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The Method of Linear Determining Equations to Evolution System and Application for Reaction-Diffusion System with Power Diffusivities

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Abstract: The method of linear determining equations is constructed to study conditional Lie–Bäcklund symmetry and the differential constraint of a two-component second-order evolution system, which generalize the determining equations used in the search for classical Lie symmetry. As an application of the approach, the two-component reaction-diffusion system with power diffusivities is considered. The conditional Lie–Bäcklund symmetries and differential constraints admitted by the reaction-diffusion system are identified. Consequently, the reductions of the resulting system are established due to the compatibility of the corresponding invariant surface conditions and the original system.

Keywords: linear determining equation; conditional Lie–Bäcklund symmetry; differential constraint; evolution system; reaction-diffusion system

1. Introduction

The method of differential constraint (DC) is pretty old, dating back at least to the time of Lagrange. Lagrange used DC to find the total integral of a first-order nonlinear equation. Darboux applied DC to integrate the partial differential equation (PDE) of second-order. Yanenko proposed the key idea of DC in [1]. The survey of this method was presented by Sidorvo, Shapeev and Yanenko in [2], where the method of DC was successfully introduced into practice on gas dynamics.

The general formulation of the method of DC requires that the original system of PDEs

$$\tilde{F}^{(1)} = 0,\ \tilde{F}^{(2)} = 0,\ \cdots,\tilde{F}^{(m)} = 0 \tag{1}$$

be enlarged by appending additional equations

$$h_1 = 0,\ h_2 = 0,\ \cdots,h_p = 0 \tag{2}$$

such that the over-determined system (1), (2) is compatible. The differential equations in (2) are called DCs. The requirements for the compatibility of system (1), (2) are so general that the method of DC does not allow us to find all the forms of DCs for the system of PDEs in question. A number of different names for the parent notions of DC (2) leads to many methods for finding exact particular solutions of PDEs can be unified within the general framework of the method of DC.

The "side condition" is proposed to unify different methods for constructing particular solutions of PDEs by Olver and Rosenau in [3], where it is stated that appending of a suitable "side condition" is responsible for different kinds of methods for obtaining explicit solutions, including Lie's classical method [4,5], Bluman and Cole's nonclassical method [6] and Ovsiannikov's partial invariance method [7]. "The invariant surface condition" is used as a unifying theme for finding special solutions to PDEs by Pucci and Saccomandi in [8], where it is shown that "the invariant surface condition" and "its general integral" are the key to understanding the link between the so-called direct method [9], separation method [10,11], nonclassical symmetry [6] and weak symmetry [12,13]. The "additional generating condition" first raised by Cherniha [14,15] is exactly the linear case of DC, which is very effective to study reductions of variant forms of diffusion equations and diffusion systems [15–17]. The method of invariant subspace initially presented by Galaktionov and his collaborators [18] can also be understood within the framework of DC due to certain linear DC. The DCs that are responsible for Clarkson and Kruskal's first-order direct reduction [9] and Galaktionov's higher-order direct reduction [11] are discussed by Olver in [19]. The equivalence relationship between weak symmetry and DC is studied by Olver and Rosenau in [13].

The method of conditional Lie–Bäcklund symmetry (CLBS) provides an appropriate symmetry background for the method of DC. The base of symmetry reduction for CLBS is the fact that the corresponding invariant surface condition is formally compatible with the governing system, which is extensively discussed in [20,21], where it is shown that the problem of discussing the DC of the evolution system is equivalent to studying the CLBS of this system.

CLBS for the scalar evolution equation is introduced by Zhdanov [22], and another term for CLBS is used by Fokas and Liu [23,24]. A family of physically important exact solutions including the multi-shock solution and multi-soliton solution is constructed for a large class of non-integrable evolution equations by using the method of CLBS [23–26]. The CLBS for the evolution system is studied by Sergyeyev in [27] and independently by Qu et al. in [28].

The procedure for determining whether or not a given DC is compatible with the original equations is straightforward. However, for a given system of differential equations, one can never know in full detail the entire range of possible DCs since the associated determining equations are an over-determined nonlinear system. Nevertheless, as is known, even finding particular DCs can lead to new explicit solutions of the considered system. In practice, the principal direction of such research is to content oneself with finding DCs in some classes, and these classes must be chosen using additional considerations. From the symmetry point of view, CLBSs related to sign-invariants [29–33], separation of variables [34] and invariant subspaces [35–37] are proved to be very effective to study the classifications and reductions of second-order nonlinear diffusion equations. These particular subclasses related to sign-invariants [29–33] and invariant subspaces [35–37] are also extended to consider CLBSs of nonlinear diffusion systems in [21] and [28,38,39].

The purpose of this paper is to construct a practical way for finding the general form of DCs

$$\begin{cases} \eta_1 = u_n + g\left(t, x, u, u_1, u_2, \cdots, u_{n-1}\right) = 0, \\ \eta_2 = v_n + h\left(t, x, v, u_1, v_2, \cdots, v_{n-1}\right) = 0 \end{cases} \tag{3}$$

compatible with a two-component second-order evolution system

$$\begin{cases} u_t = F(t, x, u, v, u_1, v_1, u_2, v_2), \\ v_t = G(t, x, u, v, u_1, v_1, u_2, v_2), \end{cases} \tag{4}$$

which is equivalent to presenting an effective method to find the general form of CLBS with the characteristics

$$\begin{cases} \eta_1 = u_n + g\left(t, x, u, u_1, u_2, \cdots, u_{n-1}\right), \\ \eta_2 = v_n + h\left(t, x, v, u_1, v_2, \cdots, v_{n-1}\right) \end{cases} \tag{5}$$

admitted by evolution system (4). It is noted that $u_k = \partial^k u/\partial x^k$ and $v_k = \partial^k v/\partial x^k$ with $k = 1, 2, \cdots, n$ in (3)–(5) and hereafter.

The constructive method of the additional generating condition is presented by Cherniha in [14], where exact solutions of the variant form of the system (4) are derived by appending an additional condition in the form of a linear system of ordinary differential equations to the original system. Here, we will present the linear determining equations to identify DC (3) and CLBS (5) in the general form of second-order evolution system (4), which is exactly the extension of the results for the scalar evolution equation in [40–42].

The method of linear determining equations is proposed for finding the general form of DC

$$\eta = u_n + g(t, x, u, u_1, u_2, \cdots, u_{n-1}) = 0 \tag{6}$$

to evolution equation

$$u_t = F(t, x, u, u_1, u_2, \cdots, u_N) \tag{7}$$

by Kaptsov in [40]. The linear determining equation

$$D_t\eta = \sum_{i=0}^{N}\sum_{k=0}^{i} b_{ik} D_x^{i-k}(F_{u_{N-k}}) D_x^{N-i}(\eta) \tag{8}$$

presented there generalizes the classical determining equations within the framework of Lie's classical symmetry. It is clear that it is workable to find the DC with the general form (6) of evolution Equation (7) by solving linear determining equation (8) about η.

The principal direction of the research on applying the method to second-order nonlinear diffusion equations [40–42] gains an appreciation of its usefulness. The two-component reaction-diffusion (RD) system with power law diffusivities

$$\begin{cases} u_t = (u^k u_x)_x + P(u, v), \\ v_t = (v^l v_x)_x + Q(u, v) \end{cases} \tag{9}$$

will be considered here to demonstrate the applicability of this method for a two-component second-order evolution system.

The RD system (9) generalizes many well-known nonlinear second-order models and is used to describe various processes in physics, chemistry and biology. A complete description of Lie symmetries of the system is presented in [16]. The conditional symmetries for (9) are studied in [43–46]. The second-order CLBS (DC) admitted by the system (9) is discussed in [21]. Once the symmetries of the considered system (9) have been identified, one can algorithmically implement the reduction procedure and thereby determine all solutions that are invariant under the resulting symmetries. In [16,21,43–46], a wide range of exact solutions has been established due to various symmetry reductions therein.

The structure of this paper is organized as follows. The necessary definitions and notations about CLBS and DC of evolution system are displayed in Section 2. In Section 3, the linear determining equations to second-order evolution system (4) are constructed. The DCs (3) and CLBSs (5) of the system (9) are identified by solving the linear determining equation for the RD system (9) in Section 4. The exact solutions of the resulting RD system (9) are constructed due to the compatibility of the DC (3) and the governing system (9) in Section 5. The last section is devoted to the final discussions and conclusions.

