - \frac{1}{2} M^2 + \frac{n}{2} \right\} \left[ \left( \frac{n}{2} T^{\frac{n}{2}-1} - \frac{1}{2} T^\alpha \right) (-1) \right. \\
&\left. + \frac{n}{2} (\frac{n}{2} - 1) T^\alpha - 2 - \frac{n}{4} T^\alpha \right] = -\frac{1}{2n}.
\end{align*}
It is obvious that there holds
\begin{align*}
AC - B^2 &= \frac{1}{2n} - 0 > 0.
\end{align*}
By a sufficient condition of two-dimensional extreme points, we can know \( R \) has a strict maximum at the point \( (M_0, T_0) \). Thus, for the constant \( C \), there exists a region \( D \) so that \( P \{ R \geq C \} = P \{ (M, T) \in D \} \).
According to the meaning of likelihood ratio, let \( P \{ R \geq C \} = 1 - \alpha \), then \( P \{ (M, T) \in D \} = 1 - \alpha \). Since
\begin{align*}
M \sim N(0, 1), \ T \sim \chi^2_{\alpha-1},
\end{align*}
there exist three positive constants \( a, b \) and \( c (b < c) \), so that
\begin{align*}
P \{ (M, T) \in D \} = P \{ |M| \leq a, b \leq T \leq c \} = P \{ \frac{x - \mu}{\sigma / \sqrt{n}} \leq a, b \leq \frac{ns^2}{\sigma^2} \leq c \} = 1 - \alpha.
\end{align*}
Since two statistics \( M, T \) are \( i.i.d \), there holds
\begin{align*}
P \{ \frac{x - \mu}{\sigma / \sqrt{n}} \leq a, b \leq \frac{ns^2}{\sigma^2} \leq c \} = P \{ \frac{x - \mu}{\sigma / \sqrt{n}} \leq a \} \cdot P \{ b \leq \frac{ns^2}{\sigma^2} \leq c \} = 1 - \alpha
\end{align*}
(10)
That is
\begin{align*}
(2\Phi(a) - 1)(\chi^2_{\alpha-1}(c) - \chi^2_{\alpha-1}(b)) = 1 - \alpha,
\end{align*}
where $\chi^2_{n-1}(b)$ is the function value of $\chi^2_{n-1}$ distribution whose freedom is $n - 1$ at the point $b$. Arrange (10) and get the joint confidence region of $(\mu, \sigma^2)$ as below

$$\{(\mu, \sigma^2) : x - \frac{a \sigma}{\sqrt{n}} \leq \mu \leq x + \frac{a \sigma}{\sqrt{n}}, \frac{ns^2}{c} \leq \sigma^2 \leq \frac{ns^2}{b}\}$$

(11)

Obviously, there are unknowns $a$, $b$, and $c$ in the joint confidence domain under the confidence level required by this paper. Therefore, to find these three unknowns, we can determine the specific joint confidence domain after given $n$ and sample variance.

**Definition 2.** Assume that $X \in \mathbb{R}^n$ is a random vector, and $(\mu, \sigma^2)$ are two index parameters, for given confidence level $\alpha$ ($0 \leq \alpha \leq 1$), one can define double-VaR of $X \in \mathbb{R}^n$ with respect to indexes $(\mu, \sigma^2)$ as

$$\text{VaR}_{(\mu, \sigma^2)}^\alpha(X) = \partial\{(\mu, \sigma^2) : x - \frac{a \sigma}{\sqrt{n}} \leq \mu \leq x + \frac{a \sigma}{\sqrt{n}}, \frac{ns^2}{c} \leq \sigma^2 \leq \frac{ns^2}{b}\}.$$

In the above definition, $a$, $b$, $c$ are unknown. However, in Section 3.2.2, we use the ideal method to find the joint confidence domain. Thus, the unknown parameters $a$, $b$, $c$ are solved. We can look up Tables 1 and 2 to find the value of $a$, $b$, $c$.

**Remark 3.** Noticing that the joint confidence region (see Figure 1 below) and combining with Markowitz’s portfolio theory, the right boundary of the banded region in Figure 1 can be regarded the double-VaR, $\text{VaR}_{(\mu, \sigma^2)}^\alpha(X)$, with $(\mu, \sigma^2)$.

