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Convergence Rate of a Stable, Monotone and Consistent Scheme for the Monge-Ampère Equation

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Abstract: We prove a rate of convergence for smooth solutions of the Monge-Ampère equation of a stable, monotone and consistent discretization. We consider the Monge-Ampère equation with a small low order perturbation. With such a perturbation, we can prove uniqueness of a solution to the discrete problem and stability of the discrete solution. The discretization considered is then known to converge to the viscosity solution but no rate of convergence was known.

Keywords: rate of convergence; Monge-Ampère; monotone scheme; smooth solution

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1. Introduction

We obtain a rate of convergence, in the case of smooth convex solutions, of the finite difference schemes introduced in [1,2] for the elliptic Monge-Ampère equation

$$\det D^2 u = f \text{ in } \Omega, u = g \text{ on } \partial\Omega \tag{1}$$

Here, for a smooth function u, $D^2 u = \Big((\partial^2 u)/(\partial x_i \partial x_j) \Big)_{i,j=1,\ldots,d}$ is the Hessian of u, a symmetric matrix field. We assume that Ω is a bounded convex domain of $\mathbb{R}^d, d \geq 2$, $g \in C(\partial\Omega)$ can be extended to a function $\tilde{g} \in C(\overline{\Omega})$ which is convex in Ω and $f > 0 \in C(\overline{\Omega})$. We consider in this paper a finite difference scheme $F_h(u^h) = 0$ which is stable, monotone and consistent for the perturbed Monge-Ampère equation

$$\det D^2 u + \delta u = f \text{ in } \Omega, u = g \text{ on } \partial\Omega \tag{2}$$

Here $\delta > 0$ is a small parameter and, for the discretizations we consider, both the term δu and its discretization are often omitted by an abuse of notation. The scheme we consider was introduced in [2] but the approach we take also applies to the one introduced in [1]. The consistency error is $O(h^2 + d\theta)$ where h is the spatial resolution and $d\theta$ the directional resolution. Our error estimates are in terms of $O(h^2 + d\theta)$. The stability of the schemes for smooth solutions is a direct consequence of our error estimates. See also Remark 3.3 for a proof of stability in the general case, which seems to indicate that the perturbation δu is needed for the stability of the scheme we consider.

Rate of convergence for smooth solutions were previously established in the context of finite elements [3–5] or the standard finite difference method [6,7]. The rate of convergence proven in this paper for a stable, monotone and consistent scheme is a key component of the theory developed in [7] for the convergence of finite difference discretizations to the Aleksandrov solution of the Monge-Ampère equation. It follows from the approach taken therein and the results of this paper, that the discretizations proposed in [1,2] have approximations which converge to the weak solution, as defined in [8], of an approximate problem to Equation (2), even in the general case where Equation (2) does not have a smooth solution. For convergence results in the classical sense, the rate of convergence given here is expected to help establish a rate of convergence for the scheme without a dependence on the smoothness of the solution. Such a result would eliminate the need for convergence to an approximate problem, instead of Equation (2), for the theory developed in [7]. See also [9] for a different approach.

2. Notations and Preliminaries

We make the usual convention of using the letter C for various constants independent of the discretization. We make the assumption that $\Omega = (0,1)^d \subset \mathbb{R}^d$. Let $h > 0$ denote the mesh size. We assume without loss of generality that $1/h \in \mathbb{Z}$. Put

$$\mathbb{Z}_h = \{x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n : x_i/h \in \mathbb{Z}\}$$
$$\Omega_0^h = \Omega \cap \mathbb{Z}_h, \Omega^h = \overline{\Omega} \cap \mathbb{Z}_h, \partial\Omega^h = \partial\Omega \cap \mathbb{Z}_h = \Omega^h \setminus \Omega_0^h$$

For $x \in \mathbb{R}^d$, we denote the maximum norm of x by $|x|_\infty = \max_{i=1,\ldots,d} |x_i|$. We will use the notation $|.|$ for the Euclidean norm.

Let $\mathcal{M}(\Omega^h)$ denote the set of real valued functions defined on Ω^h, *i.e.* the set of mesh functions. For a subset T_h of Ω^h, and $v^h \in \mathcal{M}(\Omega^h)$ we define

$$|v^h|_{\infty,T_h} = \max_{x \in T_h} |v^h(x)|_\infty$$

Let v be a continuous function on Ω and let $r_h(v)$ denote the unique element of $\mathcal{M}(\Omega^h)$ defined by

$$r_h(v)(x) = v(x), x \in \Omega^h$$

We extend the operator r_h canonically to vector fields and matrix fields. For a function g defined on $\partial\Omega$, $r_h(g)$ defines the analogous restriction on $\partial\Omega^h$.

