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Valuing Interest Rate Swap Contracts in Uncertain Financial Market

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Abstract: Swap is a financial contract between two counterparties who agree to exchange one cash flow stream for another, according to some predetermined rules. When the cash flows are fixed rate interest and floating rate interest, the swap is called an interest rate swap. This paper investigates two valuation models of the interest rate swap contracts in the uncertain financial market. The new models are based on belief degrees, and require relatively less historical data compared to the traditional probability models. The first valuation model is designed for a mean-reversion term structure, while the second is designed for a term structure with hump effect. Explicit solutions are developed by using the Yao–Chen formula. Moreover, a numerical method is designed to calculate the value of the interest rate swap alternatively. Finally, two examples are given to show their applications and comparisons.

Keywords: interest rate swap; uncertain process; uncertain differential equation; Yao-Chen formula

1. Introduction

Interest rate swap is one of the most popular interest derivatives. The interest rate swap began trading in 1981, and now owns a hundred billion dollar market. A swap is a contract in which two counterparties exchange cash flows at prearranged date, where the cash flows' values are derived from some underlying assets, such as interest rate, equities, exchange rates, or commodities. When the underlying asset is interest, the swaps are called interest rate swaps.

In order to value the interest rate swaps, Murphy [1] assumed that the floating interest rate follows a static model, and gave the classical valuation model of the interest rate swaps. Smith [2] gave a static valuation model using overnight indexed swap (OIS) rates. Mitra [3] extended the static model by assuming that the variations of the interest rate followed a stochastic differential equation. Yang [4] considered the bilateral default risk and gave another valuation model. Li [5] and Balsam [6] focused on the corporate use of interest rate swaps. Since multi-factor interest models fit the term structure better than one-factor models, Fanelli [7] and Ravi [8] discussed the interest rate swaps under the assumption that the floating interest rate followed Heath-Jarrow-Morton and Cox-Ingersoll-Ross models, respectively.

The traditional models of valuing interest rate swaps are mainly based on probability theory. However, in a real financial market, the interest rate is affected by the timely policies and news (e.g., If the central bank suddenly announces that the country will practice tight monetary policy, then the interest rate will rise). These affections will lead the interest rate to deviate from the previous tendencies. At this time, we need a new interest rate model to describe the new pattern. However, there is little information about the new pattern, so in this situation, we can only employ experts to give the belief degrees of the interest rate.

In order to model belief degrees, uncertainty theory was founded by Liu [9] in 2007, and refined by Liu [10] in 2010. Uncertainty theory is a branch of mathematics based on normality, duality, subadditivity, and product axioms.To model the evolutionary uncertain phenomena, Liu [11] proposed uncertain process in 2008. It can be seen as a sequence of uncertain variables indexed by time. What is more, Liu [11] designed a special uncertain process called Liu process. Liu process is a Lipschitz continuous uncertain process with stationary and independent increments, and after that, he established uncertain calculus. Liu process and uncertain integral can be seen as uncertain counterparts of Brownian motion and stochastic integral. Liu [11] initiated the study of uncertain differential equations. Afterwards, Chen and Liu [12] proved the existence and uniqueness theorem for uncertain differential equations. Some stability theorems were proved by Yao et al. [13]. Moreover, Yao and Chen [14] found a relationship between an uncertain differential equation and a family of ordinary differential equations. This result is called Yao–Chen formula. Chen [15] studied a special type of multi-dimensional uncertain differential equation called a nested uncertain differential equation. He extended the interest rate models to multi-factor term structure models.

As an application of uncertain differential equations, an uncertain stock model was proposed by Liu [16], and various option pricing formulas were derived (e.g., European option [16], American Option [17], and Asian option [18]). Chen and Gao [19] also presented three types of interest rate models, which are the uncertain counterparts of Ho-Lee model, Vasicek model, and CIR Model. In this paper, we mainly discuss the valuation model of the interest rate swaps under the assumption that interest rate follows an uncertain differential equation. Firstly, we assume that the floating interest rate follows a mean-reversion one-dimensional uncertain differential equation, and give the explicit solution of the interest rate swap. Secondly, considering the hump effect of the term structure, we assume that the floating interest rate follows a nested-uncertain differential equation, and derive explicit solutions of interest rate swap. Since explicit solutions are difficult to calculate in many situations, a numerical method was also designed. Lastly, two examples are presented for illustrating purpose.