2. Preliminaries

Let us review some theoretical elements of the tools about CLBS method and DC method of evolution system. Set

$$
\begin{aligned}
V = \sum_{i=1}^{m} \Big[& h_i \left(t, x, u^{(1)}, u^{(2)}, \cdots, u^{(m)}, u_1^{(1)}, u_1^{(2)}, \cdots, u_1^{(m)}, \cdots \right) \frac{\partial}{\partial u^{(i)}} \\
& + D_x h_i \left(t, x, u^{(1)}, u^{(2)}, \cdots, u^{(m)}, u_1^{(1)}, u_1^{(2)}, \cdots, u_1^{(m)}, \cdots \right) \frac{\partial}{\partial u_1^{(i)}} \\
& + D_t h_i \left(t, x, u^{(1)}, u^{(2)}, \cdots, u^{(m)}, u_1^{(1)}, u_1^{(2)}, \cdots, u_1^{(m)}, \cdots \right) \frac{\partial}{\partial u_t^{(i)}} \\
& + D_x^2 h_i \left(t, x, u^{(1)}, u^{(2)}, \cdots, u^{(m)}, u_1^{(1)}, u_1^{(2)}, \cdots, u_1^{(m)}, \cdots \right) \frac{\partial}{\partial u_2^{(i)}} \\
& + \cdots \Big]
\end{aligned} \tag{10}
$$

to be a certain smooth Lie–Bäcklund vector field (LBVF) and

$$
u_t^{(i)} = F^{(i)} \left(t, x, u^{(1)}, u^{(2)}, \cdots, u^{(m)}, u_1^{(1)}, u_1^{(2)}, \cdots, u_1^{(m)}, \cdots \right), \ \ i = 1, 2, \cdots, m \tag{11}
$$

to be a nonlinear evolution system, where $u_k^{(i)} = \partial^k u^{(i)} / \partial x^k$ with $i = 1, 2, \cdots, m$ and $k = 1, 2, \cdots$.

Definition 1. *[4,5] The evolutionary vector field (10) is said to be a Lie–Bäcklund symmetry of the evolution system (11) if*

$$
V \left(u_t^{(i)} - F^{(i)}\right) |_S = 0, \ \ i = 1,\ 2,\ \cdots,\ m,
$$

where S denotes the set of all differential consequences of the system (11).

Definition 2. *[27,28] The evolutionary vector field (10) is said to be a CLBS of (11) if*

$$
V \left(u_t^{(i)} - F^{(i)}\right) |_{S \cap H_x} = 0, \ \ i = 1,\ 2,\ \cdots,\ m, \tag{12}
$$

where H_x denotes the set of all differential consequences of the invariant surface condition

$$
h_i \left(t, x, u^{(1)}, u^{(2)}, \cdots, u^{(m)}, u_1^{(1)}, u_1^{(2)}, \cdots, u_1^{(m)}, \cdots \right) = 0 \ (i = 1,\ 2,\ \cdots,\ m) \tag{13}
$$

with respect to x.

A direct computation will yield that the conditional invariant criterion (12) can be reduced to [27,28]

$$
D_t h_i |_{S \cap H_x} = 0, \ \ i = 1, 2, \cdots, m. \tag{14}
$$

The fact that LBVF (10) is a CLBS of system (11) leads to the compatibility of the invariant surface condition (13) and the governing system (11).

Definition 3. *[47] The differential constraints (13) and the evolution system (11) satisfy the compatibility condition if*

$$
D_t h_i |_{S_x \cap H_x} = 0, \ \ i = 1, 2, \cdots, m, \tag{15}
$$

where S_x denotes the set of all differential consequences of the system (11) with respect to x.

The compatibility condition (15) is nothing but the conditional invariance criterion (14).

3. Linear Determining Equations for the DC (3) and CLBS (5) of Two-Component Second-Order Evolution System (4)

In this section, we discuss the method of linear determining equations to construct the DC (3) and CLBS (5) for second-order evolution system (4). The compatibility condition (15) can be reformulated as nonlinear equations. We now prove this result for DC (3) of the system (4), which is a natural generalization of what was the case for scalar evolution equation (7) in [40,41]. Let E_x be the union of all differential consequences of the second-order evolution system (4) with respect to x and M_x be the union of all differential consequences of DC (3) with respect to x.

Theorem 1. *The DC (3) with $n \geq 4$ is compatible with two-component second-order evolution system (4) if and only if η_1 and η_2 satisfy the following equations*

$$\begin{aligned} D_t\eta_1|_{E_x} =& F_{u_2}D_x^2\eta_1 + F_{v_2}D_x^2\eta_2 + (F_{u_1} + nD_xF_{u_2})D_x\eta_1 \\ &+ \left[F_{v_1} + nD_xF_{v_2} + \left(\eta_{1u_{n-1}} - \eta_{2v_{n-1}}\right)F_{v_2}\right]D_x\eta_2 \\ &+ \left[F_u + nD_xF_{u_1} + \frac{n(n-1)}{2}D_x^2F_{u_2} - \eta_{1u_{n-1}}D_xF_{u_2}\right. \\ &\left.- \left(2D_x\eta_{1u_{n-1}} - \eta_1\eta_{1u_{n-1}u_{n-1}}\right)F_{u_2}\right]\eta_1 + \left[F_v + nD_xF_{v_1}\right. \\ &+ \frac{n(n-1)}{2}D_x^2F_{v_2} + \left(\eta_{1u_{n-1}} - \eta_{2v_{n-1}}\right)(F_{v_1} + nD_xF_{v_2}) \\ &- \eta_{1u_{n-1}}D_xF_{v_2} + \left(\eta_{1u_{n-2}} - \eta_{2v_{n-2}}\right)F_{v_2} - \eta_{2v_{n-1}}(\eta_{1u_{n-1}} \\ &\left.- \eta_{2v_{n-1}})F_{v_2} - \left(2D_x\eta_{2v_{n-1}} - \eta_2\eta_{2v_{n-1}v_{n-1}}\right)F_{v_2}\right]\eta_2 \end{aligned} \tag{16}$$

and

$$\begin{aligned} D_t\eta_2|_{E_x} =& G_{v_2}D_x^2\eta_2 + G_{u_2}D_x^2\eta_1 + (G_{v_1} + nD_xG_{v_2})D_x\eta_2 \\ &+ \left[G_{u_1} + nD_xG_{u_2} + \left(\eta_{2v_{n-1}} - \eta_{1u_{n-1}}\right)G_{u_2}\right]D_x\eta_1 \\ &+ \left[G_v + nD_xG_{v_1} + \frac{n(n-1)}{2}D_x^2G_{v_2} - \eta_{2v_{n-1}}D_xG_{v_2}\right. \\ &\left.- \left(2D_x\eta_{2v_{n-1}} - \eta_2\eta_{2v_{n-1}v_{n-1}}\right)G_{v_2}\right]\eta_2 + \left[G_u + nD_xG_{u_1}\right. \\ &+ \frac{n(n-1)}{2}D_x^2G_{u_2} + \left(\eta_{2v_{n-1}} - \eta_{1u_{n-1}}\right)(G_{u_1} + nD_xG_{u_2}) \\ &- \eta_{2v_{n-1}}D_xG_{u_2} + \left(\eta_{2v_{n-2}} - \eta_{1u_{n-2}}\right)G_{u_2} - \eta_{1u_{n-1}}(\eta_{2v_{n-1}} \\ &\left.- \eta_{1u_{n-1}})G_{u_2} - \left(2D_x\eta_{1u_{n-1}} - \eta_1\eta_{1u_{n-1}u_{n-1}}\right)G_{u_2}\right]\eta_1. \end{aligned} \tag{17}$$

Proof. Assume that η_1 and η_2 satisfy (16) and (17). It is easy to see that all terms on the right-hand side of (16) and (17) vanish on M_x. Hence

$$D_t\eta_1|_{E_x \cap M_x} = 0 \tag{18}$$

and

$$D_t\eta_2|_{E_x \cap M_x} = 0,$$

that is, the DC (3) is compatible with the second-order evolution system (4). We now prove the converse result. Let $\alpha \simeq \beta$ indicate that there are no terms containing $u_n, v_n, u_{n+1}, v_{n+1}, u_{n+2}$ and v_{n+2} in the difference $\alpha - \beta$. Then, we can derive that

$$D_t\eta_1|_{E_x} \simeq D_x^n F + \eta_{1u_{n-1}} D_x^{n-1} F + \eta_{1u_{n-2}} D_x^{n-2} F. \tag{19}$$

Since

$$\begin{aligned}
D_x^{n-2}F &\simeq u_n F_{u_2} + v_n F_{v_2},\\
D_x^{n-1}F &\simeq u_n\left[F_{u_1} + (n-1)D_x F_{u_2}\right] + u_{n+1}F_{u_2}\\
&\quad + v_n\left[F_{v_1} + (n-1)D_x F_{v_2}\right] + v_{n+1}F_{v_2},\\
D_x^n F &\simeq u_n\left[F_u + nD_xF_{u_1} + \frac{n(n-1)}{2}D_x^2F_{u_2}\right]\\
&\quad + u_{n+1}(F_{u_1} + nD_xF_{u_2}) + u_{n+2}F_{u_2}\\
&\quad + v_n\left[F_v + nD_xF_{v_1} + \frac{n(n-1)}{2}D_x^2F_{v_2}\right]\\
&\quad + v_{n+1}(F_{v_1} + nD_xF_{v_2}) + v_{n+2}F_{v_2}
\end{aligned} \tag{20}$$

holds naturally for $n \geq 4$, (19) can be written as

$$\begin{aligned}
D_t\eta_1|_{E_x} \simeq{}& u_{n+2}F_{u_2} + u_{n+1}\left(F_{u_1} + nD_xF_{u_2} + \eta_{1u_{n-1}}F_{u_2}\right) + u_n\Big\{F_u + nD_xF_{u_1}\\
&+ \frac{n(n-1)}{2}D_x^2F_{u_2} + \eta_{1u_{n-1}}\left[F_{u_1} + (n-1)D_xF_{u_2}\right] + \eta_{1u_{n-2}}F_{u_2}\Big\}\\
&+ v_{n+2}F_{v_2} + v_{n+1}\left(F_{v_1} + nD_xF_{v_2} + \eta_{1u_{n-1}}F_{v_2}\right) + v_n\Big\{F_v + nD_xF_{v_1}\\
&+ \frac{n(n-1)}{2}D_x^2F_{v_2} + \eta_{1u_{n-1}}\left[F_{v_1} + (n-1)D_xF_{v_2}\right] + \eta_{1u_{n-2}}F_{v_2}\Big\}.
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
&D_x\eta_1 \simeq u_{n+1} + u_n\eta_{1u_{n-1}},\\
&D_x^2\eta_1 \simeq u_{n+2} + u_{n+1}\eta_{1u_{n-1}} + u_n\left(\eta_{1u_{n-2}} + 2D_x\eta_{1u_{n-1}} - u_n\eta_{1u_{n-1}u_{n-1}}\right),\\
&D_x\eta_2 \simeq v_{n+1} + v_n\eta_{2v_{n-1}},\\
&D_x^2\eta_2 \simeq v_{n+2} + v_{n+1}\eta_{2v_{n-1}} + v_n\left(\eta_{2v_{n-2}} + 2D_x\eta_{2v_{n-1}} - v_n\eta_{2v_{n-1}v_{n-1}}\right).
\end{aligned}$$