We are first interested in discrete versions of Equation (1)

$$F_h(u^h)(x) = 0, x \in \Omega_0^h, u^h(x) = r_h(g)(x), x \in \partial\Omega^h \tag{3}$$

where F_h denotes a finite difference discretization of Equation (1).

2.1. Consistency

We recall that the consistency error of the scheme is defined as $|F_h(r_h(u))|_{\infty,\Omega_0^h}$ and that the scheme is consistent if for $u \in C^2(\Omega)$ $|F_h(r_h(u))|_{\infty,\Omega_0^h} \to 0$ as $h \to 0$. We will assume that the discretization $F_h(u^h) = 0$ is consistent. For the scheme we consider, the consistency error is given in terms of the usual spatial resolution h and the directional resolution.

To introduce the directional resolution, we first note that the determinant of the Hessian of a smooth function is essentially a second order directional derivative. More precisely, if we let W denote the set of orthogonal bases of $\mathbb{R}^d$, we have [2]

$$\det D^2u(x) = \min_{(\nu_1,\ldots,\nu_d) \in W} \prod_{i=1}^d \frac{\nu_i^T (D^2u(x))\nu_i}{|\nu_i|^2}$$

where ν_i^T denotes the transpose of ν_i.

The local directional resolution at x is defined as

$$d\theta(x) = \max_{|v|=1} \min_{\alpha^h / x \pm \alpha^h \in \Omega^h} |v - \frac{\alpha^h}{|\alpha^h|}|$$

We note that as $h \to 0$, $d\theta(x) \to 0$. The directional resolution $d\theta$ is then defined as

$$d\theta = \min_{x \in \Omega_0^h} d\theta(x)$$

The directional resolution can also be defined in terms of the angles between vectors. Here we have followed the approach used in [10].

Let $x_0 \in \Omega_0^h$ such that

$$d\theta(x_0) = \min_{x \in \Omega_0^h} d\theta(x)$$

We denote by W_h the set of orthogonal bases of $\mathbb{R}^d$ such that $(\alpha_1, \ldots, \alpha_d) \in W_h$ if and only if $x_0 \pm \alpha_i \in \Omega^h, \forall i$. Let v^h be a given a mesh function and let $x \in \Omega_0^h$, such that $x + \alpha_i \in \Omega^h$ but $x - \alpha_i \notin \Omega^h$ for some i. We denote by x_{α_i} the closest to x point of intersection with $\partial\Omega$ of the line through x and $x + \alpha_i$. We then define $v^h(x - \alpha_i)$ to be the value obtained by quadratic interpolation of $v^h(x_{\alpha_i})$, $v^h(x)$ and $v^h(x + \alpha_i)$. Similarly, if $x - \alpha_i \in \Omega^h$ but $x + \alpha_i \notin \Omega^h$ for some i, we define $v^h(x + \alpha_i)$ by quadratic interpolation of $v^h(x_{\alpha_i})$, $v^h(x)$ and $v^h(x - \alpha_i)$, where now x_{α_i} is the closest to x point of intersection with $\partial\Omega$ of the line through x and $x - \alpha_i$.

We consider the discrete Monge-Ampère operator defined by

$$M[u^h](x) = \inf_{(\alpha_1, \ldots, \alpha_d) \in W_h} \prod_{i=1}^{d} \frac{u^h(x + \alpha_i) - 2u^h(x) + u^h(x - \alpha_i)}{|\alpha_i|^2} \tag{4}$$

The operator $M[r_h(u)]$ is shown in [2] to be consistent with $\det D^2 u(x)$. We give here a detailed proof.

By a Taylor series expansion

$$\begin{aligned} r_h(u)(x + \alpha) &= r_h(u)(x) + Dr_h(u)(x) \cdot \alpha + \frac{1}{2}\alpha^T D^2 r_h(u)(x)\alpha \\ &\quad + \frac{1}{6} D(\alpha^T D^2 r_h(u)(x)\alpha) \cdot \alpha + O(|\alpha|^4) \\ r_h(u)(x - \alpha) &= r_h(u)(x) - Dr_h(u)(x) \cdot \alpha + \frac{1}{2}\alpha^T D^2 r_h(u)(x)\alpha \\ &\quad - \frac{1}{6} D(\alpha^T D^2 r_h(u)(x)\alpha) \cdot \alpha + O(|\alpha|^4) \end{aligned}$$