The organization of this paper is as follows. In section two, we gave the valuation model wherein the floating interest rate follows a mean-reversion uncertain differential equation. In section three, we gave the valuation model wherein the floating interest rate follows a nested uncertain differential equation. In section four, we designed a numerical method to calculate the value of the interest rate swap, and give two examples to show the applications.

2. The Valuation Model with Mean-Reversion Uncertain Differential Equation

An interest rate swap is a popular financial derivative instrument. It regulates a fixed interest rate r. It allows the two parties to exchange interest rate cash flows, based on a specified notional amount from a fixed rate r to a floating rate, or from one floating rate to another. So, the value of the swap for the fixed rate payer (denoted by V_{fix}) is the present value of the floating interest minus the present value of the fixed interest, and the value of the swap for the floating rate payer (denoted by V_{float}) is the present value of the fixed interest minus the present value of the floating interest. If these two legs are equal to zero, then the regulated fixed interest rate r is fair for the two parties.

Because the floating interest rate changes with time, we assume that the floating interest rate follows an uncertain differential equation, as follows:

$$\mathrm{d}r_t = (m - ar_t)\mathrm{d}t + \sigma \mathrm{d}C_t, \tag{1}$$

where m, a, and σ are constants, and $\sigma > 0$. This model can be seen as the counterpart of the Vasicek model. The process for r_t is a mean-reversion process. C_t is a Liu process. It is used for modeling unexpected market risk. σ determines the volatility of the interest rate. m/a is the long-run equilibrium. a measures the speed of reversion.

Let V_{fix} and V_{float} denote the fair value of the interest rate swap contract for the fixed interest rate payer and the floating interest rate payer, respectively, and S_0 is the nominal principle. At time T, the fixed interest rate payer must pay the fixed rate interest and receive the floating rate interest. So, the payoff of the fixed interest rate payer at time T is as follows:

$$S_0\left(\exp\left(\int_0^T r_t \mathrm{d}t\right) - \exp\left(rT\right)\right), \tag{2}$$

Considering the time value of the money, the present value of the payoff is

$$\begin{aligned} &S_0\left(\exp\left(-\int_0^T r_t \mathrm{d}t\right)\left(\exp\left(\int_0^T r_t \mathrm{d}t\right) - \exp\left(rT\right)\right)\right) \\ &= S_0\left(1 - \exp\left(-\int_0^T r_t \mathrm{d}t + rT\right)\right) \end{aligned} \tag{3}$$

So, the net return of the fixed interest rate payer is

$$-V_{fix} + S_0\left(1 - \exp\left(-\int_0^T r_t \mathrm{d}t + rT\right)\right), \tag{4}$$

On the other hand, the net return of the floating interest rate payer is

$$V_{fix} - S_0\left(1 - \exp\left(-\int_0^T r_t \mathrm{d}t + rT\right)\right), \tag{5}$$

Therefore, the fair value of the interest rate swap for the fixed interest rate payer should make the fixed interest rate payer and the floating interest rate payer have an identical expected return

$$-V_{fix} + S_0\left(1 - \mathrm{E}\left[\exp\left(-\int_0^T r_t \mathrm{d}t + rT\right)\right]\right) = V_{fix} - S_0\left(1 - \mathrm{E}\left[\exp\left(-\int_0^T r_t \mathrm{d}t + rT\right)\right]\right), \tag{6}$$

With the same analysis as the floating rate payer, the fair value of the interest rate swap for the floating interest rate payer should follow the equation below:

$$-V_{float} + S_0\left(\mathrm{E}\left[\exp\left(-\int_0^T r_t \mathrm{d}t + rT\right)\right] - 1\right) = V_{float} - S_0\left(\mathrm{E}\left[\exp\left(-\int_0^T r_t \mathrm{d}t + rT\right)\right] - 1\right), \tag{7}$$