Consequently, a direct calculation will yield

$$\begin{aligned}
&D_t\eta_1|_{E_x} - F_{u_2}D_x^2\eta_1 - F_{v_2}D_x^2\eta_2 - (F_{u_1} + nD_xF_{u_2})D_x\eta_1\\
&\quad - \left[F_{v_1} + nD_xF_{v_2} + \left(\eta_{1u_{n-1}} - \eta_{2v_{n-1}}\right)F_{v_2}\right]D_x\eta_2\\
&\quad - \Big[F_u + nD_xF_{u_1} + \frac{n(n-1)}{2}D_x^2F_{u_2} - \eta_{1u_{n-1}}D_xF_{u_2}\\
&\quad - \left(2D_x\eta_{1u_{n-1}} - \eta_1\eta_{1u_{n-1}u_{n-1}}\right)F_{u_2}\Big]\eta_1 - \Big[F_v + nD_xF_{v_1}\\
&\quad + \left(\eta_{1u_{n-1}} - \eta_{2v_{n-1}}\right)(F_{v_1} + nD_xF_{v_2}) - \eta_{1u_{n-1}}D_xF_{v_2}\\
&\quad + \left(\eta_{1u_{n-2}} - \eta_{2v_{n-2}}\right)F_{v_2} - \eta_{2v_{n-1}}\left(\eta_{1u_{n-1}} - \eta_{2v_{n-1}}\right)F_{v_2}\\
&\quad - \left(2D_x\eta_{2v_{n-1}} - \eta_2\eta_{2v_{n-1}v_{n-1}}\right)F_{v_2}\Big]\eta_2\\
&\simeq 0.
\end{aligned} \tag{21}$$

Equation (18) holds since DC (3) is compatible with the system (4). Let γ denote the left-hand side of (21); it is easy to see that

$$\gamma|_{E_x \cap M_x} = 0,$$

which is equivalent to

$$\gamma|_{M_x} = 0 \tag{22}$$

since γ is independent of $u_t, v_t, u_{tx}, v_{tx}, \cdots$. As shown above, γ depends only on $u_{n-1}, v_{n-1}, u_{n-2}, v_{n-2} \cdots$. On the other hand, η_1 depends on u_n, and η_2 depends on v_n. Hence, (22) holds only for $\gamma = 0$, which yields nonlinear determining equation (16). In analogy with the discussion above, we can derive another nonlinear determining equation (17) if DC (3) is compatible with the system (4). □

In fact, the problem of solving nonlinear determining equations (16) and (17) is a very difficult, if not an impossible, problem. A practical way to identify DC (3) of the system (4) is to keep the linear part of (16) and (17). A general form of the corresponding linear determining equations will finally lead to the following definition.

Definition 4. *The linear determining equations for DCs (3) of the two-component second-order evolution system (4) are the linear equations*

$$\begin{aligned} D_t\eta_1|_{E_x} =& F_{u_2} D_x^2\eta_1 + \left(\tilde{b}_{11} F_{u_1} + \tilde{b}_{12} D_x F_{u_2}\right) D_x\eta_1 \\ &+ \left(\tilde{b}_{13} F_u + \tilde{b}_{14} D_x F_{u_1} + \tilde{b}_{15} D_x^2 F_{u_2}\right)\eta_1 \\ &+ F_{v_2} D_x^2\eta_2 + \left(\tilde{b}_{16} F_{v_1} + \tilde{b}_{17} D_x F_{v_2}\right) D_x\eta_2 \\ &+ \left(\tilde{b}_{18} F_v + \tilde{b}_{19} D_x F_{v_1} + \tilde{b}_{10} D_x^2 F_{v_2}\right)\eta_2 \end{aligned} \tag{23}$$

and

$$\begin{aligned} D_t\eta_2|_{E_x} =& G_{v_2} D_x^2\eta_2 + \left(\tilde{b}_{21} G_{v_1} + \tilde{b}_{22} D_x G_{v_2}\right) D_x\eta_2 \\ &+ \left(\tilde{b}_{23} G_v + \tilde{b}_{24} D_x G_{v_1} + \tilde{b}_{25} D_x^2 G_{v_2}\right)\eta_2 \\ &+ G_{u_2} D_x^2\eta_1 + \left(\tilde{b}_{26} G_{u_1} + \tilde{b}_{27} D_x G_{u_2}\right) D_x\eta_1 \\ &+ \left(\tilde{b}_{28} G_u + \tilde{b}_{29} D_x G_{u_1} + \tilde{b}_{20} D_x^2 G_{u_2}\right)\eta_1. \end{aligned} \tag{24}$$

Linear determining equations (23) and (24) are the sufficient condition to justify whether DC (3) is compatible with the second-order evolution system (4). This family of linear determining equations is also effective to construct CLBS (5) of evolution system (4).

4. DCs (3) and CLBSs (5) of RD system (9)

Substituting $F = u^k u_2 + k u^{k-1} u_1^2 + P(u,v)$ and $G = v^l v_2 + l v^{l-1} v_1^2 + Q(u,v)$ into linear determining equations (23) and (24), we can derive the sufficient condition to identify DCs (3) and CLBSs (5) of RD system (9)

$$\begin{aligned} D_t\eta_1|_{E_x} =& u^k D_x^2\eta_1 + k\left(2\tilde{b}_{11} + \tilde{b}_{12}\right) u^{k-1} u_1 D_x\eta_1 + \Big[k\left(\tilde{b}_{13} + 2\tilde{b}_{14} + \tilde{b}_{15}\right) u^{k-1} u_2 \\ &+ k(k-1)\left(\tilde{b}_{13} + 2\tilde{b}_{14} + \tilde{b}_{15}\right) u^{k-2} u_1^2 + \tilde{b}_{13} P_u\Big]\eta_1 + \tilde{b}_{18} P_v \eta_2 \end{aligned} \tag{25}$$

and

$$\begin{aligned} D_t\eta_2|_{E_x} =& v^l D_x^2\eta_2 + l\left(2\tilde{b}_{21}+\tilde{b}_{22}\right)v^{l-1}v_1 D_x\eta_2 + \Big[l\left(\tilde{b}_{23}+2\tilde{b}_{24}+\tilde{b}_{25}\right)v^{l-1}v_2 \\ &+ l(l-1)\left(\tilde{b}_{23}+2\tilde{b}_{24}+\tilde{b}_{25}\right)v^{l-2}v_1^2 + \tilde{b}_{23}Q_v\Big]\eta_2 + \tilde{b}_{28}Q_u\eta_1. \end{aligned} \tag{26}$$

Here, we use the general form of (25) and (26)

$$\begin{aligned} D_t\eta_1|_{E_x} =& u^k D_x^2\eta_1 + b_{11}u^{k-1}u_1 D_x\eta_1 \\ &+ \left(b_{12}u^{k-1}u_2 + b_{13}u^{k-2}u_1^2 + b_{14}P_u\right)\eta_1 + b_{15}P_v\eta_2 \end{aligned} \tag{27}$$

and

$$\begin{aligned} D_t\eta_2|_{E_x} =& v^l D_x^2\eta_2 + b_{21}v^{l-1}v_1 D_x\eta_2 \\ &+ \left(b_{22}v^{l-1}v_2 + b_{23}v^{l-2}v_1^2 + b_{24}Q_v\right)\eta_2 + b_{25}Q_u\eta_1 \end{aligned} \tag{28}$$

to construct DCs (3) and CLBSs (5) of the RD system (9).

It would be quite enlightening to give the order estimate for DCs (3) and CLBSs (5) admitted by the considered system (4). However, this is another problem, which we leave to future research. Here, we restrict our consideration to $2 \le n \le 5$.