Thus, using $|\alpha_i| = O(h)$ for $\alpha_i \in \mathbb{R}^d$ such that $x_0 \pm \alpha_i \in \Omega^h$, we have

$$\frac{\alpha_i^T D^2 u(x) \alpha_i}{|\alpha_i|^2} = \frac{r_h(u)(x + \alpha_i) - 2r_h(u)(x) + r_h(u)(x - \alpha_i)}{|\alpha_i|^2} + O(h^2) \tag{5}$$

Moreover, for $v \in \mathbb{R}^d$, $|v| = 1$, and $\alpha \in \mathbb{R}^d$ with $x_0 \pm \alpha \in \Omega^h$, a direction vector closest to v, *i.e.*, $\alpha = \arg\min_{\zeta \in \mathbb{R}^d, x_0 \pm \zeta \in \Omega^h} |v - \zeta/|\zeta||$, we have

$$\left(v + \frac{\alpha}{|\alpha|}\right)^T D^2 u(x) \left(v - \frac{\alpha}{|\alpha|}\right) = v^T D^2 u(x) v - \frac{\alpha^T D^2 u(x) \alpha}{|\alpha|^2}$$

By definition of α, $|v - \alpha/|\alpha|| \leq d\theta$. Let us assume that $u \in C^4(\Omega)$ (so that second derivatives of u are locally bounded). Since $|v + \alpha/|\alpha|| \leq 2$, for a constant $C > 0$ we have $|(v + \alpha/|\alpha|)^T D^2 u(x)(v - \alpha/|\alpha|)| \leq C d\theta$. Thus by Equation (5)

$$v^T D^2 u(x) v = \frac{r_h(u)(x + \alpha) - 2r_h(u)(x) + r_h(u)(x - \alpha)}{|\alpha|^2} + O(h^2) + O(d\theta) \tag{6}$$

Let $x \in \Omega_0^h$ and let $\nu_h = (\nu_1, \ldots, \nu_d) \in W_h$ such that

$$M[r_h(u)](x) = \prod_{i=1}^{d} \frac{r_h(u)(x + \nu_i) - 2r_h(u)(x) + r_h(u)(x - \nu_i)}{|\nu_i|^2}$$

Thus

$$M[r_h(u)](x) = \prod_{i=1}^{d} \frac{\nu_i^T D^2 u(x) \nu_i}{|\nu_i|^2} + O(h^2) + O(d\theta)$$
$$\geq \det D^2 u(x) + Ch^2 + Cd\theta$$

It follows that

$$\lim_{h \to 0} M[r_h(u)](x) \geq \det D^2 u(x) \tag{7}$$

Next, if $(\nu_1, \ldots, \nu_d)$ is a basis of eigenvectors of $D^2 u(x)$, we know from [2] that $\det D^2 u(x) = \prod_{i=1}^{d} \nu_i^T D^2 u(x) \nu_i / |\nu_i|^2$. We claim that for each $h > 0$, we can find $(\mu_1, \ldots, \mu_d) \in W_h(x)$ such that $|\nu_i/|\nu_i| - \mu_i/|\mu_i|| \leq d\theta$ for all i. This follows from the observation that for $i \neq j$, $\nu_i/|\nu_i|$ is obtained from $\nu_j/|\nu_j|$ by an orthogonal transformation. We recall that orthogonal transformations preserve inner products and that for $\mu \in \mathbb{R}^d$ such that $x \pm \mu \in \Omega_h$, the image of μ by a rotation of angle $\pi/2$ in a plane spanned by two axis vectors is a vector μ' for which $x \pm \mu' \in \Omega_h$. Thus having found μ_1 such that $|\nu_1/|\nu_1| - \mu_1/|\mu_1|| \leq d\theta$, the other vectors $\mu_i, i = 1, \ldots, d$ are obtained by orthogonal transformations.

It then follows from Equation (6) that

$$\det D^2 u(x) = \prod_{i=1}^{d} \frac{r_h(u)(x + \mu_i) - 2r_h(u)(x) + r_h(u)(x - \mu_i)}{|\mu_i|^2} + O(h^2) + O(d\theta)$$
$$\geq M[r_h(u)](x) + O(h^2) + O(d\theta)$$

We conclude that

$$\det D^2 u(x) \geq \lim_{h \to 0} M[r_h(u)](x) \tag{8}$$

It follows from Equations (7) and (8) that consistency holds for $u \in C^4(\Omega)$.

3. Rate of Convergence

The proof of the rate of convergence is an application of the combined fixed point iterative method used in [4].