That is to say, one can have the following two basic understandings:

1. For a given risk level $\frac{ns^2}{d}$ ($b \leq d \leq c$), $\text{VaR}_{(\mu, \sigma^2)}^\alpha(X) = \text{VaR}_{(\mu, \sigma^2)}^\alpha(\mu; X)$ can be regarded as VaR in the sense of Markowitz’s portfolio with a given confidence level $\alpha$;
2. For a given benefit level $\mu$ ($\geq \bar{x}$), $\text{VaR}_{(\mu, \sigma^2)}^\alpha(X) = \text{VaR}_{(\mu, \sigma^2)}^\alpha(\sigma^2; X)$ can also be regarded as VaR in the sense of Markowitz’s portfolio with a given confidence level $\alpha$.

3.2.2. Solution to Joint Confidence Region on $(\mu, \sigma^2)$

We adopt the so-called ideal point method to solve the joint confidence region.

$$\{(\mu, \sigma^2) : x - \frac{a \sigma}{\sqrt{n}} \leq \mu \leq x + \frac{a \sigma}{\sqrt{n}}, \frac{ns^2}{c} \leq \sigma^2 \leq \frac{ns^2}{b}\}.$$

(12)

It is well known that the joint confidence region of $(\mu, \sigma^2)$ is just as (12). It is now we evaluate $a$, $b$, and $c$.

Firstly, one can analyze Figure 1 of this joint confidence region of $(\mu, \sigma^2)$ as below:

![](image1.png)

**Figure 1.** The joint confidence region of $(\mu, \sigma^2)$. 
Given the confidence level $1 - \alpha$, to minimize the area of this joint confidence region, it implies that if the two straights $\sigma^2 = \frac{n \beta}{T}$ and $\sigma^2 = \frac{n \beta}{T}$ deciding the size of shaded area are closed enough, which shows $\min \frac{n \beta}{T}$ and $\max \frac{n \beta}{T}$. At the same time, we hope to make the parabola narrower, that is, one considers the problem: $\max \frac{\sigma^2}{\nu}$ and $\min \frac{\sigma^2}{\nu}$. The parabolic $\sigma^2 = \frac{\beta}{n}(x - \mu)^2$ can be regarded as an effective frontier of Markowitz portfolio in $(\mu, \sigma^2)$-coordinate system.

Then we need to solve the following optimization model

$$
\begin{align*}
\max & \frac{\sigma^2}{\nu}, \\
\min & \frac{\sigma^2}{\nu}, \\
\max & \frac{\sigma^2}{\nu}, \\
\text{s.t.} & (2\Phi(a) - 1)(\chi^2_{n-1}(c) - \chi^2_{n-1}(b)) = 1 - \alpha, \\
& a, b, c \geq 0.
\end{align*}
$$

(13)

To solve problem (13), we first refer consider three single-objective optimization models, respectively, as follows

$$
\begin{align*}
\max & \frac{\sigma^2}{\nu}, \\
\text{s.t.} & (2\Phi(a) - 1)(\chi^2_{n-1}(c) - \chi^2_{n-1}(b)) = 1 - \alpha, \\
& a, b, c \geq 0.
\end{align*}
$$

(14)

$$
\begin{align*}
\min & \frac{\sigma^2}{\nu}, \\
\text{s.t.} & (2\Phi(a) - 1)(\chi^2_{n-1}(c) - \chi^2_{n-1}(b)) = 1 - \alpha, \\
& a, b, c \geq 0.
\end{align*}
$$

(15)

$$
\begin{align*}
\max & \frac{\sigma^2}{\nu}, \\
\text{s.t.} & (2\Phi(a) - 1)(\chi^2_{n-1}(c) - \chi^2_{n-1}(b)) = 1 - \alpha, \\
& a, b, c \geq 0.
\end{align*}
$$

(16)

After knowing $n$ and the variance of samples $S^2$, these three optimization problems can be simplified as below

$$
\begin{align*}
\max & \frac{1}{\nu}, \\
\text{s.t.} & (2\Phi(a) - 1)(\chi^2_{n-1}(c) - \chi^2_{n-1}(b)) = 1 - \alpha, \\
& a, b, c \geq 0.
\end{align*}
$$

(17)

