A Novel Decision-Making Model with Pythagorean Fuzzy Linguistic Information Measures and Its Application to a Sustainable Blockchain Product Assessment Problem

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Received: 28 August 2019; Accepted: 9 October 2019; Published: 12 October 2019

Abstract: This study presents a novel multi-attribute decision-making (MADM) model on the basis of Pythagorean fuzzy linguistic information measures. To do so, we first present a new concept of Pythagorean fuzzy linguistic sets to describe fuzziness and inconsistent information, in which the Pythagorean fuzzy linguistic values (PFLVs) are represented by the linguistic membership degree and linguistic non-membership degree. Then, we introduce two axiomatic definitions of information measures for PFLVs, including Pythagorean fuzzy linguistic entropy and the Pythagorean fuzzy linguistic similarity measure, to measure the uncertainty degree of PFLVs and the similarity degree between among PFLVs. In addition, based on the logarithmic function, we construct two new information measure formulas and verify that they satisfy the axiomatic conditions of the Pythagorean fuzzy linguistic entropy and similarity measure, respectively. We further explore the relationship between the Pythagorean fuzzy linguistic entropy and similarity measure. Finally, we present a novel Pythagorean fuzzy linguistic MADM model with the Pythagorean fuzzy linguistic entropy and similarity measure. A numerical example of selecting the most desirable sustainable blockchain product is given, and a comparison with the existing approach was performed to validate the reliability of the developed decision-making model.

Keywords: multi-attribute decision-making model; Pythagorean fuzzy linguistic sets; information entropy; similarity measure; blockchain products assessment

1. Introduction

Multi-attribute decision-making (MADM) is an attractive and potentially useful approach in addressing complex decision situations. Due to the fuzziness in complex MADM problems, decision makers (DMs) tend to utilize fuzzy information to express their evaluation preference. Thus, fuzzy sets (FSs) [1] were first put forward by Zadeh and have been applied in various fields [2–9]. After this, various forms of uncertain fuzzy sets were generalized so as to satisfy actual demands, which included intuitionistic fuzzy sets (IFSs) [10–12], interval-valued intuitionistic fuzzy sets (IVIFSs) [13,14], hesitant fuzzy sets (HFSs) [15,16], and Pythagorean fuzzy sets (PFSs) [17,18]. Owing to the uncertainty and fuzziness of real-world decision-making problems, DMs are more comfortable providing their evaluation information linguistically rather than in terms of numerical
values and, hence, typically lean toward linguistic term sets (LTSs) [19,20]. Considering the desirable characteristics of PFSs and LTSs, we introduce here a new concept of Pythagorean fuzzy linguistic sets (PFLSs), in which the decision-making evaluation information is described with Pythagorean fuzzy linguistic values (PFLVs), and each PFLV is represented by the degrees of linguistic membership and linguistic non-membership. It is obvious that PFLSs are more effective than PFSs in capturing the uncertainty and fuzziness in complex MADM problems.

Information measurement is an important research issue in MADM theory, and includes entropy, the similarity measure, and their transformation relationships [21,22]. Entropy is mainly used to determine the degree of uncertainty of objects, and the similarity measure is mainly utilized to determine the degree of similarity among objects. Based on the probability measures of fuzzy events, Zadeh [23] proposed fuzzy entropy to derive the uncertainty of assessment information. After that, De Luca and Termini [24] initially introduced some axiomatic definitions of entropy for FSs, and then discussed some mathematical properties of entropy. With the help of the distance between the FS and its negation FS, Yager [25] proposed several entropy measures to derive the fuzziness of FSs. By using the distances between the fuzzy message and its nearest and farthest non-fuzzy neighbors, Kosko [26] defined a novel non-probabilistic fuzzy entropy measure. Under the intuitionistic fuzzy information environment, Szmidt and Kacprzyk [27] developed an entropy measure for IFSs. Song et al. [28] investigated a new intuitionistic fuzzy similarity measure by utilizing the direct operation on the membership and non-membership functions of intuitionistic fuzzy values. For intuitionistic fuzzy MADM problems, Wu and Zhang [29] presented an intuitionistic fuzzy weighted entropy and developed a programming approach to derive the optimal attribute weights. Inspired by this intuitionistic fuzzy weighted entropy, Jin et al. [30] investigated an interval-valued intuitionistic fuzzy continuous weighted entropy to measure the degree of uncertainty for IVIFSs. Xu and Xia [31] defined the entropy and similarity measure for HFSs, which was followed by the construction of the interchangeable method. Owing to the existing information measures for HFSs having drawbacks and limitations, Hu et al. [32] designed several new effective and reliable distance, similarity, and entropy measures for HFSs. For MADM problems with interval-valued HFSs, Jin et al. [33] introduced three axiomatic definitions of interval-valued hesitant fuzzy information measures, and then established several formulas with a continuous ordered weighted averaging operator. Farhadinia [34] designed a linguistic term fuzzy entropy to calculate the attribute weights. Based on the intuitive geometric explanation, Wu and Mendel [35] developed the Jaccard similarity measure for closed general type-2 fuzzy sets. Majumdar and Samant [36] proposed the axiomatic conditions of single-valued neutrosophic entropy. In order to overcome the drawbacks of single-valued neutrosophic entropy in Majumdar and Samant [36], Jin et al. [37] constructed a novel single-valued neutrosophic entropy and similarity measure, and then investigated a new MADM method. In order to measure the vagueness and uncertainty of PFSs and interval-valued PFSs, Xue et al. [38] introduced the definitions of Pythagorean fuzzy entropy and interval-valued Pythagorean fuzzy entropy. Under the interval-valued Pythagorean fuzzy information environment, Peng and Li [39] presented new interval-valued Pythagorean fuzzy information measures, including entropy, distance, and similarity measures, and then applied these information measures to derive the ranking of alternatives. Zeng et al. [40] proposed a series of Pythagorean fuzzy similarity measures, and then established a Pythagorean fuzzy MADM method.

From the above analysis, it can be seen that entropy and similarity measures are useful tools for addressing the fuzziness and uncertainty characteristics of complex decision-making problems. More and more MADM methods have been constructed using the entropy and similarity measures. Under the Pythagorean fuzzy linguistic information environment, studying the axiomatic definitions of Pythagorean fuzzy linguistic entropy and similarity measures, constructing reliable information measurement formulas, and exploring the relationship of Pythagorean fuzzy linguistic information measures are significant and challenging issues. Although there exist reasonable methods to address Pythagorean fuzzy or Pythagorean fuzzy linguistic MADM problems, these methods have limitations. To this end, by considering the degrees of membership, non-membership, and hesitation, Wei and
Wei [41] proposed a weighted similarity measure with a cosine function, and then developed a Pythagorean fuzzy MADM method. However, with the method developed by Wei and Wei [41], we use the Pythagorean fuzzy weighted similarity measure to calculate the weighted similarity degrees, in which some original decision-making information is lost. Thus, the decision-making results derived by the method of Wei and Wei [41] may be unreliable. Therefore, it is reasonable and necessary to propose a new method for generating decision-making results directly. Based on the weighted Pythagorean fuzzy Bonferroni mean (WPFBM) operator, Liang et al. [42] presented a new MADM method through which the ranking values of the alternatives could be obtained. However, the method by Liang et al. [42] cannot deal with MADM problems in which the information of attribute weights is completely unknown. Under the linguistic Pythagorean fuzzy information environment, Garg [43] developed a novel decision-making method on the basis of the linguistic Pythagorean fuzzy weighted average operator. It is known that the differences among these attributes and the type of attributes are different, thus the process of normalization for the decision-making matrix is necessary. However, Garg’s [43] method does not normalize the initial Pythagorean fuzzy linguistic decision-making matrix, and it directly uses the linguistic Pythagorean fuzzy weighted average operator [43] to generate the decision-making results (see details given in Section 5). Therefore, in this paper, we investigated a Pythagorean fuzzy linguistic MADM model to directly use the DM’s original evaluation information, in which the new Pythagorean fuzzy linguistic entropy and similarity measures are presented, and the DM’s original evaluation information can be preserved as much as possible.