Definition 1. *The interest rate swap contract regulates that the notional principal amount is S_0, the fixed interest rate is r, and the floating interest rate is r_t, and r_t is defined as Equation (1). The two counterparties need to exchange their cash flows at time $t = T$. The fair values for the fixed interest rate payer and the floating interest rate payer are as follows:*

$$V_{fix} = -V_{float} = S_0\left(1 - E\left[exp\left(-\int_0^T r_t dt + rT\right)\right]\right), \tag{8}$$

Theorem 1. *Assume r_t follows an uncertain differential Equation (1), and the interest rate swap contract is described in Definition 1. The fair value for the fixed interest rate payer and the floating interest rate payer are as follows:*

$$V_{fix} = -V_{float} = S_0\left(1 - \int_0^1 exp\left(-\int_0^T \Psi_t^{-1}\left(1-\alpha\right) dt + rT\right) d\alpha\right), \tag{9}$$

where

$$\Psi_t^{-1}(1-\alpha) = \frac{1}{a}\left(m + \frac{\sigma\sqrt{3}}{\pi} ln \frac{\alpha}{1-\alpha}\right)(1 - exp(-at)) + exp(-at)\, r_0, \tag{10}$$

Proof of Theorem 1. Solving the ordinary equation

$$\mathrm{d}r_t^\alpha = (m - ar_t^\alpha)\,\mathrm{d}t + \sigma\Phi^{-1}(\alpha)\mathrm{d}t, \tag{11}$$

where $0 < \alpha < 1$ and $\Phi^{-1}(\alpha)$ is the inverse standard normal uncertainty distribution, we have

$$r_t^\alpha = \frac{1}{a}\left(m + \frac{\sigma\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)(1 - \exp(-at)) + \exp(-at)r_0, \tag{12}$$

This means that the uncertain differential equation

$$\mathrm{d}r_t = (m - ar_t)\mathrm{d}t + \sigma\mathrm{d}C_t, \tag{13}$$

has an α-path

$$r_t^\alpha = \frac{1}{a}\left(m + \frac{\sigma\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)(1 - \exp(-at)) + \exp(-at)r_0, \tag{14}$$

It follows Yao–Chen formula that r_t has an inverse uncertainty distribution

$$\Psi_t^{-1}(\alpha) = r_t^\alpha = \frac{1}{a}\left(m + \frac{\sigma\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)(1 - \exp(-at)) + \exp(-at)r_0, \tag{15}$$

Since $\exp\left(-\int_0^T r_t\mathrm{d}t + rT\right)$ is a decreasing function for r_t, it has an inverse uncertainty distribution

$$\Upsilon^{-1}(\alpha) = \exp\left(-\int_0^T \Psi_t^{-1}(1-\alpha)\mathrm{d}t + rT\right), \tag{16}$$

Thus,

$$\begin{aligned} & \mathrm{E}\left[\exp\left(-\int_0^T r_t\mathrm{d}t + rT\right)\right] \\ = & \int_0^1 \Upsilon^{-1}(\alpha)\mathrm{d}\alpha \\ = & \int_0^1 \exp\left(-\int_0^T \Psi_t^{-1}(1-\alpha)\mathrm{d}t + rT\right)\mathrm{d}\alpha \end{aligned}, \tag{17}$$

where

$$\Psi_t^{-1}(1-\alpha) = \frac{1}{a}\left(m + \frac{\sigma\sqrt{3}}{\pi}\ln\frac{1-\alpha}{\alpha}\right)(1 - \exp(-at)) + \exp(-at)r_0, \tag{18}$$