Now assume that the solutions of these three problems above are respectively denoted by $\bar{a} = \bar{a}$, $\bar{b} = \bar{b}$, $\bar{c} = \bar{c}$. We set $(\bar{a}, \bar{b}, \bar{c})$ and call it an ideal point of (13), but it is obvious that the ideal point is not necessarily a solution to (13). In fact, we want to get the Euclidean distance between this extremum point and the ideal point is as small as possible. That is, we wish to consider the following

$$
\begin{align*}
\min & \sqrt{(a - \bar{a})^2 + (b - \bar{b})^2 + (c - \bar{c})^2}, \\
\text{s.t.} & (2\Phi(a) - 1)(\chi^2_{n-1}(c) - \chi^2_{n-1}(b)) = 1 - \alpha, \\
& a, b, c \geq 0.
\end{align*}
$$
There is also a point that deserves to be noticed in the theory of probability and statistics when the variance of population is unknown; estimating the mean of population can be divided into two cases:

(1) When the number of sample observations is less than 30, we often choose $t$ distribution to make a parameter estimation of the mean; when the number of sample observations is more than 30, we chose the normal distribution to estimate the parameters. To have a more accurate estimation the number of observations of more than 30 in this article, so we will use the normal distribution to estimate the mean in the process of solving the joint confidence region.

In addition, we notice that freedom $n$ is usually less than 45 in any $\chi^2$ distribution, because when $n$ is more than 45 it is close to a normal distribution $N(n, 2n)$. Thus, this article first studies the case when $30 < n < 45$.

(2) When the number of sample observations is $n > 45$, we will make a further study.

Now we can solve the above optimization problem (17) with fmincon function in MATLAB and get the unknowns $a, b$, and $c$ when the confidence level is respectively 99%, 95%, and 90% for $30 < n < 45$. By the way, the value $zz$ ($zz = a(b^{-\frac{1}{2}} - c^{-\frac{1}{2}})$) used later is also gained as below Tables 1 and 2.

**Table 1.** Ideal point method (confidence level is 99% and 95%).

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3.3. Double-VaR Model Based on \((\mu, \text{VaR}^2)\)

According to the theory of VaR, we will consider the joint confidence region of \((\mu, \text{VaR}^2)\) is reasonable and significant.

3.3.1. Accuracy Measurement of VaR

By the description in Section 2.1.1, we can know \(\text{VaR} = w_0 \sigma \xi^*\), that is, if the initial value of asset and the confidence level has been known, VaR is only related to standard deviation \(\sigma\). Because \(\sigma\) is decided by the choice of sample observations there exits statistical errors in the solving of VaR. Different lengths of confidence sections provide different accuracy of VaR. It is very important to get the length of the confidence section.

3.3.2. \((\mu, \text{VaR}^2)\)-Model

According to the definition of joint confidence region we have

\[
P\{ (\mu, \text{VaR}^2) \in D \} = 1 - \alpha
\]

where \(D\) is the joint confidence region of \((\mu, \text{VaR}^2)\) when the confidence level \(1 - \alpha\) has been known. In particular, the given confidence level of the joint confidence region of \((\mu, \text{VaR}^2)\) is not related to the confidence level of VaR itself \(1 - \beta\). If the confidence level of VaR is 99% and the confidence level of the joint confidence region of \((\mu, \text{VaR}^2)\) is 95%, the object studied in this article can be explained as: the return of an asset in one day \(\mu\) and the biggest 99% loss of VaR are in the area \(D\) with the probability 95%.

When the total \(\xi \sim N(\mu, \sigma^2)\), \((x_1, \cdots, x_n)\) are a set of sample values and the confidence level is \(1 - \alpha\), the joint confidence region of \((\mu, \text{VaR}^2)\) can be gained by the likelihood ratio method.