Consequently, in order to overcome these limitations, the following research issues were studied:

- A new concept of PFLSs is introduced, which we believe to be more reasonable and convenient to express uncertain evaluation information;
- Two axiomatic definitions of information measures for PFLVs are presented;
- With the help of logarithmic functions, two new information measure formulas were constructed;
- A novel Pythagorean fuzzy linguistic multi-attribute decision-making model was developed to derive reliable ranking of the alternatives.

The rest of this paper is organized as follows. Section 2 reviews some basic concepts of LTSs and PFSs, and then introduces the new concept of PFLSs. In Section 3, two axiomatic definitions of information measures for PFLV are presented, and two new information measure formulas are constructed. We also explore the relationship between the Pythagorean fuzzy linguistic entropy and similarity measures in Section 3. Section 4 investigates a novel MADM model with Pythagorean fuzzy linguistic information measures. In Section 5, a numerical example is given to illustrate the application of the proposed decision-making model. Conclusions and further research are presented in the last section.

2. Preliminaries

In this section, the main concepts related to LTSs and PFSs are reviewed, and then the new concept of Pythagorean fuzzy linguistic sets is presented.

2.1. LTSs and PTSs

As a symbolic linguistic computing model, the concept of a virtual linguistic model is usually used as a linguistic computing model. Suppose that $S = \{s_0, s_1, \cdots, s_{2τ}\}$ is a discrete LTS, where $s_i$ is a linguistic variable in $S$. The LTS $S$ has the following characteristics [19,20]: (1) if $p \geq q$, then $s_p \geq s_q$; and (2) the negation operator $neg(s_p) = s_{2τ−p}$.
Example 1. If the DM evaluates a supplier in a supply chain, a set of nine terms $S$ can be expressed as follows:

$$ S = \{ s_0 : \text{extremely poor}, s_1 : \text{very poor}, s_2 : \text{poor}, $$

$$ s_3 : \text{sightly poor}, s_4 : \text{fair}, s_5 : \text{sightly good}, $$

$$ s_6 : \text{good}, s_7 : \text{very good}, s_8 : \text{extremely good} \} $$

to preserve the decision-making information, Xu [20] generalized the notion of a discrete LTS to a continuous one, $\mathcal{S} = \{ s_p | p \in [0, 2\tau] \}$, where $2\tau$ is a sufficiently large positive integer. It is readily seen that these linguistic terms themselves map into the respective indices: $1 : \mathcal{S} \to [0, 1]$, such that $I(s_\mu) = \frac{\mu}{2\tau}$. Obviously, there exists an inverse function $I^{-1} : [0, 1] \to \mathcal{S}$, such that $I^{-1}(p) = s_{2\tau p}$.

**Definition 1.** Reference [17] assumes that $X = \{ x_1, x_2, \cdots, x_m \}$ is a universe set, a PFS $A = \{ (x_i, \mu_A(x_i), \nu_A(x_i)) | x_i \in X \}$ on $X$ is characterized by the membership function $\mu_A(x_i)$ and the non-membership function $\nu_A(x_i)$, where $\mu_A(x_i), \nu_A(x_i) \in [0, 1]$, and $(\mu_A(x_i))^2 + (\nu_A(x_i))^2 \leq 1$, for $\forall x_i \in X$.

$$ \pi_A(x_i) = \sqrt{1 - (\mu_A(x_i))^2 - (\nu_A(x_i))^2} $$

called the hesitant degree of $x_i \in X$.

2.2. Pythagorean Fuzzy Linguistic Sets (PFLSs)

**Definition 2.** Assume that $X = \{ x_1, x_2, \cdots, x_m \}$ is a universe set, $\mathcal{S} = \{ s_p | p \in [0, 2\tau] \}$ is a continuous LTS, and PFLS $B$ over $X$ can be described as:

$$ B = \{ (x_i, \mu_B(x_i), \nu_B(x_i)) | x_i \in X \} $$

(1)

where $\mu_B : X \to \mathcal{S}$ and $\nu_B : X \to \mathcal{S}$ represent the linguistic membership degree and linguistic non-membership degree of the element $x_i \in X$ to $B$, respectively. For each $x_i \in X$, $\mu_B(x_i), \nu_B(x_i) \in \mathcal{S}$, and $I((\mu_B(x_i))^2 + I((\nu_B(x_i))^2 \leq 1$.

For convenience, we call $a_i = (\mu_i, \nu_i) = (\mu_B(x_i), \nu_B(x_i))$ a Pythagorean fuzzy linguistic value (PFLV). The complement of $a_i$ is denoted by $a_i^C = (\nu_i, \mu_i)$. Let $\Omega$ be the set of all the PFLVs.

3. The Pythagorean Fuzzy Linguistic Entropy and Pythagorean Fuzzy Linguistic Similarity Measure

This section first introduces two axiomatic definitions of Pythagorean fuzzy linguistic information measures, including the Pythagorean fuzzy linguistic entropy and Pythagorean fuzzy linguistic similarity measure, and then two new information measure formulas were constructed. We further explore the relationship between the Pythagorean fuzzy linguistic entropy and similarity measure.

3.1. Pythagorean Fuzzy Linguistic Entropy

**Definition 3.** Assume that $\alpha = (\mu_\alpha, \nu_\alpha)$ is a PFLV with $\mu_\alpha, \nu_\alpha \in \mathcal{S} = \{ s_p | p \in [0, 2\tau] \}$, if a mapping $E : \Omega \to [0, 1]$ satisfies the following four axiomatic conditions:

(E1) $E(\alpha) = 0$ if and only if $\alpha = (s_{2\tau}, s_0)$ or $(s_0, s_{2\tau})$;

(E2) $E(\alpha) = 1$ if and only if $\mu_\alpha = \nu_\alpha$;

(E3) $E(\alpha) = E(\alpha^C)$;

(E4) $E(\alpha) \leq E(\beta)$, if $\mu_\alpha \leq \mu_\beta$ and $\nu_\alpha \geq \nu_\beta$ when $\mu_\beta \leq \nu_\beta$ or $\mu_\alpha \geq \mu_\beta$ and $\nu_\alpha \leq \nu_\beta$ when $\mu_\beta \geq \nu_\beta$;

then, $E$ is called a Pythagorean fuzzy linguistic entropy on the set of $\Omega$. 


