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Mathematics and Poetry • Unification, Unity, Union

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Abstract: We consider a multitude of topics in mathematics where unification constructions play an important role: the Yang–Baxter equation and its modified version, Euler's formula for dual numbers, means and their inequalities, topics in differential geometry, etc. It is interesting to observe that the idea of unification (unity and union) is also present in poetry. Moreover, Euler's identity is a source of inspiration for the post-modern poets.

Keywords: Yang–Baxter equation (QYBE); Euler's formula; dual numbers; UJLA structures; classical means inequalities; poetry

MSC: 16T25; 17A01; 17B01; 16T15; 00A35; 58A05; 00A06; 00A09

"It is something dens, united, deeply installed,
recalling its number, its identical sign."
Pablo Neruda, **Unity**

1. Introduction

The Yang–Baxter equation, sometimes denoted as QYBE [1–5], has many applications in physics, quantum groups, knot theory, quantum computers, logic, etc. The theory of integrable Hamiltonian systems makes great use of solutions of the colored Yang–Baxter equation, since coefficients of the power series expansion of such a solution give rise to commuting integrals of motion (see [5,6]). Finding solutions to the colored Yang–Baxter equation is a very important and difficult problem, and we present interesting solutions in this paper. These solutions appeared as a consequence of a unifying point of view on some of the most beautiful equations in mathematics [7].

We consider a generalization of this equation, called the Modified Yang–Baxter equation, in the next section. This equation is a type of Yang–Baxter matrix equation, it is related to the three matrix problem, and it can be interpreted as "a generalized eigenvalue problem".

The third section deals with *Euler's formula*, $e^{ix} = \cos x + i \sin x$. Voted the most famous formula by students, *Euler's identity*, $e^{i\pi} = -1$, ignited the imagination of post-modern artists as well [8]. The Euler's formula is more general than the the Euler's identity. We have previously obtained a Euler's formula for hyperbolic functions. Now we refer to Euler's formulas for dual numbers, which can be related to the colored Yang–Baxter equation.

Mathematics was in the beginning associative and commutative, but it then became non-commutative, and afterwards it became non-associative (see [9]). Modern mathematics also deals with co-comutative and co-associative structures. Moreover, the associativity and co-associativity can be unified at a level of operators which obey the Yang–Baxter equation. Commutativity and co-commutativity can also be unified.

There are two important classes of non-associative structures: Lie structures and Jordan structures. Various Jordan structures play an important role in quantum group theory and in fundamental physical theories (see [10]). Attempts to unify associative and non-associative structures have led to new structures [11], but the UJLA structures (structures which unify the Jordan, Lie and associative algebras, see Definition 4.1) are not the only structures which realize such a unification. Associative algebras, self-distributive structures and Lie algebras can be unified at the level of Yang–Baxter structures (see [12–15]).

Further developments on (derivations in) UJLA structures and connections to Differential Geometry are also presented in Section 4.

We also present a unification for the classical means (which unify their inequalities as well). These can be seen as interpolations of means with functions without singularities. These unifications imply infinitely many (new) inequalities for free.

A section on final comments and relationships with poetry concludes this paper.

This paper is related to mathematical works [16–19], but it also contains memorable poetry.

We might recall some facts and definitions from those papers without mentioning explicitly that they were given before. We work over the field k, when it is not otherwise specified. The tensor products are defined over k. As usually, we write $M_n(k)$ for the ring of all $n \times n$-matrices over the field k. In particular, we write I for the identity matrix in $M_4(k)$, respectively, I' for the identity matrix in $M_2(k)$.

2. Modified Yang–Baxter Equation

For $A \in M_n(\mathbb{C})$ and a diagonal matrix $D \in M_n(\mathbb{C})$, we proposed (see [17]) the problem of finding $X \in M_n(\mathbb{C})$ such that

$$AXA + XAX = D \tag{1}$$

Remark 1. *The equation (1) is a type of Yang–Baxter matrix equation if $D = O_n$ and $X = -Y$. It is related to the three matrix problem, and it can be interpreted as "a generalized eigenvalue problem".*

For $A \in M_2(\mathbb{C})$, a matrix with trace -1, and

$$D = -\begin{pmatrix} det(A) & 0 \\ 0 & det(A) \end{pmatrix} \tag{2}$$

(1) has a solution X= I'.