In fact, by the argument of maximum likelihood estimations, the mean \(\mu\) and variance \(\sigma^2\) are estimated as \(\hat{\mu} = \bar{x}, \hat{\sigma}^2 = S^2 = \frac{1}{n} \sum \limits_{i=1}^{n} (x_i - \bar{x})^2\). Since \(\text{VaR} = w_0 \sigma \xi^*\) (where \(\xi^*\) is the quantile of standard normal distribution when the confidence level of VaR itself is \(1 - \beta\) and \(\xi^*\) can be known by checking

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the corresponding table if $\beta$ is given), if $w_0$ and $\xi^*$ are known we can get the maximum likelihood estimation of $VaR^2$ as below

$$\hat{VaR}^2 = w_0^2 \xi^* = (w - 0\xi^*)^2 \sigma^2 = (w_0 \xi^*)^2 S^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2.$$  

Likelihood function is

$$L(\mu, \sigma^2) = \left( \frac{2 \pi \sigma^2}{2} \right)^{-\frac{n}{2}} \exp\left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2 \right\}$$

and likelihood ratio is

$$R = \frac{L(\mu, \sigma^2)}{L(\hat{\mu}, \hat{\sigma}^2)} = \left( \frac{S^2}{\hat{\sigma}^2} \right)^{\frac{n}{2}} \exp\left\{ -\frac{1}{2\hat{\sigma}^2} \sum_{i=1}^{n} (x_i - \mu)^2 + \frac{1}{2S^2} \sum_{i=1}^{n} (x_i - \bar{x})^2 \right\} \quad (18)$$

Similarly, we get the joint confidence region $D$ of $(\mu, VaR^2)$ as below.

**Definition 3.** Assume that $X \in \mathbb{R}^n$ is a random vector, and $(\mu, VaR^2)$ are two index parameters, for given confidence level $\alpha (0 \leq \alpha \leq 1)$, one can define double-VaR of $X \in \mathbb{R}^n$ with respect to indexes $(\mu, VaR^2)$ as

$$VaR^\alpha_{(\mu, VaR^2)}(X) = \partial \left\{ (\mu, VaR^2) : \bar{x} - \frac{\alpha \sigma}{\sqrt{n}} \leq \mu \leq \bar{x} + \frac{\alpha \sigma}{\sqrt{n}}, (w_0 \xi^*)^2 \frac{nS^2}{c} \leq VaR^2 \leq (w_0 \xi^*)^2 \frac{nS^2}{b} \right\} \quad (19)$$

From (18), one can define the so-called double-VaR with respect to $(\mu, VaR^2)$ as below.

**Remark 4.** Noticing the joint confidence region (see Figure 2) and combining with Markowitz’s portfolio theory, the right boundary of the banded region in Figure 2 can be regarded the double-VaR, $VaR^\alpha_{(\mu, VaR^2)}(X)$, with $(\mu, VaR^2)$. That is to say, one can have the following two basic understandings:

![Figure 2. The joint confidence region of $(\mu, VaR^2)$.](image-url)
(1) For a given risk level \((w_0 \xi)^2 \frac{nS^2}{\xi} (b \leq d \leq c)\), \(\text{VaR}^n_{(\mu, \text{VaR})^2}(X) \equiv \text{VaR}^n_{(\mu, \text{VaR})^2}(\mu; X)\) can be regarded as \(\text{VaR}\) in the sense of Markowitz’s portfolio with a given confidence level \(\alpha\);

(2) For a given benefit level \(\mu (\geq \xi)\), \(\text{VaR}^n_{(\mu, \text{VaR})^2}(X) \equiv \text{VaR}^n_{(\mu, \text{VaR})^2}(\mu; X)\) can also be regarded as \(\text{VaR}\) in the sense of Markowitz’s portfolio with a given confidence level \(\alpha\).

Clearly, we have

\[
P\{(\mu, \text{VaR})^2 \in D\} = P\left\{\frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \leq a, b \leq \frac{nS^2}{\sigma^2} \leq c\right\}
= P\left\{\frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \leq a\right\} \cdot P\left\{b \leq \frac{nS^2}{\sigma^2} \leq c\right\} = 1 - \alpha
\]

That is, there holds

\[
(2\Phi(a) - 1)(\chi^2_{n-1}(c) - \chi^2_{n-1}(b)) = 1 - \alpha,
\]

where \(\chi^2_{n-1}(b)\) (or \(\chi^2_{n-1}(c)\)) is the function value of \(\chi^2\) distribution at the point \(b\) (or \(c\)) when the freedom is \(n - 1\). Now we solve the unknowns \(a, b\) and \(c\) by area minimization method.