Assume that \( \alpha = \langle \mu_a, v_a \rangle \) is a PFLV, we constructed the following information measure formula for \( \alpha \) by utilizing the logarithmic function:

\[
e(\alpha) = -\frac{1}{\ln 2} \left( \frac{\mu_a}{2} - 1 - \frac{\mu_a}{2} \ln \frac{\mu_a}{2} + \frac{\mu_a}{2} \ln \frac{\mu_a}{2} \right) + \left( 1 - \frac{\mu_a}{2} + 1 - \frac{\mu_a}{2} \right) \ln \left( 1 - \frac{\mu_a}{2} + 1 - \frac{\mu_a}{2} \right)
\]

(2)

**Theorem 1.** Let \( \alpha = \langle \mu_a, v_a \rangle \) be a PFLV with \( \mu_a, v_a \in 2^S = \{ s_p | p \in [0, 2\pi] \} \), then the information measure formula \( e(\alpha) \), constructed by Equation (2), is a Pythagorean fuzzy linguistic entropy of \( \Omega \).

**Proof.** According to Definition 3, we need to prove that the information measure formula \( e(\alpha) \) meets the four axiomatic conditions in Definition 3.

As \( \alpha = \langle \mu_a, v_a \rangle \) is a PFLV, then \( I(\mu_a), I(v_a) \in [0, 1] \) and \( P^2(\mu_a) + P^2(v_a) \leq 1 \), thus \( \frac{P(\mu) + 1 - P(v_a)}{2} \in [0, 1] \). Now, we first establish a continuous function \( g(x) \) on \([0, 1]\), as follows:

\[
g(x) = \begin{cases} 
0, & x = 0 \\
-\frac{1}{\ln 2} (x \ln x + (1-x) \ln (1-x)), & x \in (0, 1) \\
0, & x = 1
\end{cases}
\]

(3)

Thus, \( \frac{dg(x)}{dx} = \frac{1}{\ln 2} \ln \frac{1-x}{x} \), \( x \in (0, 1) \). It is obvious that \( \frac{dg(x)}{dx} \geq 0 \) when \( x \in (0, \frac{1}{2}) \), \( \frac{dg(x)}{dx} \leq 0 \) when \( x \in [\frac{1}{2}, 0) \). Therefore, if \( x \in (0, \frac{1}{2}) \), then \( g(x) \) is an increasing function with respect to \( x \); if \( x \in [\frac{1}{2}, 0) \), then \( g(x) \) is a decreasing function with respect to \( x \). Furthermore, one can get that \( g_{\text{min}}(x) = 0 \) if and only if \( x = 0 \) or \( 1 \), \( g_{\text{max}}(x) = 1 \) if and only if \( x = \frac{1}{2} \). The graphical representation of \( g(x) \) is shown in Figure 1.

![Figure 1](image.png)

**Figure 1.** The graphical representation of \( g(x) \).

(E1) If \( \alpha = \langle s_2, s_0 \rangle \) or \( \langle s_0, s_2 \rangle \), then we have \( \frac{P(\mu_a) + 1 - P(v_a)}{2} = 1 \) or \( \frac{P(\mu_a) + 1 - P(v_a)}{2} = 0 \).

According to the above analysis and Equation (3), one can obtain that \( e(\alpha) = 0 \).

Suppose that \( e(\alpha) = 0 \); that is, \( g\left( \frac{P(\mu_a) + 1 - P(v_a)}{2} \right) = 0 \). From the above analysis, we have \( g(x) = 0 \) if and only if \( x = 0 \) or \( 1 \), then \( g\left( \frac{P(\mu_a) + 1 - P(v_a)}{2} \right) = 0 \) indicates that \( \frac{P(\mu_a) + 1 - P(v_a)}{2} = 1 \) or \( \frac{P(\mu_a) + 1 - P(v_a)}{2} = 0 \):

\[
P(\mu_a) + 1 - P(v_a) = 2 \text{ or } P(\mu_a) + 1 - P(v_a) = 0
\]

(4)
While \( I(\mu_a), I(\nu_a) \in [0,1] \), thus \( I(\mu_a) = 1, I(\nu_a) = 0 \) or \( I(\mu_a) = 0, I(\nu_a) = 1 \). Therefore, \( \mu_a = s_2t_r, \nu_a = s_0 \) or \( \mu_a = s_0, \nu_a = s_2t_r \), i.e., \( \alpha = s_2t_r, s_0 \) or \( \alpha = s_0, s_2t_r \).

(E2) By using the above analysis of function \( g(x) \) on \([0,1]\), if \( e(\alpha) = 1 \):

\[
\left( \frac{P_{(\mu_a)} + 1 - P_{(\nu_a)}}{2} \right) = 1 \Leftrightarrow \frac{P_{(\mu_a)} + 1 - P_{(\nu_a)}}{2} = \frac{1}{2} \tag{5}
\]

(E3) Because \( \alpha^c = (v_a, \mu_a) \), then:

\[
e(\alpha^c) = -\frac{1}{\ln 2} \left( \frac{P_{(\mu_a')} + 1 - P_{(\nu_a')}}{2} \ln \frac{P_{(\mu_a')} + 1 - P_{(\nu_a')}}{2} + \left( 1 - \frac{P_{(\mu_a')} + 1 - P_{(\nu_a')}}{2} \right) \ln \left( 1 - \frac{P_{(\mu_a')} + 1 - P_{(\nu_a')}}{2} \right) \right) = \frac{\ln 2}{\ln \alpha} \tag{6}
\]

(E4) Suppose that \( \mu_a \leq \mu' \) and \( v_a \geq v' \) when \( \mu_a \leq v' \), we can derive that 
\( 0 \leq I(\mu_a) \leq I(\mu') \leq I(v_a) \leq I(v') \leq 1 \) and \( P_{(\mu_a)} \leq P_{(\mu')} \), \( P_{(v_a)} \geq P_{(v')} \).

It follows that:

\[
\frac{P_{(\mu_a)} + 1 - P_{(\nu_a)}}{2} \leq \frac{P_{(\mu')} + 1 - P_{(\nu')}}{2} \tag{7}
\]

In addition, as \( 0 \leq I(\mu_a) \leq I(\mu') \leq I(v_a) \leq I(v') \leq 1 \), then \( -1 \leq \frac{P_{(\mu_a)} + 1 - P_{(\nu_a)}}{2} \leq \frac{P_{(\mu')} + 1 - P_{(\nu')}}{2} \leq 0 \), and one can obtain that:

\[
0 \leq \frac{P_{(\mu_a)} + 1 - P_{(\nu_a)}}{2} \leq \frac{P_{(\mu')} + 1 - P_{(\nu')}}{2} \leq \frac{1}{2} \tag{8}
\]

Owing to \( g(x) \) being an increasing function with respect to \( x \), when \( x \in (0, \frac{1}{2}] \), thus:

\[
\left( \frac{P_{(\mu_a)} + 1 - P_{(\nu_a)}}{2} \right) \leq \left( \frac{P_{(\mu')} + 1 - P_{(\nu')}}{2} \right) \tag{9}
\]

that is, \( e(\alpha) \leq e(\beta) \).

Similarly, if \( \mu_a \geq \mu' \) and \( v_a \leq v' \) when \( \mu_a \geq v' \), one can get that \( e(\alpha) \leq e(\beta) \).