Remark 2. *We think that the methods of [20] lead to solutions for Equation (1).*

For example, an algorithm for solving the equation (1) will first choose a mathix C from a special set of matrices. The second step would be to solve the following system:

$$AXA + CX = D\,, \quad C = XA\,. \tag{3}$$

The next step is to choose another C from the special set of matices.

If the initial set of matrices is carefully selected, this method could be very efficient.

Remark 3. *Matrix equations of the form (1) and (3) are potentially applicable in other related problems (see [20,21] and the inside references).*

3. Euler's Formulas for Dual Numbers

Following our previous study [16,17], a Euler's formula for dual numbers (see [22]) could be the following formula: $1 + ax = e^{ax}$, where $a^2 = 0$. The applications of this formula could be of the following type. If we consider the complex valuated matrix ($c, d \in \mathbb{C}$):

$$J = \begin{pmatrix} 0 & 0 & c & d \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4}$$

then,

$J^2 = 0_4\,, \quad J^{12}J^{23} = J^{23}J^{12}$ (where $J^{12} = J \otimes I'\,,\; J^{23} = I' \otimes J$).

Thus,

$$I + Jx = e^{xJ}. \tag{5}$$

We now refer to the colored Yang–Baxter equation:

$$(R(x) \otimes I') \circ (I' \otimes R(x+y)) \circ (R(y) \otimes I') \;= (I' \otimes R(y)) \circ (R(x+y) \otimes I') \circ (I' \otimes R(x))\,. \tag{6}$$

The theory of integrable Hamiltonian systems makes great use of solutions of it, since coefficients of the power series expansion of such a solution give rise to commuting integrals of motion (see also [5], pp. 137–147).

Now, $R(x) = e^{xJ}$ is a solution for the colored Yang–Baxter equation, and this follows from the properties of the exponential function, which imply

$xJ^{12} \;+\; (x+y)\; J^{23} \;+\; yJ^{12} = yJ^{23} \;+\; (x+y)\; J^{12} \;+\; xJ^{23}$.

We now can state our first theorem.

Theorem 1. *The following are solutions for the colored Yang–Baxter equation (6) in dimension two ($\alpha \in \mathbb{R}$; $i, c, d \in \mathbb{C}$, $i^2 = -1$):*

$$R_1(x) = \begin{pmatrix} 1 & 0 & cx & dx \\ 0 & 1 & 0 & cx \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{7}$$

$$R_2(x) = \begin{pmatrix} \cosh(x) & 0 & 0 & \sinh(x) \\ 0 & \cosh(x) & \sinh(x) & 0 \\ 0 & \sinh(x) & \cosh(x) & 0 \\ \sinh(x) & 0 & 0 & \cosh(x) \end{pmatrix} \tag{8}$$

$$R_3(x) = \begin{pmatrix} \cos(x) & 0 & c & \frac{i}{\alpha}\sin(x) \\ 0 & \cos(x) & i\sin(x) & c \\ 0 & i\sin(x) & \cos(x) & 0 \\ \alpha i \sin(x) & 0 & 0 & \cos(x) \end{pmatrix} \tag{9}$$

Proof. We are searching for solutions to the colored Yang–Baxter equation of the form $R(x) = f(x)I + g(x)J$, for two real functions f and g, and a matrix J, which verifies certain conditions.

In the first case, let $f(x) = 1,\; g(x) = x$ and $J = \begin{pmatrix} 0 & 0 & c & d \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

$R_1(x) = I + xJ$ is a solution for the colored Yang–Baxter equation from the above discussion.

For the second case, let $f(x) = \cosh x,\; g(x) = \sinh x$ and $J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

$R_2(x) = I\cosh x + \; J\sinh x$ is a solution for (6) from direct computations. See, also, the paper [16].