To ensure convergence to a convex solution of Equation (1), it is natural to use a suitable notion of discrete convexity. We require that at an interior grid point x and for $\alpha \in \mathbb{R}^d$ such that $x_0 \pm \alpha \in \Omega^h$

$$u^h(x + \alpha) - 2u^h(x) + u^h(x - \alpha) \geq 0$$

with the usual assumption of the value of $u^h(x + \alpha)$ (resp. $u^h(x - \alpha)$) obtained by quadratic interpolation of $u^h(x_\alpha)$, $u^h(x)$ and $u^h(x - \alpha)$ (resp. $u^h(x_\alpha)$, $u^h(x)$ and $u^h(x + \alpha)$) when $x + \alpha \notin \Omega^h$ (resp. $x - \alpha \notin \Omega^h$). Here we have denoted by x_α the closest to x point of intersection with $\partial\Omega$ of the line through x and $x - \alpha$ (resp. the line through x and $x + \alpha$).

As with [2], the discrete convexity conditions can be combined with a discretization of the differential operator in a single equation. Recall that $x^+ = \max(x, 0)$ and define

$$M^+[u^h](x) = \inf_{(\alpha_1, \ldots, \alpha_d) \in W_h} \prod_{i=1}^{d} \max\left(\frac{u^h(x + \alpha_i) - 2u^h(x) + u^h(x - \alpha_i)}{|\alpha_i|^2}, 0 \right)$$

Put

$$F_h(u^h)(x) = M^+[u^h](x) + \delta u_h - r_h(f)(x)$$

In practice the term δu_h is not used. It makes the discretization proper as defined below. And guarantees uniqueness of the discrete solution. For consistency, we are forced to consider the perturbed Equation (2).

The discrete Monge-Ampère equation is given by

$$\begin{aligned} M^+[u^h](x) + \delta u_h - r_h(f)(x) &= 0, x \in \Omega_0^h \\ u^h(x) &= r_h(g)(x) \text{ on } \partial\Omega^h \end{aligned} \tag{9}$$

Let $N(x)$ denote the set of points $x \pm \alpha, \alpha \in W_h$ and let $\#N(x)$ denote the cardinality of the set $N(x)$. We note that the discretization takes the form

$$F_h(u^h)(x) \equiv \hat{F}_h(u^h(x), u^h(y) - u^h(x)|_{y \neq x, y \in N(x)})$$

where for $x \in \Omega_0^h$, $\hat{F}_h$ is a real valued map defined on $\mathbb{R} \times \mathbb{R}^{\#N(x)}$. For convenience we do not write explicitly the dependence of $\hat{F}_h$ on x.

A scheme is proper if there is $\delta > 0$ such that for $x \in \Omega_0^h$ and for all $a_0, a_1 \in \mathbb{R}$ and $b \in \mathbb{R}^{\#N(x)}$, $a_0 \leq a_1$ implies $\hat{F}_h(a_0, b) - \hat{F}_h(a_1, b) \leq \delta(a_0 - a_1)$.

Thus our scheme is proper and the constant $\delta > 0$ can be chosen independently of h.

The scheme is degenerate elliptic since if it is nondecreasing in each of the variables $u^h(x)$ and $u^h(y) - u^h(x), y \in N(x), y \neq x$.

A scheme $F_h(u^h) = 0$ is Lipschitz continuous if there is $K > 0$ such that for all $x \in \Omega_0^h$ and $(a_0, b_0), (a_1, b_1) \in \mathbb{R}^{\#N(x)+1}$

$$|\hat{F}_h((a_0, b_0)) - \hat{F}_h((a_1, b_1))| \leq K|(a_0, b_0) - (a_1, b_1)|_\infty$$

We claim that our scheme is Lipschitz continuous with Lipschitz constant $K = C/h^{2d}$. For $(a, b) \in \mathbb{R}^{\#N(x)+1}$, we can find $r \in \mathbb{R}^{2d+1}$ such that

$$\hat{F}_h((a, b)) = -f(x) + \delta r_0 + \prod_{i=1}^{d} \frac{r_i + r_{i+d}}{h^2}$$

If $M^+[v^h](x) = 0$ we take $r_i = 0, i = 1, \ldots, 2d$. Otherwise we have

$$M^+[v^h](x) = \prod_{i=1}^{d} \frac{v^h(x + \alpha_i) - 2v^h(x) + v^h(x - \alpha_i)}{|\alpha_i|^2}$$

for an orthogonal basis $(\alpha_1, \ldots, \alpha_d)$. We then take $r_i = h^2/|\alpha_i|^2(v^h(x + \alpha_i) - v^h(x))$, and $r_{i+d} = h^2/|\alpha_i|^2(v^h(x - \alpha_i) - v^h(x))$. In both cases, $r_0 = v^h(x)$.