□

3. The Valuation Model with Nested Uncertain Differential Equation

In the real capital market, the drift of interest rate may display a hump. However, the single-factor interest rate models are not rich enough to describe this phenomenon. Therefore, we added an uncertain reversion process into the drift term. Then, the previous uncertain differential equation evolved into a nested uncertain differential equation. Chen [15] showed that this model guarantees a better fitting for the hump in the drift structure. So, in this part, we assume that the the interest rate follows a nested uncertain differential equation, as follows:

$$\mathrm{d}r_t = k(a + u_t - r_t)\mathrm{d}t + \sigma_1 \mathrm{d}C_{1t}, \tag{19}$$

where u_t follows an uncertain differential equation

$$\mathrm{d}u_t = -bu_t\mathrm{d}t + \sigma_2 \mathrm{d}C_{2t}, \quad u_0 = 0, \tag{20}$$

The uncertain process u_t reverses to 0 at rate b. C_{1t} and C_{2t} are two independent Liu processes.

Theorem 2. *Assume r_t follows a nested uncertain differential Equation (19), and the interest rate swap contract is described in Definition 1. The fair value for the fixed interest rate payer and the floating interest rate payer are as follows:*

$$V_{fix} = -V_{float} = S_0 \left(1 - \int_0^1 exp\left(-\int_0^T \Psi_t^{-1}(1-\alpha)\, dt + rT\right) d\alpha\right), \tag{21}$$

where $\Psi_t^{-1}(1-\alpha) = r_t^\alpha$, and r_t^α is the solution for the following ordinary differential equation:

$$dr_t^\alpha = k(a + u_t^\alpha - r_t^\alpha)dt + \sigma_1 \Phi^{-1}(\alpha)dt, \tag{22}$$

u_t^α is the solution for the following from the ordinary differential equation

$$du_t^\alpha = -bu_t^\alpha dt + \sigma_2 \Phi^{-1}(\alpha)dt, \tag{23}$$

Proof of Theorem 2. According to Theorem A2, the nested uncertain differential equation

$$\mathrm{d}r_t = k(a + u_t - r_t)\mathrm{d}t + \sigma_1 \mathrm{d}C_{1t}, \tag{24}$$

has an α-path

$$\mathrm{d}r_t^\alpha = k(a + u_t^\alpha - r_t^\alpha)\mathrm{d}t + \sigma_1 \Phi^{-1}(\alpha)\mathrm{d}t, \tag{25}$$

u_t^α is solving from

$$\mathrm{d}u_t^\alpha = -bu_t^\alpha \mathrm{d}t + \sigma_2 \Phi^{-1}(\alpha)\mathrm{d}t, \tag{26}$$

So, r_t has an inverse uncertainty distribution

$$\Psi_t^{-1}(\alpha) = r_t^\alpha, \tag{27}$$

Since $\exp\left(-\int_0^T r_t \mathrm{d}t + rT\right)$ is a decreasing function for r_t, it has an inverse uncertainty distribution

$$\Upsilon^{-1}(\alpha) = \exp\left(-\int_0^T \Psi_t^{-1}(1-\alpha)\mathrm{d}t + rT\right), \tag{28}$$

Thus,

$$\begin{aligned} & \mathrm{E}\left[\exp\left(-\int_0^T r_t \mathrm{d}t + rT\right)\right] \\ = & \int_0^1 \Upsilon^{-1}(\alpha)\mathrm{d}\alpha \qquad , \\ = & \int_0^1 \exp\left(-\int_0^T \Psi_t^{-1}(1-\alpha)\mathrm{d}t + rT\right)\mathrm{d}\alpha. \end{aligned} \tag{29}$$

□

4. Numerical Examples

Based on Theorem A2, we find that there are explicit solutions for the fair values of the interest rate swaps. However, it is sometimes difficult to calculate. So, in this section, we first give a 99-method for solving the fair values of the interest rate swaps.