The area of joint confidence region of \((\mu, \text{VaR})^2\) can be calculated by double integrals:

\[
\int_{\bar{x}}^{\frac{\mu}{\sigma}} \int_{\Delta + \frac{\xi^2}{\sigma^2}}^{\Delta + \frac{\xi^2}{\sigma^2}} d(\mu\Delta)d(\sigma^2\Delta) = (\Delta^2) \int_{\bar{x}}^{\frac{\mu}{\sigma}} \frac{2a\xi^2\Delta}{\sqrt{n}} d\sigma^2 = \frac{4}{3} nS^3(\Delta^4) a(b^{\frac{3}{2}} - c^{\frac{3}{2}})
\]

Thus, after knowing \(n, \sigma, S^2, w_0, \xi\), we can solve the joint confidence region of \((\mu, \text{VaR})^2\) by the following optimization problem:

\[
\begin{aligned}
\min & \quad \frac{4}{3} n(w_0 \xi)^2 S^3 a(b^{\frac{3}{2}} - c^{\frac{3}{2}}) \\
\text{s.t.} & \quad (2\Phi(a) - 1)(\chi^2_{n-1}(c) - \chi^2_{n-1}(b)) = 1 - \alpha
\end{aligned}
\]

Similarly, we can now solve the above optimization problem with fsolve function in MATLAB and get the unknowns \(a, b, c\) when the confidence level is respectively 99\%, 95\%, and 90\% for \(30 < n < 45\). By the way, the value \(zz\) (\(zz = a(b^{\frac{3}{2}} - c^{\frac{3}{2}})\)) belonging to the area \(\frac{4}{3} nS^3 a(b^{\frac{3}{2}} - c^{\frac{3}{2}})\) is also be gained as below:

At the same time, one can now solve the above optimization problem with fmincon function to solve constrained nonlinear minimization problem in MATLAB and get the unknowns \(a, b, c\) when the confidence level is respectively 99\%, 95\%, and 90\% for \(n > 45\). Incidentally, the value \(zz\) (\(zz = a(b^{\frac{3}{2}} - c^{\frac{3}{2}})\)) belonging to the area \(\frac{4}{3} nS^3 a(b^{\frac{3}{2}} - c^{\frac{3}{2}})\) can also be gained as below.

We can see the result of this model is similar to that under the area minimizing model; then there holds the following:

**Theorem 1.** If the confidence levels are respectively 99\%, 95\%, 90\%, \(30 < n < 45\). At the same time, the mean \(\bar{x}\), the variance \(S^2\) of samples, \(w_0\) and \(\xi^*\) have been known. We can inquire of the unknowns \(a, b, c\) in Tables 3 and 4 and put them into

\[
\{ (\mu, \text{VaR})^2 : \bar{x} - \frac{a\sigma}{\sqrt{n}} \leq \mu \leq \bar{x} + \frac{a\sigma}{\sqrt{n}}, (w_0 \xi^*)^2 \frac{nS^2}{c} \leq \text{VaR}^2 \leq (w_0 \xi^*)^2 \frac{nS^2}{b} \}
\]

to get the joint confidence region of \((\mu, \text{VaR})^2\).
Table 3. Minimizing area method (confidence level is 99% and 95%).

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<th>Min-Area (Confidence Level: 95%)</th>
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Table 4. Minimizing area point method (confidence level is 90%).

| \( n \) | \( a \) | \( b \) | \( c \) | \( \text{zz} \) |
| 31 | 2.0457 | 18.7961 | 53.0159 | 0.0198 |
| 32 | 2.0428 | 19.5671 | 54.2198 | 0.0185 |
| 33 | 2.0401 | 20.3418 | 55.4228 | 0.0173 |
| 34 | 2.0375 | 21.1193 | 56.6243 | 0.0162 |
| 35 | 2.0351 | 21.9005 | 57.8240 | 0.0152 |
| 36 | 2.0328 | 22.6834 | 59.0225 | 0.0143 |
| 37 | 2.0307 | 23.4694 | 60.2189 | 0.0135 |
| 38 | 2.0286 | 24.2587 | 61.4147 | 0.0128 |
| 39 | 2.0267 | 25.0498 | 62.6071 | 0.0121 |
| 40 | 2.0248 | 25.8426 | 63.7978 | 0.0114 |
| 41 | 2.0230 | 26.6392 | 64.9906 | 0.0109 |
| 42 | 2.0213 | 27.4365 | 66.1760 | 0.0103 |
| 43 | 2.0197 | 28.2368 | 67.3628 | 0.0098 |
| 44 | 2.0182 | 29.0392 | 68.5481 | 0.0093 |
| 45 | 2.0167 | 29.8436 | 69.7319 | 0.0089 |

If the confidence level is not in the chart we can get the results by changing parameters in the program.