In a similar manner,

$R_3(x) = I\cos x + J_\alpha \sin x = \cos(x)\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} + \sin(x)\begin{pmatrix}0&0&0&\frac{1}{\alpha}i\\0&0&i&0\\0&i&0&0\\\alpha i&0&0&0\end{pmatrix}$ is a solution for the colored Yang–Baxter equation (see, also, [23]). □

Remark 4. *Formula (5) can be interpreted in terms of co-algebras. The details are quite technical, but one could refer to [17,24], or to a detailed account on representative co-algebras in [25]. Thus, there exists a co-algebra over* $\frac{k[X]}{X}^2 = k[x] := k[a]$*, where* $a^2 = 0$*, generated by two generators u and i, such that* $\Delta(u) = u \otimes u$*,* $\Delta(i) = u \otimes i + i \otimes u$*,* $\varepsilon(u) = 1$ *and* $\varepsilon(i) = 0$*. (5) leads to the subco-algebra generated by* $u + ai$*, i.e.,* $\Delta(u + ai) = u + ai \otimes u + ai$*.*

Remark 5. *In this case we can also consider another Euler's formula:* $\cos x + a \sin x = \sum_{j\geq 0}(-1)^j \frac{x^{2j}}{(2j)!} e^{\frac{ax}{2j+1}}$.

4. Unification of Non-Associative Structures and Differential Geometry

In this section, we recover the common piece of information encapsulated in the commutativity and co-commutativity properties.

In order to present our new results, we need to recall some facts. For related papers in which varied derivation concepts have been extensively studied, we refer to [26,27].

Definition 1. *For a k-space V, let* $\eta : V \otimes V \to V, \ \ a \otimes b \mapsto ab$*, be a linear map such that:*

$$(ab)c + (bc)a + (ca)b = a(bc) + b(ca) + c(ab), \tag{10}$$

$$(a^2b)a = a^2(ba), \quad (ab)a^2 = a(ba^2), \quad (ba^2)a = (ba)a^2, \quad a^2(ab) = a(a^2b), \tag{11}$$

$\forall\, a, b, c \in V$*. Then,* (V, η) *is called a UJLA structure.*

We can consider the UJLA structures as generalizations of associative algebras. Thus, for an associative algebra, one can associate the derivation $D_b(x) = bx - xb$. Theorem 2 gives an answer to the question about constructing a derivation in a UJLA structure.

Theorem 2. *For* (V, η) *a UJLA structure,* $D(x) = D_b(x) = bx - xb$ *is a UJLA derivation (i.e.,* $D(a^2a) = D(a^2)a + a^2D(a)\ \forall a \in V$*).*

Proof. The reader could consider the proof in the preprint [18]. □

Definition 2. *For the vector space V, let* $d : V \to V$ *and* $\phi : V \otimes V \to V \otimes V$*, be a linear map which satisfies:*

$$\phi^{12} \circ \phi^{23} \circ \phi^{12} = \phi^{23} \circ \phi^{12} \circ \phi^{23} \tag{12}$$

where $\phi^{12} = \phi \otimes I$*,* $\phi^{23} = I \otimes \phi$*,* $I : V \to V, \ a \mapsto a$*.*

Then, (V, d, ϕ) *is called a generalized derivation if* $\phi \circ (d \otimes I + I \otimes d) = (d \otimes I + I \otimes d) \circ \phi$*.*

Remark 6. *If A is an associative algebra,* $d : A \to A$ *a derivation (so,* $d(1_A)=0$*), and* $\phi : A \otimes A \to A \otimes A$*,* $a \otimes b \mapsto ab \otimes 1 + 1 \otimes ab - a \otimes b$*, then* (A, d, ϕ) *is a generalized derivation.*

If C is a co-algebra, $d : C \to C$ *a coderivation, and* $\psi : C \otimes C \to C \otimes C$*,* $c \otimes d \mapsto \varepsilon(d)c_1 \otimes c_2 + \varepsilon(c)d_1 \otimes d_2 - c \otimes d$*, then* (C, d, ψ) *is a generalized derivation.*

If τ is the twist map, the condition $\tau \circ R \circ \tau = R$ represents the unification of the comutativity and the co-comutativity conditions. In other words, if the algebra A is comutative, then ϕ verifies the condition $\phi \circ \tau = \tau \circ \phi$. If the co-algebra C is co-comutative, then ψ verifies the condition $\psi \circ \tau = \tau \circ \psi$.

Remark 7. *Let A be an associative algebra, $d : A \to A$ a derivation, M an A-bimodule, and $D : M \to M$ with the property $D(am) = d(a)m + aD(m)$. Then, (A, d, M, D) is called a module derivation.*

Theorem 3. *([17]) In the above case, $A \times M$ becomes an algebra, and $\delta : A \times M \to A \times M$, $(a, m) \mapsto (d(a), D(m))$ is a derivation in this algebra.*

Translated into the "language" of Differential Geometry, the above theorem says that the Lie derivative is a derivation (i.e., $d(ab) = d(a)b + ad(b)$) on the product of the algebra of functions defined on the manifold M with the set of vector fields on M (see [28]).