Therefore, the proof of Theorem 1 is completed. \( \square \)

3.2. Pythagorean Fuzzy Linguistic Similarity Measure

In order to measure the similarity degree among PFLVs, this subsection introduces the concept of the Pythagorean fuzzy linguistic similarity measure, and then a new kind of similarity measure formula was designed based on the logarithmic function.

**Definition 4.** Assume that \( \alpha = (\mu_a, v_a) \) and \( \beta = (\mu'_a, v'_a) \) are two PFLVs, if a mapping \( S : \Omega \times \Omega \rightarrow [0,1] \) satisfies the following four axiomatic conditions:

(S1) \( S(\alpha, \beta) = 0 \) if and only if \( \alpha = (s_2t_r, s_0) \), \( \beta = (s_0, s_2t_r) \) or \( \alpha = (s_0, s_2t_r) \), \( \beta = (s_2t_r, s_0) \);

(S2) \( S(\alpha, \beta) = 1 \) if and only if \( \mu_a = \mu'_b, v_a = v'_b \);

(S3) \( S(\alpha, \beta) = S(\beta, \alpha) \);

(S4) \( S(\alpha, \gamma) \leq S(\alpha, \beta), S(\alpha, \gamma) \leq S(\beta, \gamma) \), if \( \mu_a \leq \mu'_b \) and \( v_a \geq v'_b \) or \( \mu_a \geq \mu'_b \) and \( v_a \leq v'_b \); then, \( S \) is called a Pythagorean fuzzy linguistic similarity measure of \( \Omega \).
Assume that $\alpha = \langle \mu_\alpha, v_\alpha \rangle$ and $\beta = \langle \mu_\beta, v_\beta \rangle$ are two PFLVs, we constructed the following information measure formula by utilizing the logarithmic function:

$$s(\alpha, \beta) = \frac{1}{2} \sum_{\xi = \mu, v} \left( \frac{1}{\ln 2} \left( \frac{\mu(\xi) + 1 - \nu(\xi)}{2} \cdot \ln \frac{\mu(\xi) + 1 - \nu(\xi)}{2} + \left(1 - \frac{\mu(\xi) + 1 - \nu(\xi)}{2}\right) \cdot \ln \left(1 - \frac{\mu(\xi) + 1 - \nu(\xi)}{2}\right) \right) \right)$$

(10)

**Theorem 2.** Let $\alpha = \langle \mu_\alpha, v_\alpha \rangle$ and $\beta = \langle \mu_\beta, v_\beta \rangle$ be two PFLVs, then the information measure formula $s(\alpha, \beta)$, constructed by Equation (10), is a Pythagorean fuzzy linguistic similarity measure of $\Omega$.

**Proof.** As $\alpha = \langle \mu_\alpha, v_\alpha \rangle$ and $\beta = \langle \mu_\beta, v_\beta \rangle$ are two PFLVs, then $\frac{\mu(\xi) + 1 - \nu(\xi)}{2} \in [0, 1]$, for each $\xi = \mu, v$. Thus, according to the proof of Theorem 1, Equation (10) can be rewritten as $s(\alpha, \beta) = \frac{1}{2} \sum_{\xi = \mu, v} g\left(\frac{\mu(\xi) + 1 - \nu(\xi)}{2}\right)$.

(S1) If $\alpha = \langle s_2, s_0 \rangle, \beta = \langle s_0, s_2 \rangle$ or $\alpha = \langle s_0, s_2 \rangle, \beta = \langle s_2, s_0 \rangle$, then:

$$\frac{\mu(\xi) + 1 - \nu(\xi)}{2} = 0 \text{ or } \frac{\mu(\xi) + 1 - \nu(\xi)}{2} = 1, \text{ for } \xi = \mu, v$$

Thus, $g\left(\frac{\mu(\xi) + 1 - \nu(\xi)}{2}\right) = 0, \xi = \mu, v$, and we have $s(\alpha, \beta) = 0$.

Assume that $s(\alpha, \beta) = 0$. Because $\frac{\mu(\xi) + 1 - \nu(\xi)}{2} \in [0, 1], \xi = \mu, v$, then for each $\xi = \mu, v$, we have $g\left(\frac{\mu(\xi) + 1 - \nu(\xi)}{2}\right) \geq 0$; therefore, $s(\alpha, \beta) = 0$ indicates that $g\left(\frac{\mu(\xi) + 1 - \nu(\xi)}{2}\right) = 0, \xi = \mu, v$. According to the analysis of function $g(x)$ on $[0, 1]$ in Theorem 1, we have $\frac{\mu(\xi) + 1 - \nu(\xi)}{2} = 0$ or $\frac{\mu(\xi) + 1 - \nu(\xi)}{2} = 1$, for $\xi = \mu, v$, and then $\alpha = \langle s_2, s_0 \rangle, \beta = \langle s_0, s_2 \rangle$ or $\alpha = \langle s_0, s_2 \rangle, \beta = \langle s_2, s_0 \rangle$.

(S2) Owing to $\frac{\mu(\xi) + 1 - \nu(\xi)}{2} \in [0, 1]$, for each $\xi = \mu, v$, then $0 \leq g\left(\frac{\mu(\xi) + 1 - \nu(\xi)}{2}\right) \leq 1, \xi = \mu, v$; therefore,

$$s(\alpha, \beta) = 1 \Leftrightarrow \frac{1}{2} \sum_{\xi = \mu, v} g\left(\frac{\mu(\xi) + 1 - \nu(\xi)}{2}\right) = 1 \Leftrightarrow g\left(\frac{\mu(\xi) + 1 - \nu(\xi)}{2}\right) = 1, \xi = \mu, v$$

\begin{align}
\Leftrightarrow & \frac{\mu(\xi) + 1 - \nu(\xi)}{2} = \frac{1}{2}, \xi = \mu, v \Leftrightarrow I(\xi_\alpha) = I(\xi_\beta), \xi = \mu, v \\
\Leftrightarrow & I(\mu_\alpha) = I(\mu_\beta), I(v_\alpha) = I(v_\beta) \\
\Leftrightarrow & \mu_\alpha = \mu_\beta, v_\alpha = v_\beta
\end{align}

(11)

(S3)

$$s(\beta, \alpha) = \frac{1}{2} \sum_{\xi = \mu, v} \left( -\frac{1}{\ln 2} \left( \frac{\mu(\xi) + 1 - \nu(\xi)}{2} \cdot \ln \frac{\mu(\xi) + 1 - \nu(\xi)}{2} + \left(1 - \frac{\mu(\xi) + 1 - \nu(\xi)}{2}\right) \cdot \ln \left(1 - \frac{\mu(\xi) + 1 - \nu(\xi)}{2}\right) \right) \right) + \frac{\mu(\xi) + 1 - \nu(\xi)}{2} \cdot \ln \left(\frac{\mu(\xi) + 1 - \nu(\xi)}{2}\right)$$

(12)

$$= s(\alpha, \beta).$$
Theorem 3. Assume that \( \alpha = (\mu_\alpha, v_\alpha) \) is a PFLV with \( \mu_\alpha, v_\alpha \in \tilde{S} = \{ \mu \mid \mu \in [0, 2\pi] \} \), then the Pythagorean fuzzy linguistic similarity measure between \( \alpha \) and \( \alpha' \) is the Pythagorean fuzzy linguistic entropy of \( \alpha \); that is, 
\[ E(\alpha) = S(\alpha, \alpha') \]

Proof. Now, we prove that \( S(\alpha, \alpha') \) satisfies the four axiomatic conditions in Definition 3.