**Theorem 2.** If the confidence levels are respectively 99%, 95%, 90%, \( n > 45 \). At the same time the mean \( \bar{x} \), the variance \( S^2 \) of samples, \( w_0 \) and \( \xi^* \) have been known. We can inquire the unknowns \( a, b, c \) in the Tables 5 and 6 and put them into

\[
\left\{ (\mu, VaR^2) : \bar{x} - \frac{a \sigma}{\sqrt{n}} \leq \mu \leq \bar{x} + \frac{a \sigma}{\sqrt{n}} (w_0 \xi^*)^2 \frac{nS^2}{c} \leq VaR^2 \leq (w_0 \xi^*)^2 \frac{nS^2}{b} \right\}
\]

to get the joint confidence region of \( (\mu, VaR^2) \).
Table 5. Minimizing area method (confidence level is 99% and 95%).

<table>
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Table 6. Minimizing area point method (confidence level is 90%).

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<th>b</th>
<th>c</th>
<th>zz</th>
</tr>
</thead>
<tbody>
<tr>
<td>46</td>
<td>2.0912</td>
<td>30.298</td>
<td>69.251</td>
<td>0.0089</td>
</tr>
<tr>
<td>47</td>
<td>2.0884</td>
<td>31.107</td>
<td>70.450</td>
<td>0.0085</td>
</tr>
<tr>
<td>48</td>
<td>2.0857</td>
<td>31.919</td>
<td>71.648</td>
<td>0.0081</td>
</tr>
<tr>
<td>49</td>
<td>2.0831</td>
<td>32.731</td>
<td>72.842</td>
<td>0.0078</td>
</tr>
<tr>
<td>50</td>
<td>2.0806</td>
<td>33.543</td>
<td>74.035</td>
<td>0.0074</td>
</tr>
<tr>
<td>60</td>
<td>2.0598</td>
<td>41.772</td>
<td>85.867</td>
<td>0.0050</td>
</tr>
<tr>
<td>70</td>
<td>2.0447</td>
<td>50.130</td>
<td>97.556</td>
<td>0.0036</td>
</tr>
<tr>
<td>80</td>
<td>2.0333</td>
<td>58.594</td>
<td>109.128</td>
<td>0.0027</td>
</tr>
<tr>
<td>90</td>
<td>1.0479</td>
<td>24.094</td>
<td>27.545</td>
<td>0.0016</td>
</tr>
<tr>
<td>100</td>
<td>5.5482</td>
<td>23.979</td>
<td>26.310</td>
<td>0.0061</td>
</tr>
</tbody>
</table>

4. Empirical Analysis

4.1. Description of Sample Data

This subsection analyzes the sample data from the historical stock quotes in Merchants Bank (600036) of the Sohu securities network. The holding period is a month and there are a total of 40 pieces of data regarding closing prices in the observation period from 23 January 2009 to 27 April 2012. Now we want to predict at a given confidence level the mean and range of risks to China Merchants Bank stock yields at the end of May 2012, that is to solve the joint confidence region of \((\mu, \sigma^2)\).

Here we use the logarithmic gain: \(R_t = \ln(P_t/P_{t-1})\), where \(P_t\) is the closing price of stock at time \(t\). Thus, we can get 39 yields as below Table 7.
Table 7. Sample data year and month closing price yield.