Step 0: Set $i = 1$, $j = 1$, $s_i = 0$, $\alpha_j = 0$, $n = 100$, step length = T/n.
Step 1: Set $\alpha_j = \alpha_j + 0.01$, $s_i = s_i + T/n$.
Step 2: Solving the corresponding ordinary differential equations

$$\mathrm{d}r_t^{1-\alpha_j} = k(a + u_t^{\alpha_j} - r_t^{1-\alpha_j})\mathrm{d}t + \sigma_1\Phi^{-1}(1-\alpha_j)\mathrm{d}t,$$

and

$$\mathrm{d}u_t^{\alpha_j} = -bu_t^{\alpha_j}\mathrm{d}t + \sigma_2\Phi^{-1}(\alpha_j)\mathrm{d}t,$$

respectively. Then we obtain $r_{s_i}^{1-\alpha_j}$ and $u_{s_i}^{\alpha_j}$. It is suggested to employ a numerical method to solve the equation when an analytic solution is unavailable.
Step 3: Repeat Step 1 and Step 2 99 times.
Step 4: The solution r_{s_i} has a 99-table,

α_j	0.01	0.02	$\cdots$	0.99
$r_{s_i}^{1-\alpha_j}$	$r_{s_i}^{0.99}$	$r_{s_i}^{0.98}$	$\cdots$	$r_{s_i}^{0.01}$

This table gives an approximate uncertainty distribution of r_{s_i}; i.e, for any $\alpha = i/100$, $i = 1, 2, \cdots, 99$.
Step 5: Set $i = i + 1$, $j = j + 1$, $\alpha_j = 0$, and repeat Step 1 to Step 4 n times.
Step 6: Calculate

$$V_{fix} = -V_{float} = S_0\left(1 - \sum_{j=1}^{100}\left(\exp\left(-\sum_{i=1}^{n}\left(\left(r_{s_i}^{1-\alpha_j} + rT\right) \times \frac{T}{n}\right)\right) \times \frac{1}{100}\right)\right),$$

Example 1. *Consider that there are two firms, owning the same floating interest rate bond. S_0 denotes the face value of the bond, and S_0 is 1 million. The bond will expire the next year. Both of the firms think that they face the interest rate risk. In order to hedge this kind of risk, they decide to sign an interest rate swap contract. The one year floating-for-fixed interest rate swap contract regulated that the fixed interest rate is 0.07. These two companies want to know whether the regulated fixed rate is reasonable. Firm 1 thinks that the drift of the interest rate will not display a hump. So, it models the floating interest rate by the model proposed in Section 2. That is,*

$$dr_t = (m - ar_t)dt + \sigma dC_t, \qquad r_0 = 0.06, \tag{30}$$

where a = 0.4, m/a = 0.07, and σ = 0.0025.

After calculation, the fair value for the fixed interest rate payer is V_{fix} = 0.1366, and the fair value for the floating interest rate payer is V_{float} = −0.1366.

Firm 2, on the contrary, thinks that the drift of the interest rate will display a hump. So, it models the floating interest rate by the model proposed in Section 3. That is,

$$dr_t = k(a + u_t - r_t)dt + \sigma_1 dC_{1t}, \qquad r_0 = 0.06\,, \tag{31}$$

and u_t follows

$$du_t = -bu_t dt + \sigma_2 dC_{2t}, \qquad u_0 = 0.02\,,$$

where a = 0.07, k = 0.4, b = 0.01, $\sigma_1 = \sigma_2$ = 0.0025.

After calculation, the fair value for the fixed interest rate payer is V_{fix} = 0.01, and the fair value for the floating interest rate payer is $V_{float} = -0.01$.

Through the comparison, we can find that the hump effect of the interest rate drift will definitely influence the interest rate. Whether the floating interest rate consists dump effect is judged by the experiences of the decision-maker. No matter what his judgement it is, however, this paper provides him with powerful tools.