Firstly, we consider the case of $n = 3$. A direct computation will give

$$\begin{aligned} D_t\eta_1|_{E_x} =& u^k u_5 + \left(5ku^{k-1}u_1 + u^k g_{u_2}\right)u_4 + \Big[4ku^{k-1}u_1 g_{u_2} + u^k g_{u_1} + P_u \\ &+ 10ku^{k-1}u_2 + 10k(k-1)u^{k-2}u_1^2\Big]u_3 + P_v v_3 + \Big[u_1^2 P_{uu} + v_1^2 P_{vv} \\ &+ 2u_1v_1P_{uv} + u_2P_u + v_2P_v + 3ku^{k-1}u_2^2 + 6k(k-1)u^{k-2}u_2u_1^2 \\ &+ k(k-1)(k-2)u^{k-3}u_1^4\Big]g_{u_2} + \left[u_1P_u + v_1P_v + 3ku^{k-1}u_2u_1\right. \\ &\left.+ k(k-1)u^{k-2}u_1^3\right]g_{u_1} + \left(u^k u_2 + ku^{k-1}u_1^2 + P\right)g_u + u_1^3P_{uuu} \\ &+ v_1^3P_{vvv} + 3u_1^2v_1P_{uuv} + 3u_1v_1^2P_{uvv} + 3u_1u_2P_{uu} + 3v_1v_2P_{vv} \\ &+ 3(u_1v_2 + u_2v_1)P_{uv} + k(k-1)(k-2)(k-3)u^{k-4}u_1^5 \\ &+ 10k(k-1)(k-2)u^{k-3}u_2u_1^3 + 15k(k-1)u^{k-2}u_2^2u_1 + g_t \end{aligned}$$

and

$$\begin{aligned} &u^k D_x^2\eta_1 + b_{11}u^{k-1}u_1 D_x\eta_1 + \left(b_{12}u^{k-1}u_2 + b_{13}u^{k-2}u_1^2 + b_{14}P_u\right)\eta_1 + b_{15}P_v\eta_2 \\ =& u^k u_5 + \left(b_{11}u^{k-1}u_1 + u^k g_{u_2}\right)u_4 + u^k g_{u_2u_2}u_3^2 + (2u^k u_2 g_{u_1u_2} + 2u^k u_1 g_{uu_2} \\ &+ 2u^k g_{xu_2} + b_{11}u^{k-1}u_1 g_{u_2} + u^k g_{u_1} + b_{14}P_u + b_{12}u^{k-1}u_2 + b_{13}u^{k-2}u_1^2)u_3 \\ &+ b_{15}P_v v_3 + u^k u_2^2 g_{u_1u_1} + u^k u_1^2 g_{uu} + 2u^k u_1u_2 g_{uu_1} + 2u^k u_1 g_{xu} + u^k g_{xx} \\ &+ 2u^k u_2 g_{xu_1} + b_{11}u^{k-1}u_1u_2 g_{u_1} + b_{11}u^{k-1}u_1^2 g_u + u^k u_2 g_u + b_{11}u^{k-1}u_1 g_x \\ &+ \left(b_{12}u^{k-1}u_2 + b_{13}u^{k-2}u_1^2 + b_{14}P_u\right)g + b_{15}P_v h. \end{aligned}$$

Since the left-hand side and right-hand side of (27) are both polynomials about u_5, u_4, u_3, v_3, equating the coefficients of similar terms will give $b_{11} = 5k$, $b_{15} = 1$ and

$$\begin{aligned}
&g_{u_2u_2} = 0,\\
&2u^k u_2 g_{u_1u_2} + 2u^k u_1 g_{uu_2} + 2u^k g_{xu_2} + (b_{11} - 4k)u^{k-1}u_1 g_{u_2} + (b_{14} - 1)P_u\\
&+ (b_{12} - 10k)u^{k-1}u_2 + [b_{13} - 10k(k-1)]u^{k-2}u_1^2 = 0,\\
&u^k u_2^2 g_{u_1u_1} + u^k u_1^2 g_{uu} + 2u^k u_1 u_2 g_{uu_1} + 2u^k u_1 g_{xu} + u^k g_{xx} + 2u^k u_2 g_{xu_1}\\
&- \Big[u_1^2 P_{uu} + v_1^2 P_{vv} + 2u_1 v_1 P_{uv} + u_2 P_u + v_2 P_v + 6k(k-1)u^{k-2}u_2 u_1^2\\
&+ 3ku^{k-1}u_2^2 + k(k-1)(k-2)u^{k-3}u_1^4\Big] g_{u_2} - \big[(3k - b_{11})u^{k-1}u_2 u_1\\
&+ k(k-1)u^{k-2}u_1^3 + u_1 P_u + v_1 P_v\big] g_{u_1} - \big[(k - b_{11})u^{k-1}u_1^2 + P\big] g_u\\
&+ b_{11}u^{k-1}u_1 g_x - g_t + \Big(b_{12}u^{k-1}u_2 + b_{13}u^{k-2}u_1^2 + b_{14}P_u\Big) g + b_{15}P_v h\\
&- \Big[u_1^3 P_{uuu} + v_1^3 P_{vvv} + 3u_1^2 v_1 P_{uuv} + 3u_1 v_1^2 P_{uvv} + 3(u_1 v_2 + u_2 v_1)P_{uv}\\
&+ 3u_1 u_2 P_{uu} + 3v_1 v_2 P_{vv} + k(k-1)(k-2)(k-3)u^{k-4}u_1^5\\
&+ 10k(k-1)(k-2)u^{k-3}u_2 u_1^3 + 15k(k-1)u^{k-2}u_2^2 u_1\Big] = 0.
\end{aligned} \tag{29}$$

Similar discussion about (28) will yield $b_{21} = 5l$, $b_{25} = 1$ and

$$\begin{aligned}
&h_{v_2v_2} = 0,\\
&2v^l v_2 h_{v_1v_2} + 2v^l v_1 h_{vv_2} + 2v^l h_{xv_2} + (b_{21} - 4l)v^{l-1}v_1 h_{v_2} + (b_{24} - 1)Q_v\\
&+ (b_{22} - 10l)v^{l-1}v_2 + [b_{23} - 10l(l-1)]v^{l-2}v_1^2 = 0,\\
&v^l v_2^2 h_{v_1v_1} + v^l v_1^2 h_{vv} + 2v^l v_1 v_2 h_{vv_1} + 2v^l v_1 h_{xv} + v^l h_{xx} + 2v^l v_2 h_{xv_1}\\
&- \Big[v_1^2 Q_{vv} + u_1^2 Q_{uu} + 2u_1 v_1 Q_{uv} + u_2 Q_u + v_2 Q_v + 6l(l-1)v^{l-2}v_2 v_1^2\\
&+ 3lv^{l-1}v_2^2 + l(l-1)(l-2)v^{l-3}v_1^4\Big] h_{v_2} - \big[(3l - b_{21})v^{l-1}v_2 v_1\\
&+ l(l-1)v^{l-2}v_1^3 + u_1 Q_u + v_1 Q_v\big] h_{v_1} - \big[(l - b_{21})v^{l-1}v_1^2 + Q\big] h_v\\
&+ b_{21}v^{l-1}v_1 h_x - h_t + \Big(b_{22}v^{l-1}v_2 + b_{23}v^{l-2}v_1^2 + b_{24}Q_v\Big) h + b_{25}Q_u g\\
&- \Big[u_1^3 Q_{uuu} + v_1^3 Q_{vvv} + 3u_1^2 v_1 Q_{uuv} + 3u_1 v_1^2 Q_{uvv} + 3(u_1 v_2 + u_2 v_1)Q_{uv}\\
&+ 3u_1 u_2 Q_{uu} + 3v_1 v_2 Q_{vv} + l(l-1)(l-2)(l-3)v^{l-4}v_1^5\\
&+ 10l(l-1)(l-2)v^{l-3}v_2 v_1^3 + 15l(l-1)u^{l-2}v_2^2 v_1\Big] = 0.
\end{aligned} \tag{30}$$

It is easy to know that g and h can be represented as

$$g(t, x, u, u_1, u_2) = g_1(t, x, u, u_1)u_2 + g_2(t, x, u, u_1)$$

and

$$h(t, x, v, v_1, v_2) = h_1(t, x, v, v_1)v_2 + h_2(t, x, v, v_1).$$

Substituting g into the second one of (29), we will derive that

$$\begin{aligned}&\left[2ug_{1u_1}+(b_{12}-10k)\right]u^{k-1}u_2+\left[(2ug_{1u}+kg_1)u_1+2ug_{1x}\right]u^{k-1}\\&+\left[b_{13}-10k(k-1)\right]u^{k-2}u_1^2+(b_{14}-1)P_u=0.\end{aligned}$$

The vanishing of the coefficient of u_2 will yield

$$g_1(t,x,u,u_1)=\frac{10k-b_{12}}{2u}u_1+g_3(t,x,u).$$

As a consequence, (29) can be simplified as

$$\left(b_{13}-5k^2-\frac{1}{2}kb_{12}+b_{12}\right)u^{k-2}u_1^2+(2ug_{3u}+kg_3)\,u^{k-1}u_1+2u^kg_{3x}+(b_{14}-1)P_u=0,$$

which is a polynomial about u_1. Thus, $b_{13}=5k^2+\frac{1}{2}kb_{12}-b_{12}$ and $g_3(t,x,u)=g_4(t,x)u^{-\frac{k}{2}}$ can be derived by equating the coefficients of this polynomial to be zero. Subsequently, (29) finally becomes

$$2u^{\frac{k}{2}}g_{4x}+(b_{14}-1)P_u=0.$$

Since $P(u,v)$ must depend on v, we will arrive at $g_4(t,x)=g_5(t)$ and $b_{14}=1$ or $P(u,v)=P_1(v)$ from the above equality.