Since the map $r \to \prod_{i=1}^d (r_i + r_{i+d})/h^2$ is multilinear, $\hat{F}_h$ is Lipschitz continuous with Lipschitz constant $\max(\delta, C/h^{2d})$ which we can take as C/h^{2d} for h sufficiently small.

Next we define the mapping

$$S : \mathcal{M}(\Omega^h) \to \mathcal{M}(\Omega^h), \qquad S(v^h)(x) = v^h(x) - \nu F_h(v^h)(x) \tag{10}$$

for $\nu > 0$. We have ([11] Theorem 7)

Lemma 3.1. *There exists a positive constant $a < 1$ such that for all $v^h, w^h \in \mathcal{M}(\Omega^h)$, we have*

$$|S(v^h) - S(w^h)|_{\infty, \Omega_0^h} \leq a|v^h - w^h|_{\infty, \Omega_0^h}$$

for $C_0 \leq \nu \leq C_1$ where C_0 and C_1 are positive constants.

The proof of ([11] Theorem 7) shows that under the assumption $\nu K < 1$, the constant a takes the form $\max(1 - \nu\delta, \nu K)$. If necessary, by taking ν smaller we may assume that $\nu K < 1/2$ and $\nu\delta < 1/2$. Thus the constant a takes the form

$$a = 1 - \nu\delta$$

and since $K = C/h^{2d}$, we have $\nu \leq Ch^{2d}$.

We can now state the main result of this paper.

Theorem 3.2. *For a solution u^h of Equation (9) and for $u \in C^4(\overline{\Omega})$ we have*

$$|u^h - r_h(u)|_{\infty, \Omega_0^h} \leq \frac{C}{\delta}(h^2 + d\theta)$$

for a constant C which is a scalar multiple of the maximum of the derivatives of u up to order 4 on $\overline{\Omega}$ and d.

We recall that is it is proven in [2], see [11] for details, that Equation (9) has a solution u^h which is a fixed point of the mapping S. We have

$$\begin{aligned} |u^h - r_h(u)|_{\infty,\Omega_0^h} &= |S(u^h) - r_h(u)|_{\infty,\Omega_0^h} \\ &\leq |S(u^h) - S(r_h(u))|_{\infty,\Omega_0^h} + |S(r_h(u)) - r_h(u)|_{\infty,\Omega_0^h} \\ &\leq a|u^h - r_h(u)|_{\infty,\Omega_0^h} + |S(r_h(u)) - r_h(u)|_{\infty,\Omega_0^h} \\ &= a|u^h - r_h(u)|_{\infty,\Omega_0^h} + \nu|F_h(r_h u)|_{\infty,\Omega_0^h} \\ &= a|u^h - r_h(u)|_{\infty,\Omega_0^h} + \nu|M[r_h(u)] - r_h(\det D^2 u)|_{\infty,\Omega_0^h} \\ &\leq a|u^h - r_h(u)|_{\infty,\Omega_0^h} + C\nu(h^2 + d\theta) \end{aligned}$$

by the consistency of the scheme, the observation that $r_h(u)$ is discrete convex and Equation (2). The constant C is a consistency error constant and depends on the maximum of derivatives of u up to order 4 on $\overline{\Omega}$ and d.

We therefore have

$$\begin{aligned} |u^h - r_h(u)|_{\infty,\Omega_0^h} &\leq C\frac{\nu}{1-a}(h^2 + d\theta) \\ &= \frac{C}{\delta}(h^2 + d\theta) \end{aligned}$$

This completes the proof.

Remark 3.3. *The approach in the proof of the previous theorem also gives stability of a solution of Equation* (9) *in the general case, when f is uniformly bounded. We have*

$$\begin{aligned} |u^h|_{\infty,\Omega_0^h} = |S(u^h)|_{\infty,\Omega_0^h} &\leq |S(u^h) - S(0)|_{\infty,\Omega_0^h} + |S(0)|_{\infty,\Omega_0^h} \\ &\leq a|u^h|_{\infty,\Omega_0^h} + \nu|f|_{\infty,\Omega_0^h} \end{aligned}$$

Therefore

$$|u^h|_{\infty,\Omega_0^h} \leq \frac{\nu}{1-a}|f|_{\infty,\Omega_0^h} \leq \frac{1}{\delta}|f|_{\infty,\Omega_0^h}$$

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