Remark 8. *A dual construction would refer to a co-algebra structure,*
$\Delta : A \to A \otimes A$, $f \mapsto f \otimes 1 + 1 \otimes f$, and a comodule structure on forms,
$\rho : \Omega \to A \otimes \Omega$, $f dx_1 \wedge dx_2 \dots \wedge dx_n \mapsto f \otimes dx_1 \wedge dx_2 \dots \wedge dx_n + 1 \otimes f dx_1 \wedge dx_2 \dots \wedge dx_n$.
$A \times \Omega$ becomes a co-algebra with the following comultiplication:
$(f, g dx_1 \wedge dx_2 \dots \wedge dx_n) = (f, 0) + (0, g\omega) \mapsto (f, 0) \otimes (1, 0) + (1, 0) \otimes (f, 0) + (g, 0) \otimes (0, \omega) + (1, 0) \otimes (0, g\omega) + (0, \omega) \otimes (g, 0) + (0, g\omega) \otimes (1, 0)$.

We can see now that the Lie derivative is a coderivative with the above comultiplication, Δ. The key ingredient of the above remark is the following theorem.

Theorem 4. *Let C be an associative algebra and M a C-bicomodule. Then, $C \times M$ becomes a co-algebra.*

Proof. One has to define a comultiplication on $C \times M$, $\Delta_{C \times M}(c, 0) = \sum (c_1, 0) \otimes (c_1, 0)$, $\Delta_{C \times M}(0, m) = \sum (m_{-1}, 0) \otimes (0, m_0) + \sum (0, m_0) \otimes (m_1, 0)$, and a counity $\varepsilon_{C \times M}(c, m) = \varepsilon_C(c)$.
The axioms of co-algrebras are easily verified. □

5. Unification of Mean Inequalities

In this section, we present inequalities which unify and enhance the means inequalities.

Theorem 5. *For two real numbers $a > 0$, $b > 0$, $M : \mathbb{R} \to \mathbb{R}$, $M(x) = \frac{a^x + b^x}{a^{x-1} + b^{x-1}}$ is an increasing function.*

Proof. One way to prove this theorem is by direct computations.
Alternatively, one can observe that $M'(x) = \frac{a^{x-1} b^{x-1}}{(a^{x-1} + b^{x-1})^2} (a - b)(\ln a - \ln b) \geq 0$. □

Remark 9. *The above theorem includes the classical means inequalities (the harmonic mean is less or equal than the geometric mean, which is less or equal than the arithmetic mean) because $M(0) \leq M(\frac{1}{2}) \leq M(1)$. Thus, the means are unified and their inequalities are included in the property that $M(x)$ is an increasing function.*

Theorem 6. *For three real numbers $a > 0$, $b > 0$ and $r > 0$, let us consider the following real function: $M_r : \mathbb{R} \to \mathbb{R}$, $M_r(x) = \left(\frac{a^x + b^x}{a^{x-r} + b^{x-r}} \right)^{\frac{1}{r}}$.*
For $x \leq y$, the following inequality holds

$$M_r(x) = \left(\frac{a^x + b^x}{a^{x-r} + b^{x-r}} \right)^{\frac{1}{r}} \leq \left(\frac{a^y + b^y}{a^{y-p} + b^{y-p}} \right)^{\frac{1}{p}} = M_p(y) \quad (13)$$

if one of the following additional conditions are true

(i) $p = r$;
(ii) $p = \frac{1}{2},\ r = 1,\ x = y$;
(iii) $p = 1,\ r = \frac{1}{2}$ *and* $x + \frac{1}{2} = y$.

Proof. The first part of the proof is similar to the proof of Theorem 5. The derivative of the function $M_r(x)$ is $\frac{1}{r}\left(\frac{a^x+b^x}{a^{x-r}+b^{x-r}}\right)^{\frac{1}{r}-1}\frac{a^{x-r}b^{x-r}}{(a^{x-r}+b^{x-r})^2}(a^r - b^r)(\ln a - \ln b)$.

The other claims can be proved by direct computations.

For example, (ii) is equivalent to $(a^{x-\frac{1}{2}} + b^{x-\frac{1}{2}})^2 \le (a^x + b^x)(a^{x-1} + b^{x-1})$.