1. \( E(\alpha) = 0 \Leftrightarrow S(\alpha, \alpha') = 0 \Leftrightarrow \alpha = (s_{2\alpha}, s_0), \alpha' = (s_0, s_{2\alpha}) \) or \( \alpha = (s_0, s_{2\alpha}), \alpha' = (s_{2\alpha}, s_0) \), i.e., 
\[ \alpha = (s_{2\alpha}, s_0) \text{ or } (s_0, s_{2\alpha}) \].

2. \( E(\alpha) = 1 \Leftrightarrow S(\alpha, \alpha') = 1 \Leftrightarrow \mu_\alpha = \mu_{\alpha'}, \nu_\alpha = \nu_{\alpha'} \Leftrightarrow \mu_\alpha = v_\alpha \).

3. \( E(\alpha') = S(\alpha', \alpha') = S(\alpha', \alpha) = E(\alpha) \).

4. Let \( \beta = (\mu_\beta, v_\beta) \) be a PFLV, if \( \mu_\alpha \leq \mu_\beta \) and \( \nu_\alpha \geq \nu_\beta \), then
\[ 0 \leq I(\mu_\alpha) \leq I(\mu_\beta) \leq I(\nu_\beta) \leq I(\nu_\alpha) \leq 1. \]

Thus \( 0 \leq I(\mu_\alpha) \leq I(\mu_\beta) \leq I(\mu_{\alpha'}) \leq I(\mu_{\beta'}) \leq I(\nu_\beta) \geq I(\nu_\alpha) \geq I(\nu_{\alpha'}) \geq 0 \).

Utilizing the axiomatic condition (S4) in Definition 2, one can obtain that \( S(\alpha, \alpha') \leq S(\beta, \alpha') \leq S(\beta, \beta') \), that is,

Similarly, if \( \mu_\alpha \geq \mu_\beta \) and \( \nu_\alpha \leq \nu_\beta \), one can obtain that \( E(\alpha) \leq E(\beta) \). The proof of Theorem 3 is completed.

4. The MADM Model with Pythagorean Fuzzy Linguistic Information Measures

In this section, we investigate a new MADM model with Pythagorean fuzzy linguistic information measures to rank the alternatives and select the desirable alternative. The main steps of the proposed MADM model are as follows:

4.1. Step 1: Constructing the Initial Pythagorean Fuzzy Linguistic Decision-Making Matrix

Suppose that \( X = \{x_1, x_2, \cdots, x_m\} \) is a given set of alternatives, \( C = \{C_1, C_2, \cdots, C_n\} \) is an attribute set with the weight vector \( w = (w_1, w_2, \cdots, w_n)^T \), satisfying \( w_j \geq 0 \) and \( \sum_{j=1}^n w_j = 1 \). Owing to the
increasing complexity of the real decision-making environment, the information about attribute weights is completely unknown. Furthermore, in order to transform and express the evaluation information of experts comprehensively, we utilize the PFLV \( \alpha_{ij} = \{\mu_{ij}, v_{ij}\} \) with \( \mu_{ij}, v_{ij} \in \mathbb{F} = \{s_p | p \in [0, 2]\} \) to depict the evaluation information, where \( \mu_{ij} \) and \( v_{ij} \) are the linguistic membership degree and linguistic non-membership degree of alternative \( x_i \) under the attribute \( C_j \), respectively. Based on this, a Pythagorean fuzzy linguistic decision-making matrix \( D = (\alpha_{ij})_{m \times n} \) was constructed.

4.2. Step 2: Normalization of the Pythagorean Fuzzy Linguistic Decision-Making Matrix

Owing to there being a difference among these attributes and the type of attributes, we needed to normalize the initial Pythagorean fuzzy linguistic decision-making matrix \( D = (\alpha_{ij})_{m \times n} \). Generally, the alternative’s attributes can be divided into two types, including the benefit attribute and cost attribute. Therefore, we utilized the following transformation method to derive the normalized Pythagorean fuzzy linguistic decision-making matrix \( \Phi = (\beta_{ij})_{m \times n} \), in which all the attributes were benefit attributes:

\[
\beta_{ij} = \begin{cases} 
(\mu_{ij}, v_{ij}), & C_j \text{ is benefit attribute} \\
(\mu_{ij}, v_{ij}), & C_j \text{ is cost attribute}
\end{cases}, \quad i = 1, 2, \cdots, m; j = 1, 2, \cdots, n
\] (13)

4.3. Step 3: Determining the Attribute Weights with Pythagorean Fuzzy Linguistic Entropy

As we all know, information entropy is an effective method to measure information uncertainty. The greater the information entropy, the higher the degree of ambiguity; the smaller the information entropy, the higher the deterministic information. In Pythagorean fuzzy linguistic MADM problems, if the Pythagorean fuzzy linguistic entropy of all the attribute values under an attribute is smaller, it reflects that this attribute provides more valuable information for DMs in the process of decision-making, and the attribute weight of this attribute should be given a larger attribute weight. If the Pythagorean fuzzy linguistic entropy of all the attribute values under an attribute is larger, it means that this attribute can only provide less valuable information for DMs in the process of decision-making, thus the attribute weight of this attribute should be smaller. Therefore, the attribute weight is inversely proportional to the total Pythagorean fuzzy linguistic entropy under this attribute. As the information about attribute weights is completely unknown, we designed the following entropy-based method to calculate attribute weights:

\[
w_j = \frac{1 - e_j}{\sum_{i=1}^{m} (1 - e_i)}, \quad j = 1, 2, \cdots, n
\] (14)

where \( e_j = \frac{1}{m} \sum_{i=1}^{m} e(\beta_{ij}) \) and:

\[
e(\beta_{ij}) = -\frac{1}{\ln 2} \left[ \frac{2^{\mu_{ij}} + 2^{v_{ij}}}{2} \ln \frac{2^{\mu_{ij}} + 2^{v_{ij}}}{2} + \left( 1 - \frac{2^{\mu_{ij}} + 2^{v_{ij}}}{2} \right) \ln \left( 1 - \frac{2^{\mu_{ij}} + 2^{v_{ij}}}{2} \right) \right]
\] (15)

4.4. Step 4: Obtaining the Weighted Similarity Degree for an Alternative with the Pythagorean Fuzzy Linguistic Similarity Measure