A similar computational procedure for the first one and second one of (30) will give

$$\begin{aligned}&h(t,x,,v,v_1,v_2)=h_1(t,x,v,v_1)v_2+h_2(t,x,v,v_1),\\&h_1(t,x,v,v_1)=\frac{10l-b_{22}}{2v}v_1+h_3(t,x,v),\\&h_3(t,x,v)=h_4(t,x)v^{-\frac{l}{2}},\ h_4(t,x)=h_5(t),\\&b_{23}=5l^2+\frac{1}{2}lb_{22}-b_{22}\end{aligned}$$

and

$$b_{24}=1\ or\ Q(u,v)=Q_1(u).$$

We will consider four different cases, including

$$\begin{aligned}&(i)\ b_{14}=1,\ b_{24}=1;\\&(ii)\ b_{14}=1,\ Q(u,v)=Q_1(u);\\&(iii)\ P(u,v)=P_1(v),\ b_{24}=1;\\&(iv)\ P(u,v)=P_1(v),\ Q(u,v)=Q_1(u)\end{aligned}$$

to further study. Further research about the last ones of (29) and (30) will finally identify DCs (3) and CLBSs (5) of RD system (9). The comprehensive computational procedure is omitted here, and the obtained results are listed in Table 1. The procedure to identify DCs (3) and CLBSs (5) of RD system (4) for $n=2$, $n=4$ and $n=5$ is almost the same as that for the case of $n=3$. We just list the obtained results in Table 1. It is noted that the results for $n=2$ are all presented in [21], so we will not list these cases in Table 1.

Table 1. conditional Lie–Bäcklund symmetry (CLBS) (5) of reaction-diffusion (RD) System (9).

No.	RD System (9)	CLBS (5)
1	$\begin{cases} u_t = \left(u^{-\frac{3}{2}}u_x\right)_x - \frac{s}{r}b_1 u + a_1 u^{\frac{5}{2}} + b_1 u^{\frac{5}{2}} v^{-\frac{3}{2}}, \\ v_t = \left(v^{-\frac{3}{2}}v_x\right)_x - \frac{r}{s}b_2 v + a_2 v^{\frac{5}{2}} + b_2 v^{\frac{5}{2}} u^{-\frac{3}{2}} \end{cases}$	$\begin{cases} \eta_1 = u_3 - \frac{15}{2u}u_1u_2 + \frac{35}{4u^2}u_1^3 + ru^{\frac{5}{2}}, \\ \eta_2 = v_3 - \frac{15}{2v}v_1v_2 + \frac{35}{4v^2}v_1^3 + sv^{\frac{5}{2}} \end{cases}$
2	$\begin{cases} u_t = \left(u^{-\frac{4}{3}}u_x\right)_x + a_1 u + b_1 u^{\frac{5}{3}} - \frac{3s}{4}u^{-\frac{1}{3}} + c_1 u^{\frac{5}{3}} v^{-\frac{2}{3}}, \\ v_t = \left(v^{-\frac{4}{3}}v_x\right)_x + a_2 v + b_2 v^{\frac{5}{3}} - \frac{3s}{4}v^{-\frac{1}{3}} + c_2 v^{\frac{5}{3}} u^{-\frac{2}{3}} \end{cases}$	$\begin{cases} \eta_1 = u_3 - \frac{5}{u}u_1u_2 + \frac{40}{9u^2}u_1^3 + su_1, \\ \eta_2 = v_3 - \frac{5}{v}v_1v_2 + \frac{40}{9v^2}v_1^3 + sv_1 \end{cases}$
3	$\begin{cases} u_t = \left(u^{-\frac{4}{3}}u_x\right)_x + a_1 u + b_1 u^{\frac{5}{3}} - \frac{3s}{4}u^{-\frac{1}{3}} + c_1 u^{\frac{5}{3}} v^{l}, \\ v_t = \left(v^{l}v_x\right)_x + a_2 v + b_2 v^{1-l} + \frac{(l+1)s}{l^2}v^{1+l} + c_2 v^{1-l} u^{-\frac{2}{3}} \end{cases}$	$\begin{cases} \eta_1 = u_3 - \frac{5}{u}u_1u_2 + \frac{40}{9u^2}u_1^3 + su_1, \\ \eta_2 = v_3 + \frac{3(l-1)}{v}v_1v_2 + \frac{(l-1)(l-2)}{v^2}v_1^3 + sv_1 \end{cases}$
4	$\begin{cases} u_t = \left(u^{k}u_x\right)_x + a_1 u + b_1 u^{1-k} + \frac{(k+1)s}{k^2}u^{1+k} + c_1 u^{1-k} v^{l}, \\ v_t = \left(v^{l}v_x\right)_x + a_2 v + b_2 v^{1-l} + \frac{(l+1)s}{l^2}v^{1+l} + c_2 v^{1-l} u^{k} \end{cases}$	$\begin{cases} \eta_1 = u_3 + \frac{3(k-1)}{u}u_1u_2 + \frac{(k-1)(k-2)}{u^2}u_1^3 + su_1, \\ \eta_2 = v_3 + \frac{3(l-1)}{v}v_1v_2 + \frac{(l-1)(l-2)}{v^2}v_1^3 + sv_1 \end{cases}$
5	$\begin{cases} u_t = \left(u^{-\frac{3}{2}}u_x\right)_x + a_1 u + b_1 u^{\frac{5}{2}} + c_1 u^{\frac{5}{2}} v^{-\frac{3}{2}}, \\ v_t = \left(v^{-\frac{3}{2}}v_x\right)_x + a_2 v + b_2 v^{\frac{5}{2}} + c_2 v^{\frac{5}{2}} u^{-\frac{3}{2}} \end{cases}$	$\begin{cases} \eta_1 = u_4 - \frac{10}{u}u_1u_3 - \frac{15}{2u}u_2^2 + \frac{105}{2u^2}u_1^2u_2 - \frac{315}{8u^3}u_1^4, \\ \eta_2 = v_4 - \frac{10}{v}v_1v_3 - \frac{15}{2v}v_2^2 + \frac{105}{2v^2}v_1^2v_2 - \frac{315}{8v^3}v_1^4 \end{cases}$
6	$\begin{cases} u_t = \left(u^{-\frac{4}{3}}u_x\right)_x + a_1 u + b_1 u^{\frac{7}{3}} - \frac{3s}{20}u^{-\frac{1}{3}} + c_1 u^{\frac{7}{3}} v^{-\frac{4}{3}}, \\ v_t = \left(v^{-\frac{4}{3}}v_x\right)_x + a_2 v + b_2 v^{\frac{7}{3}} - \frac{3s}{20}v^{-\frac{1}{3}} + c_2 v^{\frac{7}{3}} u^{-\frac{4}{3}} \end{cases}$	$\begin{cases} \eta_1 = u_5 - \frac{35}{3u}u_1u_4 + \left(-\frac{70}{3u}u_2 + \frac{700}{9u^2}u_1^2 + s\right)u_3 \\ \quad + \frac{350}{3u^2}u_1u_2^2 - \left(\frac{7s}{u}u_1 + \frac{9100}{27u^3}u_1^3\right)u_2 + \frac{14560}{81u^4}u_1^5 \\ \quad + \frac{70s}{9u^2}u_1^3 + \frac{4s^2}{25}u_1, \\ \eta_2 = v_5 - \frac{35}{3v}v_1v_4 + \left(-\frac{70}{3v}v_2 + \frac{700}{9v^2}v_1^2 + s\right)v_3 \\ \quad + \frac{350}{3v^2}v_1v_2^2 - \left(\frac{7s}{v}v_1 + \frac{9100}{27v^3}v_1^3\right)v_2 + \frac{14560}{81v^4}v_1^5 \\ \quad + \frac{70s}{9v^2}v_1^3 + \frac{4s^2}{25}v_1, \end{cases}$

5. Reductions of RD System (9)

The compatibility of the RD system (9) and the invariant surface condition (DC) (3) is the basic reduction idea of CLBS. Therefore, the evolution system (9) and the admitted DC (3) share a common manifold of solutions. We first solve the DC (3) to identify the form of u and v and then substitute the obtained results into (9) to finally determine the solutions. Here, we will construct the reductions of the resulting systems (9) in Table 1.