<table>
<thead>
<tr>
<th>Sample Data</th>
<th>Sample Data</th>
<th>Sample Data</th>
</tr>
</thead>
<tbody>
<tr>
<td>Y-M price</td>
<td>yield</td>
<td>Y-M price</td>
</tr>
<tr>
<td>0901 13.51</td>
<td>-</td>
<td>1001 15.17</td>
</tr>
<tr>
<td>0902 14.27</td>
<td>0.054729</td>
<td>1002 15.90</td>
</tr>
<tr>
<td>0903 15.93</td>
<td>0.110045</td>
<td>1003 16.28</td>
</tr>
<tr>
<td>0904 15.50</td>
<td>-0.027364</td>
<td>1004 14.27</td>
</tr>
<tr>
<td>0905 16.86</td>
<td>0.084104</td>
<td>1005 13.23</td>
</tr>
<tr>
<td>0906 22.41</td>
<td>0.284563</td>
<td>1006 13.01</td>
</tr>
<tr>
<td>0907 19.64</td>
<td>-0.131939</td>
<td>1007 14.52</td>
</tr>
<tr>
<td>0908 13.63</td>
<td>-0.365295</td>
<td>1008 13.54</td>
</tr>
<tr>
<td>0909 14.78</td>
<td>0.081002</td>
<td>1009 12.95</td>
</tr>
<tr>
<td>0910 17.77</td>
<td>0.184237</td>
<td>1010 14.57</td>
</tr>
<tr>
<td>0911 17.29</td>
<td>-0.027383</td>
<td>1011 13.05</td>
</tr>
<tr>
<td>0912 18.05</td>
<td>0.043017</td>
<td>1012 12.81</td>
</tr>
<tr>
<td>- - - - -</td>
<td>1201 12.65</td>
<td>0.063643</td>
</tr>
<tr>
<td>- - - - -</td>
<td>1202 12.87</td>
<td>0.017242</td>
</tr>
<tr>
<td>- - - - -</td>
<td>1203 11.90</td>
<td>-0.078361</td>
</tr>
<tr>
<td>- - - - -</td>
<td>1204 12.20</td>
<td>0.024898</td>
</tr>
</tbody>
</table>

By AVERAGE and VAR function in Excel we can get the MLE: $\bar{x}$ and $S^2$ of mean $\mu$ and variance $\sigma^2$ are, respectively, $-0.002615$ and $0.011898$.

4.2. $(\mu, \sigma^2)$-Model

Effect between Ideal Point Method and the Method of the Smallest

Now we respectively use ideal point method and area minimization method to solve the joint confidence region of at the given confidence level and make a contrast of the results from these two methods.

By checking the corresponding tables when $n = 39$ in this paper we have the following Tables 8 and 9:

Table 8. $n = 39$ (confidence level 99% and 95%).

<table>
<thead>
<tr>
<th>Method</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$z$</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>ideal point</td>
<td>3.6797</td>
<td>18.4736</td>
<td>63.0094</td>
<td>0.0390</td>
<td>3.2882</td>
<td>22.0311</td>
<td>55.8159</td>
<td>0.0239</td>
</tr>
<tr>
<td>Min-area</td>
<td>2.9024</td>
<td>19.6137</td>
<td>74.1233</td>
<td>0.0289</td>
<td>2.3212</td>
<td>23.1191</td>
<td>66.3364</td>
<td>0.0166</td>
</tr>
</tbody>
</table>

Table 9. $n = 39$ (confidence level is 90%).

<table>
<thead>
<tr>
<th>Method</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>ideal point</td>
<td>3.1106</td>
<td>24.0351</td>
<td>63.0094</td>
<td>0.0390</td>
</tr>
<tr>
<td>Min-area</td>
<td>2.0267</td>
<td>25.0498</td>
<td>62.6071</td>
<td>0.0121</td>
</tr>
</tbody>
</table>
Put these values into the following formula and get the joint confidence region of \((\mu, \sigma^2)\) as below

\[
\{(\mu, \sigma^2) : \bar{x} - \frac{a\sigma}{\sqrt{n}} \leq \mu \leq \bar{x} + \frac{a\sigma}{\sqrt{n}} (w_0\xi^*)^2 \frac{nS^2}{c} \leq \sigma^2 \leq (w_0\xi^*)^2 \frac{nS^2}{b}\}
\]

whose figure is the area enclosed by two horizontal lines and a parabola line

(1) \(\alpha = 1\%\) (confidence level is 99\%) (Figures 3–5)

Figure 3. The joint confidence region of \((\mu, \sigma^2)\) by ideal point method.

Figure 4. The joint confidence region of \((\mu, \sigma^2)\) by area minimization method.
Figure 5. The contrast of two methods.

where the solid line is the result of area minimization method and the broken line is the result of ideal point method.

(2) $\alpha = 5\%$ (confidence level is 95\%) (Figures 6–8)

Figure 6. The joint confidence region of $(\mu, \sigma^2)$ by ideal point method.