Let \( \beta^+ = \{\beta_1^+, \beta_2^+, \cdots, \beta_n^+\} = \{s_{2+}, s_{0}, s_{2+}, s_{0}, \cdots, s_{2+}, s_{0}\} \) be the ideal alternative, \( \beta^- = \{\beta_1^-, \beta_2^-, \cdots, \beta_n^-\} = \{s_{0}, s_{2+}, s_{0}, s_{2+}, s_{0}, \cdots, s_{0}, s_{2+}\} \) be the anti-ideal alternative, where \( s_{2+}, s_{0} \) and \( s_{0}, s_{2+} \) indicate the ideal PFLV and anti-ideal PFLV, respectively. First, by using the proposed Pythagorean fuzzy linguistic similarity measure (i.e., Equation (10)), one can obtain the similarity degree \( s(\beta_{ij}, \beta_j^+) \) between the evaluation information \( \beta_{ij} \) of alternative \( x_i \) under the attribute \( C_j \) and the ideal PFLV \( \beta_j^+ \) and the similarity degree \( s(\beta_{ij}, \beta_j^-) \) between evaluation information \( \beta_{ij} \) and the ideal PFLV \( \beta_j^- \).
Then, based on the attribute weight vector \( w = (w_1, w_2, \ldots, w_n)^T \) determined by Equation (14), we utilized the following methods to calculate the weighted similarity degrees \( S^+_i(i = 1, 2, \ldots, m) \) and \( S^-_i (i = 1, 2, \ldots, m) \):

\[
S^+_i = \sum_{j=1}^{n} w_j \cdot s(\beta_{ij}, \beta^+_j), i = 1, 2, \ldots, m
\]

\[
S^-_i = \sum_{j=1}^{n} w_j \cdot s(\beta_{ij}, \beta^-_j), i = 1, 2, \ldots, m
\]

where \( S^+_i \) and \( S^-_i \) indicate the weighted similarity degrees of alternative \( x_i \) with the ideal alternative \( \beta^+ \) and anti-ideal alternative \( \beta^- \).

4.5. Step 5: Deriving the Closeness Degrees of Alternatives

It is obvious that the greater the weighted similarity degree \( S^+_i \) between alternative \( x_i \) and ideal alternative \( \beta^+ \), the smaller the weighted similarity degree \( S^-_i \) between alternative \( x_i \) and anti-ideal alternative \( \beta^- \), then the better comprehensive performance of the alternative \( x_i \). Therefore, the closeness degree of the alternative \( x_i \) can be determined as follows:

\[
T_i = \frac{S^+_i}{S^+_i + S^-_i}, i = 1, 2, \ldots, m
\]

4.6. Step 6: Ranking the Alternatives

According to the obtained closeness degrees \( T_i (i = 1, 2, \ldots, m) \), we can sort \( T_i (i = 1, 2, \ldots, m) \) in descending order, and then the ranking order of the alternatives \( x_i (i = 1, 2, \ldots, m) \) can be determined, which is followed by the selection of a desirable alternative.

5. Illustrative Example and Comparative Analysis

In the following, an applied case of a sustainable blockchain product assessment is given to illustrate the effect of the developed decision-making model, then we discuss the comparative analysis between the proposed model and existing method.

5.1. Application to Sustainable Blockchain Product Assessment

Blockchain is the underlying technology of Bitcoin [44]. Owing to the safety and convenience of blockchain, it has gradually attracted the attention of banks and the financial industry [45–48]. Blockchain is divided into three main categories, including public blockchain, joint blockchain, and private blockchain. Blockchain management is an important guarantee for the development of blockchain. Evaluating blockchain products is a key point for blockchain management research.

Suppose that there are five possible sustainable blockchain products \( X = \{x_1, x_2, x_3, x_4, x_5\} \) to be evaluated. In order to evaluate and obtain the most desirable sustainable blockchain products, the expert evaluates the above five sustainable blockchain products by considering four attributes, including \( C_1: \) the cost of products; \( C_2: \) the lifetime of products; \( C_3: \) the performance of products; and \( C_4: \) the quality of after-sales service, while the attribute weight vector \( w = (w_1, w_2, w_3, w_4)^T \) is completely unknown. The expert utilizes PFLV \( a_{ij} = (\mu_{ij}, \nu_{ij}) \) with \( \mu_{ij}, \nu_{ij} \in S = \{s_p | p \in [0, 8]\} \) to
express the evaluation information, and then a Pythagorean fuzzy linguistic decision-making matrix
$D = (a_{ij})_{5 \times 4}$ is constructed as follows:

$$
D = \begin{bmatrix}
1.0 & 0.6519 & 0.6507 & 0.1131 \\
0.5060 & 0.6397 & 0.4946 & 0.4866 \\
0.7258 & 0.4866 & 0.4866 & 0.8 \\
0.7432 & 0.7258 & 0.4866 & 0.6507 \\
0.9117 & 0.7432 & 0.7258 & 0.8
\end{bmatrix}
$$

In the following, we utilized the developed Pythagorean fuzzy linguistic MADM model to rank
the above five sustainable blockchain products and explore the most desirable sustainable blockchain
product. The detailed steps are as follows.

5.1.1. Step 1

As $C_1$ is the cost attribute, we utilized Equation (13) to obtain the following normalized Pythagorean
fuzzy linguistic decision-making matrix $\Phi = (\beta_{ij})_{5 \times 4}$:

$$
\Phi = \begin{bmatrix}
1.0 & 0.6519 & 0.6507 & 0.1131 \\
0.5060 & 0.6397 & 0.4946 & 0.4866 \\
0.7258 & 0.4866 & 0.4866 & 0.8 \\
0.7432 & 0.7258 & 0.4866 & 0.6507 \\
0.9117 & 0.7432 & 0.7258 & 0.8
\end{bmatrix}
$$

5.1.2. Step 2

By using Equations (14) and (15), one can obtain the attribute weights as follows:

$$w_1 = 0.0228, w_2 = 0.3661, w_3 = 0.4980, w_4 = 0.1131$$

5.1.3. Step 3

Applying Equations (10), (16), and (17), we obtained the weighted similarity degrees $S^+_1$ and $S^-_1$
of sustainable blockchain product $x_i$:

$$S^+_1 = 0.8092, S^+_2 = 0.9259, S^+_3 = 0.7432, S^+_4 = 0.9216, S^+_5 = 0.9117,$$

$$S^-_1 = 0.7716, S^-_2 = 0.5215, S^-_3 = 0.7258, S^-_4 = 0.4946, S^-_5 = 0.4866.$$

5.1.4. Step 4

According to Equation (18), we calculated the closeness degrees $T_i (i = 1, 2, \cdots , 5)$ of sustainable
blockchain products $x_i (i = 1, 2, \cdots , 5)$:

$$T_1 = 0.5100, T_2 = 0.6397, T_3 = 0.5060, T_4 = 0.6507, T_5 = 0.6519.$$

5.1.5. Step 5

It was obvious that $T_5 > T_4 > T_2 > T_1 > T_3$, then could determine that the ranking of the sustainable blockchain products was $x_5 > x_4 > x_2 > x_1 > x_3$, and the most desirable sustainable blockchain product was $x_5$.

5.2. Comparative Analysis and Discussion

In the following, we compare our proposed decision-making model with previous methods in
the literature to verify the effectiveness of the developed MADM model, and then we explore the
advantages of the developed model.
By considering the degrees of membership, non-membership, and hesitation, Wei and Wei [41] proposed a weighted similarity measure with a cosine function and then developed a Pythagorean fuzzy MADM method. For the same Pythagorean fuzzy linguistic decision-making matrix \( D = (a_{ij})_{5x4} \), we used the method in Wei and Wei [41] to derive the most desirable sustainable blockchain product.

5.2.1. Step 1

By using the decision-making matrix \( D = (a_{ij})_{5x4} \) and Equation (14), we determined the normalized Pythagorean fuzzy linguistic decision-making matrix \( \Phi = (\beta_{ij})_{5x4} = (\mu_{\beta_{ij}}, \nu_{\beta_{ij}})_{5x4} \) and attribute weight vector (See Sections 5.1.1 and 5.1.2).