Example 1. *RD system*

$$\begin{cases} u_t = \left(u^{-\frac{3}{2}}u_x\right)_x - \frac{s}{r}b_1 u + a_1 u^{\frac{5}{2}} + b_1 u^{\frac{5}{2}} v^{-\frac{3}{2}}, \\ v_t = \left(v^{-\frac{3}{2}}v_x\right)_x - \frac{r}{s}b_2 v + a_2 v^{\frac{5}{2}} + b_2 v^{\frac{5}{2}} u^{-\frac{3}{2}} \end{cases}$$

admits CLBS

$$\begin{cases} \eta_1 = u_3 - \frac{15}{2u}u_1 u_2 + \frac{35}{4u^2}u_1^3 + r u^{\frac{5}{2}}, \\ \eta_2 = v_3 - \frac{15}{2v}v_1 v_2 + \frac{35}{4v^2}v_1^3 + s v^{\frac{5}{2}}. \end{cases}$$

The solutions of this system are listed as

$$\begin{cases} u(x,t) = \left[\frac{r}{4}x^3 + C_1^{(1)}(t)x^2 + C_2^{(1)}(t)x + C_3^{(1)}(t)\right]^{-\frac{2}{3}}, \\ v(x,t) = \left[\frac{s}{4}x^3 + C_1^{(2)}(t)x^2 + C_2^{(2)}(t)x + C_3^{(2)}(t)\right]^{-\frac{2}{3}}, \end{cases}$$

where $C_1^{(1)}(t), C_2^{(1)}(t), C_3^{(1)}(t), C_1^{(2)}(t), C_2^{(2)}(t)$ and $C_3^{(2)}(t)$ satisfy the six-dimensional dynamical system

$$\begin{cases} C_1^{(1)'} = -\frac{2}{3}C_1^{(1)^2} + \frac{3s}{2r}b_1 C_1^{(1)} + \frac{r}{2}C_2^{(1)} - \frac{3}{2}b_1 C_1^{(2)}, \\ C_2^{(1)'} = -\frac{2}{3}C_1^{(1)}C_2^{(1)} + \frac{3s}{2r}b_1 C_2^{(1)} + \frac{3r}{2}C_3^{(1)} - \frac{3}{2}b_1 C_2^{(2)}, \\ C_3^{(1)'} = 2C_1^{(1)}C_3^{(1)} - \frac{2}{3}C_2^{(1)^2} + \frac{3s}{2r}b_1 C_3^{(1)} - \frac{3}{2}b_1 C_3^{(2)} - \frac{3}{2}a_1, \\ C_1^{(2)'} = -\frac{2}{3}C_1^{(2)^2} + \frac{3r}{2s}b_2 C_1^{(2)} + \frac{s}{2}C_2^{(2)} - \frac{3}{2}b_2 C_1^{(1)}, \\ C_2^{(2)'} = -\frac{2}{3}C_1^{(2)}C_2^{(2)} + \frac{3r}{2s}b_2 C_2^{(2)} + \frac{3s}{2}C_3^{(2)} - \frac{3}{2}b_2 C_2^{(1)}, \\ C_3^{(2)'} = 2C_1^{(2)}C_3^{(2)} - \frac{2}{3}C_2^{(2)^2} + \frac{3r}{2s}b_2 C_3^{(2)} - \frac{3}{2}b_2 C_3^{(1)} - \frac{3}{2}a_2. \end{cases}$$

Example 2. *RD system*

$$\begin{cases} u_t = \left(u^{-\frac{4}{3}}u_x\right)_x + a_1 u + b_1 u^{\frac{5}{3}} - \frac{3s}{4}u^{-\frac{1}{3}} + c_1 u^{\frac{5}{3}} v^{-\frac{2}{3}}, \\ v_t = \left(v^{-\frac{4}{3}}v_x\right)_x + a_2 v + b_2 v^{\frac{5}{3}} - \frac{3s}{4}v^{-\frac{1}{3}} + c_2 v^{\frac{5}{3}} u^{-\frac{2}{3}} \end{cases}$$

admits CLBS

$$\begin{cases} \eta_1 = u_3 - \frac{5}{u}u_1 u_2 + \frac{40}{9u^2}u_1^3 + s u_1, \\ \eta_2 = v_3 - \frac{5}{v}v_1 v_2 + \frac{40}{9v^2}v_1^3 + s v_1. \end{cases}$$

The solutions of this system are given as below.

- *For* $s > 0$,

$$\begin{cases} u(x,t) = \left[C_1^{(1)}(t) + C_2^{(1)}(t)\sin\left(\sqrt{s}x\right) + C_3^{(1)}(t)\cos\left(\sqrt{s}x\right)\right]^{-\frac{3}{2}}, \\ v(x,t) = \left[C_1^{(2)}(t) + C_2^{(2)}(t)\sin\left(\sqrt{s}x\right) + C_3^{(2)}(t)\cos\left(\sqrt{s}x\right)\right]^{-\frac{3}{2}}, \end{cases}$$

where $C_1^{(1)}(t), C_2^{(1)}(t), C_3^{(1)}(t), C_1^{(2)}(t), C_2^{(2)}(t)$ *and* $C_3^{(2)}(t)$ *satisfy the six-dimensional dynamical system*

$$\begin{cases} C_1^{(1)\prime} = \frac{s}{2}C_1^{(1)}\left({C_1^{(1)}}^2 - {C_2^{(1)}}^2 - {C_3^{(1)}}^2\right) - \frac{2}{3}a_1C_1^{(1)} - \frac{2}{3}c_1C_1^{(2)} - \frac{2}{3}b_1, \\ C_2^{(1)\prime} = \frac{s}{2}C_2^{(1)}\left({C_1^{(1)}}^2 - {C_2^{(1)}}^2 - {C_3^{(1)}}^2\right) - \frac{2}{3}a_1C_2^{(1)} - \frac{2}{3}c_1C_2^{(2)}, \\ C_3^{(1)\prime} = \frac{s}{2}C_3^{(1)}\left({C_1^{(1)}}^2 - {C_2^{(1)}}^2 - {C_3^{(1)}}^2\right) - \frac{2}{3}a_1C_3^{(1)} - \frac{2}{3}c_1C_3^{(2)}, \\ C_1^{(2)\prime} = \frac{s}{2}C_1^{(2)}\left({C_1^{(2)}}^2 - {C_2^{(2)}}^2 - {C_3^{(2)}}^2\right) - \frac{2}{3}a_2C_1^{(2)} - \frac{2}{3}c_2C_1^{(1)} - \frac{2}{3}b_2, \\ C_2^{(2)\prime} = \frac{s}{2}C_2^{(2)}\left({C_1^{(2)}}^2 - {C_2^{(2)}}^2 - {C_3^{(2)}}^2\right) - \frac{2}{3}a_2C_2^{(2)} - \frac{2}{3}c_2C_2^{(1)}, \\ C_3^{(2)\prime} = \frac{s}{2}C_3^{(2)}\left({C_1^{(2)}}^2 - {C_2^{(2)}}^2 - {C_3^{(2)}}^2\right) - \frac{2}{3}a_2C_3^{(2)} - \frac{2}{3}c_2C_3^{(1)}. \end{cases}$$

- *For* $s = 0$,

$$\begin{cases} u(x,t) = \left[C_1^{(1)}(t)x^2 + C_2^{(1)}(t)x + C_3^{(1)}(t)\right]^{-\frac{3}{2}}, \\ v(x,t) = \left[C_1^{(2)}(t)x^2 + C_2^{(2)}(t)x + C_3^{(2)}(t)\right]^{-\frac{3}{2}}, \end{cases}$$

where $C_1^{(1)}(t), C_2^{(1)}(t), C_3^{(1)}(t), C_1^{(2)}(t), C_2^{(2)}(t)$ *and* $C_3^{(2)}(t)$ *satisfy the six-dimensional dynamical system*

$$\begin{cases} C_1^{(1)\prime} = 2{C_1^{(1)}}^2C_3^{(1)} - \frac{1}{2}C_1^{(1)}{C_2^{(1)}}^2 - \frac{2}{3}a_1C_1^{(1)} - \frac{2}{3}c_1C_1^{(2)}, \\ C_2^{(1)\prime} = 2C_1^{(1)}C_2^{(1)}C_3^{(1)} - \frac{1}{2}{C_2^{(1)}}^3 - \frac{2}{3}a_1C_2^{(1)} - \frac{2}{3}c_1C_2^{(2)}, \\ C_3^{(1)\prime} = 2C_1^{(1)}{C_3^{(1)}}^2 - \frac{1}{2}{C_2^{(1)}}^2C_3^{(1)} - \frac{2}{3}a_1C_3^{(1)} - \frac{2}{3}c_1C_3^{(2)} - \frac{2}{3}b_1, \\ C_1^{(2)\prime} = 2{C_1^{(2)}}^2C_3^{(2)} - \frac{1}{2}C_1^{(2)}{C_2^{(2)}}^2 - \frac{2}{3}a_2C_1^{(2)} - \frac{2}{3}c_2C_1^{(1)}, \\ C_2^{(2)\prime} = 2C_1^{(2)}C_2^{(2)}C_3^{(2)} - \frac{1}{2}{C_2^{(2)}}^3 - \frac{2}{3}a_2C_2^{(2)} - \frac{2}{3}c_2C_2^{(1)}, \\ C_3^{(2)\prime} = 2C_1^{(2)}{C_3^{(2)}}^2 - \frac{1}{2}{C_2^{(2)}}^2C_3^{(2)} - \frac{2}{3}a_2C_3^{(2)} - \frac{2}{3}c_2C_3^{(1)} - \frac{2}{3}b_2. \end{cases}$$