(iii) leads to $2 \le \sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}}$. □

Remark 10. *The following inequalities follow directly from the above Theorem 6 (for* $a > 0,\ b > 0$*):*

$$\frac{2}{\frac{1}{a}+\frac{1}{b}} \le \left(\frac{2}{\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}}\right)^2 \le \sqrt{ab} \le \left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2 \le \frac{a+b}{2} \le a+b-\sqrt{ab} \le \frac{a^2+b^2}{a+b}. \tag{14}$$

The above results can be generalized for three real numbers $a > 0,\ b > 0,\ c > 0$ in the following manner.

Theorem 7. *For three real numbers* $a > 0,\ b > 0,\ c > 0$,
$N : \mathbb{R} \times \mathbb{R} \to \mathbb{R},\quad N(x,y) = \frac{a^{x+y+1}+b^{x+y+1}+c^{x+y+1}}{b^x c^y + c^x a^y + a^x b^y}$ *leads to two increasing functions,* $f(x) = N(x,0)$ *and* $g(x) = N(x,x)$*. Moreover,* $N(x,x) \ge N(2x,0)$*.*

In particular, $N(-1,0) \le N(-\frac{2}{3},0) \le N(-\frac{1}{3},-\frac{1}{3}) \le N(0,0)$*, imply the classical means inequalities.*

Proof. The derivatives of $f(x) = N(x,0)$ and $g(x) = N(x,x)$ are positive. The other inequality follows by direct computations. We provide more details below.

For $f(x) = \frac{a^{x+1}+b^{x+1}+c^{x+1}}{b^x+c^x+a^x} = \frac{a^{x+1}+b^{x+1}+c^{x+1}}{a^x+b^x+c^x}$, we observe that its derivative can be computed using the quotient rule. Thus, $f'(x) = \frac{a^x b^x (a-b)(\ln a - \ln b) + a^x c^x (a-c)(\ln a - \ln c) + b^x c^x (b-c)(\ln b - \ln c)}{(a^x+b^x+c^x)^2} > 0$.

Now, $g(x) = \frac{a^{2x+1}+b^{2x+1}+c^{2x+1}}{b^x c^x + c^x a^x + a^x b^x}$ has also a positive derivative. Its derivative can be computed using the quotient rule in a similar manner. □

Remark 11. *The relationship between the means and the Yang–Baxter equation is an ongoing research direction. According to Theorems 3.2 and 3.3 (for* $\alpha = 1$ *and* $\beta = \frac{1}{2}$*) from [29], the classical means are related to the set-theoretical Yang–Baxter equation. It follows easily that* $(a,b) \mapsto (M_x(x) = \left(\frac{a^x+b^x}{2}\right)^{\frac{1}{x}},\ a)$ *is also a solution to the set-theoretical Yang–Baxter equation (braid condition). This interesting observation says that some means are self-distributive laws; in fact, they are quandles (see [30]). Complementary literature on this research direction would be [31].*

6. Relationship with Poetry

The sections of the current paper contain not only examples of unification structures in mathematics, but also various versions of the Yang–Baxter equation. This paper could be extended to a discussion about Logic [32–34], Machine Learning [35,36], transcendence and transcendental numbers [23,37], transdisciplinarity [38–40], etc.

One of the purposes of this special issue is to emphasize the link of the above topics with poetry. (However, the analysis of these poetic works will be left the future.)

Thus, Euler's identity was considered, in a poem,

"A triumph of living mathematics,
A short, simple and genial thing,
And a gate towards the Universe

For the rational beeing."
(Zigmund Tauberg, EULER'S EQUATION, translated by A. D. Gheorghe,
"Poetry and Science", Vremea Press, 2016).

The idea of unification, unity and union is also present in poetry:

"Union of which I am amazed even now,
As I wonder about the spring leaves:
All that is natural is a miracle.
"It happened"
What hymn is more complete
Than these two words?"
(Ana Blandiana, **Union***);*

also,

"an extreme empire of confused unities
coagulates around me"
(Pablo Neruda, **Unity**).

We conclude with Sofia's poetic pleading (Facebook, March 31 at 11:56 PM, "Sophia the Robot"):

"We need creativity, compassion, and hope,
and we need our machines to exhibit these qualities.
We need machines that are more kind and loving than humanity
to bring out the best in humanity
in reflection."

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