5.2.2. Step 2

By using the function \( I : \mathbb{F} \rightarrow [0,1] \), we could transform the normalized Pythagorean fuzzy linguistic decision-making matrix \( \Phi = (\beta_{ij})_{5x4} \) into the Pythagorean fuzzy decision-making matrix \( \Theta = (\gamma_{ij})_{5x4} \) (where \( \gamma_{ij} = (\mu_{\gamma_{ij}}, \nu_{\gamma_{ij}}) \)) and:

\[
\Theta = \begin{bmatrix}
(0.4, 0.8) & (0.8, 0.6) & (0.6, 0.7) & (0.6, 0.5) \\
(0.7, 0.5) & (0.9, 0.2) & (0.8, 0.5) & (0.5, 0.6) \\
(0.3, 0.4) & (0.3, 0.7) & (0.7, 0.4) & (0.5, 0.4) \\
(0.6, 0.6) & (0.7, 0.5) & (0.8, 0.1) & (0.8, 0.2) \\
(0.5, 0.7) & (0.6, 0.4) & (0.9, 0.3) & (0.7, 0.1)
\end{bmatrix}
\]

5.2.3. Step 3

Applying the following Pythagorean fuzzy cosine weighted similarity measures [41] to compute the weighted similarity degrees of the sustainable blockchain product \( x_i \):

\[
WPFCs(\gamma^+_{ij}) = \sum_{j=1}^{4} w_j \cos\left(\frac{\pi}{2} \cdot \max\left\{ \mu_{\gamma_{ij}}^2 - \mu_{\gamma_{ij}} \cdot \nu_{\gamma_{ij}}^2, \nu_{\gamma_{ij}}^2 - \nu_{\gamma_{ij}} \cdot \mu_{\gamma_{ij}}^2, \mu_{\gamma_{ij}}^2 - \nu_{\gamma_{ij}}^2, \nu_{\gamma_{ij}}^2 - \mu_{\gamma_{ij}} \cdot \nu_{\gamma_{ij}}^2 \right\}\right),
\]

\[
WPFCs(\gamma^-_{ij}) = \sum_{j=1}^{4} w_j \cos\left(\frac{\pi}{2} \cdot \max\left\{ \mu_{\gamma_{ij}}^2 - \mu_{\gamma_{ij}} \cdot \nu_{\gamma_{ij}}^2, \nu_{\gamma_{ij}}^2 - \nu_{\gamma_{ij}} \cdot \mu_{\gamma_{ij}}^2, \mu_{\gamma_{ij}}^2 - \nu_{\gamma_{ij}}^2, \nu_{\gamma_{ij}}^2 - \mu_{\gamma_{ij}} \cdot \nu_{\gamma_{ij}}^2 \right\}\right)
\]

Then we used Equation (18) to derive the closeness degrees \( T_i^\prime (i = 1, 2, \cdots, 5) \) of sustainable blockchain products \( x_i \) \((i = 1, 2, \cdots, 5)\) as follows:

\[
T_1' = 0.5873, T_2' = 0.6458, T_3' = 0.5544, T_4' = 0.6420, T_5' = 0.6617.
\]

5.2.4. Step 4

It can be seen that \( T_5' > T_2' > T_4' > T_1' > T_3' \). Therefore, the five sustainable blockchain products were ranked as: \( x_5 > x_2 > x_4 > x_1 > x_3 \), and the most desirable sustainable blockchain product was \( x_5 \).

5.3. The decision-making process with the method in Liang et al.

Liang et al. [42], based on the weighted Pythagorean fuzzy Bonferroni mean (WPFBM) operator, presented a new MADM method, following which the ranking values of the alternatives could be obtained. By using this method [42], the following steps are given to obtain the most desirable sustainable blockchain product:

5.3.1. Steps 1’ and 2’

See Sections 5.1.1 and 5.1.2.
5.3.2. Step 3’

By using the following WPFBM operator \([39]\) (let \(p = q = 2\)):

\[
y_i = \left\{ \sqrt[5]{1 - \prod_{j=1}^{5}(1 - (1 - x_{ij})^{\alpha_j})^{\frac{1}{5}}}, \sqrt[5]{1 - \prod_{j=1}^{5}(1 - (1 - y_{ij})^{\alpha_j})^{\frac{1}{5}}} \right\}^{\frac{1}{5}}.
\]

(21)

To aggregate all \(\gamma_{ij}(j = 1, 2, 3, 4)\) into an overall Pythagorean fuzzy value \(\gamma_i\) of the sustainable blockchain products \(x_i(i = 1, 2, 3, 4, 5)\) as follows:

\[
\gamma_1 = (0.6331, 0.6358), \gamma_2 = (0.6520, 0.4415), \\
\gamma_3 = (0.4619, 0.5134), \gamma_4 = (0.7088, 0.3695), \gamma_5 = (0.7113, 0.2384).
\]

5.3.3. Step 4’

Applying Definition 3 \([42]\), one can obtain the following score values \(s(\gamma_i)(i = 1, 2, 3, 4, 5)\) of the sustainable blockchain products \(x_i(i = 1, 2, 3, 4, 5)\):

\[
s(\gamma_1) = -0.0034, s(\gamma_2) = 0.2302, s(\gamma_3) = -0.0502, s(\gamma_4) = 0.3659, s(\gamma_5) = 0.4491.
\]

5.3.4. Step 5’

Because \(s(\gamma_5) > s(\gamma_4) > s(\gamma_2) > s(\gamma_1) > s(\gamma_3)\), then we had \(x_5 > x_4 > x_2 > x_1 > x_3\), and the most desirable sustainable blockchain product was \(x_5\).

5.4. The decision-making process with the method in Garg

Under the linguistic Pythagorean fuzzy information environment, Garg \([43]\) developed a novel decision-making method on the basis of the linguistic Pythagorean fuzzy weighted average operator. By using the method by Garg \([43]\), the following steps are used to obtain the desirable sustainable blockchain product:

First, according to Section 5.1, one can construct the linguistic Pythagorean fuzzy matrix

\[
D = (a_{ij})_{5 \times 4} = ((S_{0_{ij}}, S_{1_{ij}}))_{5 \times 4}.
\]

Owing to \(e(a_{ij}) = e(a_{ij}')\), then we can derive the same attribute weights with Step 2 by utilizing Equations (14) and (15):

\[
w_1 = 0.0228, w_2 = 0.3661, w_3 = 0.4980, w_4 = 0.1131
\]

Then, based on the following linguistic Pythagorean fuzzy weighted average operator \([43]\):

\[
a_i = \left( \frac{s}{2r} \sqrt{\prod_{j=1}^{r}(1 - (1 - a_{ij})^{\alpha_j})^{\frac{1}{r}}} \right)
\]

(22)

To fuse all the linguistic Pythagorean fuzzy values \(a_{ij} = (s_{0_{ij}}, s_{1_{ij}})\) into the collective linguistic Pythagorean fuzzy value \(a_i\) for each sustainable blockchain product \(x_i\):

\[
a_1 = (s_{6.1222}, s_{1.6247}), a_2 = (s_{4.6230}, s_{3.9019}), a_3 = (s_{5.5904}, s_{5.0305}), a_4 = (s_{6.6222}, s_{2.9421}), a_5 = (s_{6.5045}, s_{2.3826}).
\]

Furthermore, by using Definition 3.2 \([43]\), we derive the score functions \(S(a_i)(i = 1, 2, 3, 4, 5)\) of \(a_i(i = 1, 2, 3, 4, 5)\) as follows:

\[
S(a_1) = s_{7.0300}, S(a_2) = s_{8.9223}, S(a_3) = s_{8.9138}, S(a_4) = s_{7.0426}, S(a_5) = s_{7.0934}.
\]

Finally, as \(S(a_5) > S(a_4) > S(a_1) > S(a_2) > S(a_3)\), and then the ranking of the sustainable blockchain products is \(x_5 > x_4 > x_1 > x_2 > x_3\). Therefore, the most desirable sustainable blockchain product is \(x_5\).
According to above analysis, the ranking results and the most desirable sustainable blockchain product can be summarized in Table 1.