- *For* $s < 0$,

$$\begin{cases} u(x,t) = \left[C_1^{(1)}(t) + C_2^{(1)}(t)\sinh\left(\sqrt{-s}x\right) + C_3^{(1)}(t)\cosh\left(\sqrt{-s}x\right)\right]^{-\frac{3}{2}}, \\ v(x,t) = \left[C_1^{(2)}(t) + C_2^{(2)}(t)\sinh\left(\sqrt{-s}x\right) + C_3^{(2)}(t)\cosh\left(\sqrt{-s}x\right)\right]^{-\frac{3}{2}}, \end{cases}$$

where $C_1^{(1)}(t), C_2^{(1)}(t), C_3^{(1)}(t), C_1^{(2)}(t), C_2^{(2)}(t)$ *and* $C_3^{(2)}(t)$ *satisfy the six-dimensional dynamical system*

$$\begin{cases} C_1^{(1)\prime} = \frac{s}{2}C_1^{(1)}\left({C_1^{(1)}}^2 + {C_2^{(1)}}^2 - {C_3^{(1)}}^2\right) - \frac{2}{3}a_1C_1^{(1)} - \frac{2}{3}c_1C_1^{(2)} - \frac{2}{3}b_1, \\ C_2^{(1)\prime} = \frac{s}{2}C_2^{(1)}\left({C_1^{(1)}}^2 + {C_2^{(1)}}^2 - {C_3^{(1)}}^2\right) - \frac{2}{3}a_1C_2^{(1)} - \frac{2}{3}c_1C_2^{(2)}, \\ C_3^{(1)\prime} = \frac{s}{2}C_3^{(1)}\left({C_1^{(1)}}^2 + {C_2^{(1)}}^2 - {C_3^{(1)}}^2\right) - \frac{2}{3}a_1C_3^{(1)} - \frac{2}{3}c_1C_3^{(2)}, \\ C_1^{(2)\prime} = \frac{s}{2}C_1^{(2)}\left({C_1^{(2)}}^2 + {C_2^{(2)}}^2 - {C_3^{(2)}}^2\right) - \frac{2}{3}a_2C_1^{(2)} - \frac{2}{3}c_2C_1^{(1)} - \frac{2}{3}b_2, \\ C_2^{(2)\prime} = \frac{s}{2}C_2^{(2)}\left({C_1^{(2)}}^2 + {C_2^{(2)}}^2 - {C_3^{(2)}}^2\right) - \frac{2}{3}a_2C_2^{(2)} - \frac{2}{3}c_2C_2^{(1)}, \\ C_3^{(2)\prime} = \frac{s}{2}C_3^{(2)}\left({C_1^{(2)}}^2 + {C_2^{(2)}}^2 - {C_3^{(2)}}^2\right) - \frac{2}{3}a_2C_3^{(2)} - \frac{2}{3}c_2C_3^{(1)}. \end{cases}$$

Example 3. *RD system*

$$\begin{cases} u_t = \left(u^{-\frac{4}{3}}u_x\right)_x + a_1u + b_1u^{\frac{5}{3}} - \frac{3s}{4}u^{-\frac{1}{3}} + c_1u^{\frac{5}{3}}v^l, \\ v_t = \left(v^lv_x\right)_x + a_2v + b_2v^{1-l} + \frac{(l+1)s}{l^2}v^{1+l} + c_2v^{1-l}u^{-\frac{2}{3}} \end{cases}$$

admits CLBS

$$\begin{cases} \eta_1 = u_3 - \frac{5}{u}u_1u_2 + \frac{40}{9u^2}u_1^3 + su_1, \\ \eta_2 = v_3 + \frac{3(l-1)}{v}v_1v_2 + \frac{(l-1)(l-2)}{v^2}v_1^3 + sv_1. \end{cases}$$

The solutions of this system are given as below.

- *For* $s > 0$,

$$\begin{cases} u(x,t) = \left[C_1^{(1)}(t) + C_2^{(1)}(t)\sin\left(\sqrt{s}x\right) + C_3^{(1)}(t)\cos\left(\sqrt{s}x\right)\right]^{-\frac{3}{2}}, \\ v(x,t) = \left[C_1^{(2)}(t) + C_2^{(2)}(t)\sin\left(\sqrt{s}x\right) + C_3^{(2)}(t)\cos\left(\sqrt{s}x\right)\right]^{\frac{1}{l}}, \end{cases}$$

where $C_1^{(1)}(t), C_2^{(1)}(t), C_3^{(1)}(t), C_1^{(2)}(t), C_2^{(2)}(t)$ *and* $C_3^{(2)}(t)$ *satisfy the six-dimensional dynamical system*

$$\begin{cases} C_1^{(1)\prime} = \frac{s}{2}C_1^{(1)}\left({C_1^{(1)}}^2 - {C_2^{(1)}}^2 - {C_3^{(1)}}^2\right) - \frac{2}{3}a_1C_1^{(1)} - \frac{2}{3}c_1C_1^{(2)} - \frac{2}{3}b_1, \\ C_2^{(1)\prime} = \frac{s}{2}C_2^{(1)}\left({C_1^{(1)}}^2 - {C_2^{(1)}}^2 - {C_3^{(1)}}^2\right) - \frac{2}{3}a_1C_2^{(1)} - \frac{2}{3}c_1C_2^{(2)}, \\ C_3^{(1)\prime} = \frac{s}{2}C_3^{(1)}\left({C_1^{(1)}}^2 - {C_2^{(1)}}^2 - {C_3^{(1)}}^2\right) - \frac{2}{3}a_1C_3^{(1)} - \frac{2}{3}c_1C_3^{(2)}, \\ C_1^{(2)\prime} = \frac{(l+1)s}{l}{C_1^{(2)}}^2 + \frac{s}{l}\left({C_2^{(2)}}^2 + {C_3^{(2)}}^2\right) + la_2C_1^{(2)} + lc_2C_1^{(1)} + lb_2, \\ C_2^{(2)\prime} = \frac{(l+2)s}{l}C_1^{(2)}C_2^{(2)} + la_2C_2^{(2)} + lc_2C_2^{(1)}, \\ C_3^{(2)\prime} = \frac{(l+2)s}{l}C_1^{(2)}C_3^{(2)} + la_2C_3^{(2)} + lc_2C_3^{(1)}. \end{cases}$$

- *For* $s = 0$,

$$\begin{cases} u(x,t) = \left[C_1^{(1)}(t)x^2 + C_2^{(1)}(t)x + C_3^{(1)}(t)\right]^{-\frac{3}{2}}, \\ v(x,t) = \left[C_1^{(2)}(t)x^2 + C_2^{(2)}(t)x + C_3^{(2)}(t)\right]^{\frac{1}{l}}, \end{cases}$$

where $C_1^{(1)}(t), C_2^{(1)}(t), C_3^{(1)}(t), C_1^{(2)}(t), C_2^{(2)}(t)$ and $C_3^{(2)}(t)$ satisfy the six-dimensional dynamical system

$$\begin{cases} {C_1^{(1)}}' = 2{C_1^{(1)}}^2 C_3^{(1)} - \frac{1}{2} C_1^{(1)} {C_2^{(1)}}^2 - \frac{2}{3} a_1 C_1^{(1)} - \frac{2}{3} c_1 C_1^{(2)}, \\ {C_2^{(1)}}' = 2C_1^{(1)} C_2^{(1)} C_3^{(1)} - \frac{1}{2} {C_2^{(1)}}^3 - \frac{2}{3} a_1 C_2^{(1)} - \frac{2}{3} c_1 C_2^{(2)}, \\ {C_3^{(1)}}' = 2C_1^{(1)} {C_3^{(1)}}^2 - \frac{1}{2} {C_2^{(1)}}^2 C_3^{(1)} - \frac{2}{3} a_1 C_3^{(1)} - \frac{2}{3} c_1 C_3^{(2)} - \frac{2}{3} b_1, \\ {C_1^{(2)}}' = \frac{2(l+2)}{l} {C_1^{(2)}}^2 + l a_2 C_1^{(2)} + l c_2 C_1^{(1)}, \\ {C_2^{(2)}}' = \frac{2(l+2)}{l} C_1^{(2)} C_2^{(2)} + l a_2 C_2^{(2)} + l c_2 C_2^{(1)}, \\ {C_3^{(2)}}' = \frac{1}{l} {C_2^{(2)}}^2 + 2C_1^{(2)} C_3^{(2)} + l a_2 C_3^{(2)} + l c_2 C_3^{(1)} + l b_2. \end{cases}$$

- *For $s < 0$,*

$$\begin{cases} u(x,t) = \left[C_1^{(1)}(t) + C_2^{(1)}(t) \sinh\left(\sqrt{-s}x\right) + C_3^{(1)}(t) \cosh\left(\sqrt{-s}x\right) \right]^{\frac{3}{2}}, \\ v(x,t) = \left[C_1^{(2)}(t) + C_2^{(2)}(t) \sinh\left(\sqrt{-s}x\right) + C_3^{(2)}(t) \cosh\left(\sqrt{-s}x\right) \right]^{\frac{1}{l}}, \end{cases}$$

where $C_1^{(1)}(t), C_2^{(1)}(t), C_3^{(1)}(t), C_1^{(2)}(t), C_2^{(2)}(t)$ and $C_3^{(2)}(t)$ satisfy the six-dimensional dynamical system

$$\begin{cases} {C_1^{(1)}}' = \frac{s}{2} C_1^{(1)} \left({C_1^{(1)}}^2 + {C_2^{(1)}}^2 - {C_3^{(1)}}^2 \right) - \frac{2}{3} a_1 C_1^{(1)} - \frac{2}{3} c_1 C_1^{(2)} - \frac{2}{3} b_1, \\ {C_2^{(1)}}' = \frac{s}{2} C_2^{(1)} \left({C_1^{(1)}}^2 + {C_2^{(1)}}^2 - {C_3^{(1)}}^2 \right) - \frac{2}{3} a_1 C_2^{(1)} - \frac{2}{3} c_1 C_2^{(2)}, \\ {C_3^{(1)}}' = \frac{s}{2} C_3^{(1)} \left({C_1^{(1)}}^2 + {C_2^{(1)}}^2 - {C_3^{(1)}}^2 \right) - \frac{2}{3} a_1 C_3^{(1)} - \frac{2}{3} c_1 C_3^{(2)}, \\ {C_1^{(2)}}' = \frac{(l+1)s}{l} {C_1^{(2)}}^2 + \frac{s}{l} \left({C_3^{(2)}}^2 - {C_2^{(2)}}^2 \right) + l a_2 C_1^{(2)} + l c_2 C_1^{(1)} + l b_2, \\ {C_2^{(2)}}' = \frac{(l+2)s}{l} C_1^{(2)} C_2^{(2)} + l a_2 C_2^{(2)} + l c_2 C_2^{(1)}, \\ {C_3^{(2)}}' = \frac{(l+2)s}{l} C_1^{(2)} C_3^{(2)} + l a_2 C_3^{(2)} + l c_2 C_3^{(1)}. \end{cases}$$