Figure 7. The joint confidence region of $(\mu, \sigma^2)$ by area minimization method.
Figure 8. The contrast of two methods.

where the solid line is the result of area minimization method and the broken line is the result of ideal point method.

(3) $\alpha = 10\%$ (confidence level is 90\%) (Figures 9–11)

Figure 9. The joint confidence region of $(\mu, \sigma^2)$ by ideal point method.

Figure 10. The joint confidence region of $(\mu, \sigma^2)$ by area minimization method.
where the solid line is the result of area minimization method and the broken line is the result of ideal point method.

According to these results, we can know the area by area minimization method is smaller, so at a given confidence level area minimization method has a better effect to solve the joint confidence region of \((\mu, \sigma^2)\).

4.3. \((\mu, VaR^2)\)-Model

Suppose a customer buys shares of China Merchants Bank whose initial assets is \(w_0 = 1\) and the confidence level of VaR is 95% that is \(\zeta^* = 1.645\). By Theorem 1 and the corresponding tables we have Table 10 as below.

<table>
<thead>
<tr>
<th>(\alpha)</th>
<th>(a)</th>
<th>(b)</th>
<th>(c)</th>
<th>(zz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1%</td>
<td>2.9024</td>
<td>19.6137</td>
<td>74.1233</td>
<td>0.0289</td>
</tr>
<tr>
<td>5%</td>
<td>2.3212</td>
<td>23.1191</td>
<td>66.3364</td>
<td>0.0166</td>
</tr>
<tr>
<td>10%</td>
<td>2.0267</td>
<td>25.0498</td>
<td>62.6071</td>
<td>0.0121</td>
</tr>
</tbody>
</table>

Put these values into the following formula and get the joint confidence region of \((\mu, VaR^2)\) below

\[
\{ (\mu, VaR^2) : \bar{x} - \frac{a\sigma}{\sqrt{n}} \leq \mu \leq \bar{x} + \frac{a\sigma}{\sqrt{n}} (w_0\zeta^*)^2 \frac{nS^2}{c} \leq VaR^2 \leq (w_0\zeta^*)^2 \frac{nS^2}{b} \}
\]

whose figure is the area enclosed by two horizontal lines and a parabola line.

(1) \(\alpha = 1\%\) (confidence level is 99%) (Figure 12)
Figure 12. The joint confidence region of \((\mu, \text{VaR}^2)\) when \(\alpha = 1\%\).

(2) \(\alpha = 5\%\) (confidence level is 95\%) (Figure 13)

Figure 13. The joint confidence region of \((\mu, \text{VaR}^2)\) when \(\alpha = 5\%\).

(3) \(\alpha = 10\%\) (confidence level is 90\%) (Figure 14)

Figure 14. The joint confidence region of \((\mu, \text{VaR}^2)\) when \(\alpha = 10\%\).
5. Conclusions

VaR is just a single indicator value to characterize risk, providing less information to the user, and the risk warning function is too thin. In practice, people often need to know both the benefits they may receive and the risks they are involved with. Therefore, this paper studies and constructs the double VaR according to the definition and research methods of VaR, expanding the one-dimensional single-risk monitoring indicator-VaR into a two-dimensional revenue-risk monitoring indicator.

This paper selects \((\mu, \sigma^2)\) and \((\mu, \text{VaR}^2)\) as the models of the double-VaR. It shows the risk/maximum loss of an asset at a given time in the future and the area in which the revenue is located. Such indicators can better weigh the risk–return of assets, and deduce the joint confidence region of \((\mu, \sigma^2)\) (or \((\mu, \text{VaR}^2)\)) by virtue of the two-dimensional likelihood ratio method. Then, the ideal joint method and the area minimization method are used to solve the specific joint confidence domain, and the solution effect of the two methods is compared. The obtained area minimization method is more accurate and better. After the VaR is double-expanded, users can know more information and better evaluate assets and avoid certain financial risks.

In this paper, only the normal distribution is considered in terms of its own knowledge structure and time. In fact, the author has great interest in risk management in the case of market with fat tails and the probability of extreme events, which will have important practical significance. We will discuss this in the next article on VaR.

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Conflicts of Interest: The authors declare no conflict of interest.

References


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