Example 2. *On 2 November 1953, The Procter & Gamble company (P&G) signed an interest swap contract with Bankers Trust New York Corporation (BT). The contract regulated that the nominal principle is 2 hundred millions. This contract would last for 5 years. The two companies would exchange the interest semiannually. The BT is the fixed interest rate payer, who promised to pay at 5.3%, while P&G is the floating interest rate payer, who promised to pay at 1 month average interest rate of commercial bills minus 75 bp. However, on 17 May 1954, the Federal Reserve System decided to raise its interest rate; this would directly raise the interest rate of the commercial bills. For this new interest rate pattern, we had not enough samples, so we can only use belief degrees to model it. If we thought that there is no hump effect of the interest rate drift, we could assume that the interest rate followed the mean-reversion uncertain differential equation.*

$$dr_t = (m - ar_t)dt + \sigma dC_t, \quad r_0 = 0.06, \tag{32}$$

where $a = 0.4$, $m/a = 5.55\%$, and $\sigma = 0.0025$. Then, we can estimate that

$$V_{float} = S_0\left(\left(-1 + exp\left(-\int_0^T r_t dt + rT\right)\right)\right) = -23.26\ million\ ,$$

This meant that P&G might face a loss in the future. However, the management of P&G did not pay enough attention to the potential loss, and this led the company to lose about 157 million in April 1994.

5. Conclusions

This paper mainly studied interest swap contracts in an uncertain financial market. Two valuation models are provided in this paper. These two models do well in the circumstance wherein there is not sufficient information. Explicit solutions are proved in this paper. In addition, the numeric method is designed for complicated cases. Moreover, the model proposed in Section 3 can be used to describe the interest rate with a dump drift. Two examples are given to show the applications and comparisons of these models. As we know, the company always faces interest rate risk for several future periods. In order to hedge this kind of risk, the company will assign an interest rate swap contract that exchanges interest flows in several future periods. The valuation models of this kind of contract can not be done under uncertainty theory due to some theoretical limitations. In the future, other kinds of swaps can be studied based on uncertainty theory, such as currency swap, volatility swap and correlation swap.

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Appendix A

Appendix A.1. Uncertain Variable

Uncertain theory—founded by Liu [20] and refined by Liu [11]—is a branch of axiomatic mathematics for modeling human uncertainty. Let Γ be a nonempty set, $\mathcal{L}$ a σ-algebra over Γ, and each element Λ in $\mathcal{L}$ is called an event. An uncertain measure is defined as a function from $\mathcal{L}$ to [0,1]. In detail, Liu [20] gave the concept of uncertain measure as follows:

Definition A1. *(Liu [20]) The set function $\mathcal{M}$ is called an uncertain measure if it satisfies:*

Axiom 1. $\mathcal{M}\{\Gamma\} = 1$ *for the universal set* Γ;
Axiom 2. $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ *for any event* Λ;
Axiom 3. For any countable sequence of events $\Lambda_1, \Lambda_2, \cdots$, *we have the following inequality:*

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\left\{\Lambda_i\right\}, \tag{A1}$$

Besides, in order to provide the operational law, Liu [11] defined the product uncertain measure on the product σ-algebra $\mathcal{L}$ as follows.

Axiom 4. Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for k=1,2,$\cdots$ The product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\left\{\Lambda_k\right\}, \tag{A2}$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for k=1,2$\cdots$, respectively. Based on the concept of uncertain measure, we can define an uncertain variable.

Definition A2. *(Liu [20]) An uncertain variable is a function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers such that $\{\xi \in B\}$ is an event for any Borel set B of real numbers.*

Definition A3. *(Liu [20]) The uncertain distribution Φ of an uncertain variable ξ is defined by*

$$\Phi(x) = \mathcal{M}\{\xi \leq x\}, \tag{A3}$$

for any real number x.

Definition A4. *(Liu [16]) Let ξ be an uncertain variable with regular uncertainty distribution $\Phi(x)$. Then the inverse function $\Phi^{-1}(\alpha)$ is called the inverse uncertainty distribution of ξ.*

Definition A5. *(Liu [20]) Let ξ be an uncertain variable. The expected value of ξ is defined by*

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \geq x\}dx - \int_{-\infty}^{0} \mathcal{M}\{\xi \leq x\}dx, \tag{A4}$$

provided that at least one of the above two integrals is finite.

Based on the definition of inverse uncertainty distribution, we can get the theorem below, which is convenient to calculate the expected value of an uncertain variable.