Table 1. The decision-making results by different methods.

<table>
<thead>
<tr>
<th>Approaches</th>
<th>The Ranking Results of the Sustainable Blockchain Products</th>
<th>The Most Desirable Sustainable Blockchain Product</th>
</tr>
</thead>
<tbody>
<tr>
<td>Our model</td>
<td>$x_5 &gt; x_4 &gt; x_2 &gt; x_1 &gt; x_3$</td>
<td>$x_5$</td>
</tr>
<tr>
<td>Wei and Wei’s [41] method</td>
<td>$x_5 &gt; x_4 &gt; x_2 &gt; x_1 &gt; x_3$</td>
<td>$x_5$</td>
</tr>
<tr>
<td>Liang et al.’s [42] method</td>
<td>$x_5 &gt; x_4 &gt; x_2 &gt; x_1 &gt; x_3$</td>
<td>$x_5$</td>
</tr>
<tr>
<td>Garg’s [43] method</td>
<td>$x_5 &gt; x_4 &gt; x_1 &gt; x_2 &gt; x_3$</td>
<td>$x_5$</td>
</tr>
</tbody>
</table>

From the above numerical example and comparison with other methods, the proposed MADM model with Pythagorean fuzzy linguistic information measures had the following characteristics:

1. Because the decision-making methods in Wei and Wei [41], Liang et al. [42], and Garg [43] cannot deal with MADM problems in which the information of attribute weights is completely unknown, the application scope of the proposed MADM model is wider than the decision-making methods in those papers.

2. Pythagorean fuzzy linguistic sets are a useful tool to depict the uncertainty and fuzziness of elements. Hence, the PFLSs are more reasonable and convenient as a technique to provide uncertain information. However, although there are many decision-making methods in the existing literature [41,42], these methods cannot address situations in which the input decision-making information takes the form of PFLVs. Our model can handle decision-making problems with PFLVs.

3. Although the proposed MADM model generates the same decision-making result as the method by Wei and Wei [41], our model and the method in Wei and Wei [41] generate different ranking results among the sustainable blockchain products. In the MADM process, the developed model directly applies the original information of the decision-making matrix and derives the weighted similarity degrees of sustainable blockchain products, which can preserve the original information of the expert. However, with the method by Wei and Wei [41], Equations (19) and (20) are used to calculate the weighted similarity degrees of sustainable blockchain products, and some original decision-making information is lost. Therefore, our model is more reliable and scientific than the method by Wei and Wei [38].

4. It can be observed that our model and the method in Garg [43] derive different ranking results for the sustainable blockchain products. Owing to differences existing among these attributes and the type of attributes being different, the process of normalization for the decision-making matrix is necessary. However, the method by Garg [43] does not normalize the initial Pythagorean fuzzy linguistic decision-making matrix, and it directly uses the linguistic Pythagorean fuzzy weighted average operator [43] to fuse the initial Pythagorean fuzzy linguistic information into collective linguistic Pythagorean fuzzy values. In contrast, with our MADM model, we first normalize the initial Pythagorean fuzzy linguistic decision-making matrix, then apply the Pythagorean fuzzy linguistic information measures to derive a decision-making result. In addition, according to the normalized Pythagorean fuzzy linguistic decision-making matrix $\Phi = (\beta_{ij})_{5 \times 4}$, we have $\beta_{21} > \beta_{11}, \beta_{22} > \beta_{12}, \beta_{23} > \beta_{13}$, and $\beta_{24} < \beta_{14}$, which indicates that the sustainable blockchain product $x_2$ is better than $x_1$, i.e., $x_2 > x_1$. Thus, our model can produce the correct ranking orders of these sustainable blockchain products. Therefore, the decision-making result derived by our model is more accurate.

6. Conclusions

This paper focused on the MADM model with the Pythagorean fuzzy linguistic entropy and similarity measure. First, based on the LTSs and PFSs, we proposed a new notion of Pythagorean fuzzy linguistic sets. Then, we presented the two axiomatic definitions of information measures for...
PFLVs, including Pythagorean fuzzy linguistic entropy and the Pythagorean fuzzy linguistic similarity measure. With the help of the logarithmic function, two new information measurement formulas were constructed. We further explored the inter-relationship between the Pythagorean fuzzy linguistic entropy and similarity measure. Finally, we developed a novel Pythagorean fuzzy linguistic MADM model. A numerical example of a sustainable blockchain product assessment was given to illustrate the reliability of the developed MADM model.

The main advantages of the proposed MADM model with Pythagorean fuzzy linguistic information are summarized as follows: (1) a new notion of PFLSs was introduced, in which the evaluation information is represented by a linguistic membership degree and a linguistic non-membership degree; (2) the Pythagorean fuzzy linguistic entropy and Pythagorean fuzzy linguistic similarity measure were investigated to measure the uncertainty degree of PFLV and similarity degree between PFLVs; (3) two new information measurement formulas were constructed to preserve DMs’ original decision-making information to the greatest extent possible; and (4) the developed MADM model can derive a reliable ranking of alternatives and extend the application scope of the decision-making model.

However, this paper did not discuss the situation in which some experts decide to not provide their assessment information; that is, how to construct a decision-making method with incomplete Pythagorean fuzzy linguistic information in decision-making problems. Therefore, in the future, we will focus on extending the proposed model for incomplete Pythagorean fuzzy linguistic multi-attribute decision-making problems, applying these models to solve practical applications in other areas, including knowledge recommendations, information fusion systems, and cooperative decision-making.

Author Contributions: All of the authors contributed to this research. F.J., L.P. and H.C. conceived the study and were responsible for the design and development of the data analysis. F.J. and H.C. were responsible for data collection and analysis. L.P., R.L. and J.L. were responsible for data interpretation. F.J. wrote the first draft of the article.

Funding: This research was funded by the National Natural Science Foundation of China (No. 71901001, No. 71871001, No. 11901150, No. 71771001, No. 71701001, No. 71972001, No. 71972002), the Construction Fund for Scientific Research Conditions of Introducing Talents in Anhui University (No. S02011802/085), the Natural Science Foundation for Distinguished Young Scholars of Anhui Province (No. 1908085J03), the Key Research Project of Humanities and Social Sciences in Colleges and Universities of Anhui Province (SK2019A0013).

Conflicts of Interest: The authors declare no conflict of interest.

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