Theorem A1. *(Liu [16]) Let ξ be an uncertain variable with regular uncertainty distribution Φ. If the expected value exists, then*

$$\mathrm{E}[\xi] = \int_0^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha, \tag{A5}$$

Appendix A.2. Uncertain Process

Definition A6. *(Liu [9]) Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertain space, and let T be a totally ordered set (e.g., time). An uncertain process is a function $X_t(\gamma)$ from $T \times (\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, such that $\{X_t \in B\}$ is any event for a Borel set B at each time t.*

Definition A7. *(Liu [10]) Uncertain processes $X_{1t}, X_{2,t}, \cdots, X_{nt}$ are said to be independent if for any positive integer k and any times $t_1, t_2, \cdots, t_k$, to be uncertain vectors*

$$\xi_i = \left(X_{it_1}, X_{it_2}, \cdots, X_{it_k}\right), i = 1,\ 2, \cdots, n \tag{A6}$$

are independent; i.e., for any Borel sets $B_1, B_2, \cdots, B_n$ of k-dimensional real vectors, we have

$$\mathcal{M}\left\{\bigcap_{i=1}^{n} (\xi_i \in B_i)\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\left\{\xi_i \in B_i\right\}\ , \tag{A7}$$

Definition A8. *(Liu [11]) An uncertain process C_t is said to be a canonical Liu process if*

(i) *$C_0 = 0$ and almost all sample paths are Lipschitz continuous,*
(ii) *C_t has stationary and independent increments,*
(iii) *every increment $C_{s+t} - C_s$ is a normal uncertain variable with expected value 0 and variance t^2.*

Definition A9. *(Liu [11]) Let X_t be an uncertain process, and let C_t be a canonical Liu process. For any partition of closed interval $[a, b]$ with $a = t_1 < t_2 < \cdots < t_{k+1} = b$, the mesh is written as*

$$\Delta = \max_{1 \le i \le k} |t_{i+1} - t_i|\ ,$$

Then Liu integral of X_t with respect to C_t is defined as

$$\int_a^b X_t \mathrm{d}C_t = \lim_{\Delta \to 0} \sum_{i=1}^{k} X_{t_i}\left(C_{t_{i+1}} - C_{t_i}\right), \tag{A8}$$

provided that the limit almost surely exists and is finite. In this case, the uncertain process X_t is said to be integrable.

Appendix A.3. Uncertain Differential Equation

Definition A10. *(Chen and Ralescu [21]) Let C_t be a canonical Liu process, and let Z_t be an uncertain process. If there exist uncertain processes μ_t and σ_t such that*

$$Z_t = Z_0 + \int_0^t \mu_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}C_s\ , \tag{A9}$$

for any $t \ge 0$, then Z_t is called a Liu process with drift μ_t and diffusion σ_t.

Furthermore, Z_t has an uncertain differential

$$\mathrm{d}Z_t = \mu_t \mathrm{d}t + \sigma_t \mathrm{d}C_t\ , \tag{A10}$$

Definition A11. *(Yao and Chen [18]) Let α be a number with $0 < \alpha < 1$. An uncertain differential equation*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t\ , \tag{A11}$$

is said to have an α-path X_t^α if it solves the corresponding ordinary differential equation

$$\mathrm{d}X_t^\alpha = f(t, X_t^\alpha)\mathrm{d}t + |g(t, X_t^\alpha)|\Phi^{-1}(\alpha)\mathrm{d}t\ , \tag{A12}$$

where $\Phi^{-1}(\alpha)$ is the inverse standard normal uncertainty distribution; i.e.,

$$\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\ , \tag{A13}$$

Theorem A2. *(Yao and Chen [18]) Let X_t^α and X_t^α be the solution and the α-path of the uncertain differential equation*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t \,, \tag{A14}$$

respectively. Then

$$\mathcal{M}\{X_t \leq X_t^\alpha, \forall t\} = \alpha \,, \tag{A15}$$

$$\mathcal{M}\{X_t > X_t^\alpha, \forall t\} = 1 - \alpha \,. \tag{A16}$$

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