Example 4. *RD system*

$$\begin{cases} u_t = \left(u^k u_x \right)_x + a_1 u + b_1 u^{1-k} + \frac{(k+1)s}{k^2} u^{1+k} + c_1 u^{1-k} v^l, \\ v_t = \left(v^l v_x \right)_x + a_2 v + b_2 v^{1-l} + \frac{(l+1)s}{l^2} v^{1+l} + c_2 v^{1-l} u^k \end{cases}$$

admits CLBS

$$\begin{cases} \eta_1 = u_3 + \frac{3(k-1)}{u} u_1 u_2 + \frac{(k-1)(k-2)}{u^2} u_1^3 + s u_1, \\ \eta_2 = v_3 + \frac{3(l-1)}{v} v_1 v_2 + \frac{(l-1)(l-2)}{v^2} v_1^3 + s v_1. \end{cases}$$

The solutions of this system are given as below.

- *For $s > 0$,*

$$\begin{cases} u(x,t) = \left[C_1^{(1)}(t) + C_2^{(1)}(t) \sin\left(\sqrt{s}x\right) + C_3^{(1)}(t) \cos\left(\sqrt{s}x\right) \right]^{\frac{1}{k}}, \\ v(x,t) = \left[C_1^{(2)}(t) + C_2^{(2)}(t) \sin\left(\sqrt{s}x\right) + C_3^{(2)}(t) \cos\left(\sqrt{s}x\right) \right]^{\frac{1}{l}}, \end{cases}$$

where $C_1^{(1)}(t), C_2^{(1)}(t), C_3^{(1)}(t), C_1^{(2)}(t), C_2^{(2)}(t)$ *and* $C_3^{(2)}(t)$ *satisfy the six-dimensional dynamical system*

$$\begin{cases} C_1^{(1)\prime} = \frac{(k+1)s}{k} C_1^{(1)^2} + \frac{s}{k}\left(C_2^{(1)^2} + C_3^{(1)^2}\right) + ka_1 C_1^{(1)} + kc_1 C_1^{(2)} + kb_1, \\ C_2^{(1)\prime} = \frac{(k+2)s}{k} C_1^{(1)} C_2^{(1)} + ka_1 C_2^{(1)} + kc_1 C_2^{(2)}, \\ C_3^{(1)\prime} = \frac{(k+2)s}{k} C_1^{(1)} C_3^{(1)} + ka_1 C_3^{(1)} + kc_1 C_3^{(2)}, \\ C_1^{(2)\prime} = \frac{(l+1)s}{l} C_1^{(2)^2} + \frac{s}{l}\left(C_2^{(2)^2} + C_3^{(2)^2}\right) + la_2 C_1^{(2)} + lc_2 C_1^{(1)} + lb_2, \\ C_2^{(2)\prime} = \frac{(l+2)s}{l} C_1^{(2)} C_2^{(2)} + la_2 C_2^{(2)} + lc_2 C_2^{(1)}, \\ C_3^{(2)\prime} = \frac{(l+2)s}{l} C_1^{(2)} C_3^{(2)} + la_2 C_3^{(2)} + lc_2 C_3^{(1)}. \end{cases}$$

- *For* $s = 0$,

$$\begin{cases} u(x,t) = \left[C_1^{(1)}(t)x^2 + C_2^{(1)}(t)x + C_3^{(1)}(t)\right]^{\frac{1}{k}}, \\ v(x,t) = \left[C_1^{(2)}(t)x^2 + C_2^{(2)}(t)x + C_3^{(2)}(t)\right]^{\frac{1}{l}}, \end{cases}$$

where $C_1^{(1)}(t), C_2^{(1)}(t), C_3^{(1)}(t), C_1^{(2)}(t), C_2^{(2)}(t)$ *and* $C_3^{(2)}(t)$ *satisfy the six-dimensional dynamical system*

$$\begin{cases} C_1^{(1)\prime} = \frac{2(k+2)}{k} C_1^{(1)^2} + ka_1 C_1^{(1)} + kc_1 C_1^{(2)}, \\ C_2^{(1)\prime} = \frac{2(k+2)}{k} C_1^{(1)} C_2^{(1)} + ka_1 C_2^{(1)} + kc_1 C_2^{(2)}, \\ C_3^{(1)\prime} = \frac{1}{k} C_2^{(1)^2} + 2C_1^{(1)} C_3^{(1)} + ka_1 C_3^{(1)} + kc_1 C_3^{(2)} + kb_1, \\ C_1^{(2)\prime} = \frac{2(l+2)}{l} C_1^{(2)^2} + la_2 C_1^{(2)} + lc_2 C_1^{(1)}, \\ C_2^{(2)\prime} = \frac{2(l+2)}{l} C_1^{(2)} C_2^{(2)} + la_2 C_2^{(2)} + lc_2 C_2^{(1)}, \\ C_3^{(2)\prime} = \frac{1}{l} C_2^{(2)^2} + 2C_1^{(2)} C_3^{(2)} + la_2 C_3^{(2)} + lc_2 C_3^{(1)} + lb_2. \end{cases}$$

- *For* $s < 0$,

$$\begin{cases} u(x,t) = \left[C_1^{(1)}(t) + C_2^{(1)}(t)\sinh\left(\sqrt{-s}x\right) + C_3^{(1)}(t)\cosh\left(\sqrt{-s}x\right)\right]^{\frac{1}{k}}, \\ v(x,t) = \left[C_1^{(2)}(t) + C_2^{(2)}(t)\sinh\left(\sqrt{-s}x\right) + C_3^{(2)}(t)\cosh\left(\sqrt{-s}x\right)\right]^{\frac{1}{l}}, \end{cases}$$

where $C_1^{(1)}(t), C_2^{(1)}(t), C_3^{(1)}(t), C_1^{(2)}(t), C_2^{(2)}(t)$ *and* $C_3^{(2)}(t)$ *satisfy the six-dimensional dynamical system*

$$\begin{cases} C_1^{(1)\prime} = \frac{(k+1)s}{k} C_1^{(1)^2} + \frac{s}{k}\left(C_3^{(1)^2} - C_2^{(1)^2}\right) + ka_1 C_1^{(1)} + kc_1 C_1^{(2)} + kb_1, \\ C_2^{(1)\prime} = \frac{(k+2)s}{k} C_1^{(1)} C_2^{(1)} + ka_1 C_2^{(1)} + kc_1 C_2^{(2)}, \\ C_3^{(1)\prime} = \frac{(k+2)s}{k} C_1^{(1)} C_3^{(1)} + ka_1 C_3^{(1)} + kc_1 C_3^{(2)}, \\ C_1^{(2)\prime} = \frac{(l+1)s}{l} C_1^{(2)^2} + \frac{s}{l}\left(C_3^{(2)^2} - C_2^{(2)^2}\right) + la_2 C_1^{(2)} + lc_2 C_1^{(1)} + lb_2, \\ C_2^{(2)\prime} = \frac{(l+2)s}{l} C_1^{(2)} C_2^{(2)} + la_2 C_2^{(2)} + lc_2 C_2^{(1)}, \\ C_3^{(2)\prime} = \frac{(l+2)s}{l} C_1^{(2)} C_3^{(2)} + la_2 C_3^{(2)} + lc_2 C_3^{(1)}. \end{cases}$$

Example 5. *RD system*

$$\begin{cases} u_t = \left(u^{-\frac{3}{2}} u_x\right)_x + a_1 u + b_1 u^{\frac{5}{2}} + c_1 u^{\frac{5}{2}} v^{-\frac{3}{2}}, \\ v_t = \left(v^{-\frac{3}{2}} v_x\right)_x + a_2 v + b_2 v^{\frac{5}{2}} + c_2 u^{-\frac{3}{2}} v^{\frac{5}{2}} \end{cases}$$

admits CLBS

$$\begin{cases} \eta_1 = u_4 - \frac{10}{u}u_1u_3 - \frac{15}{2u}u_2^2 + \frac{105}{2u^2}u_1^2u_2 - \frac{315}{8u^3}u_1^4, \\ \eta_2 = v_4 - \frac{10}{v}v_1v_3 - \frac{15}{2v}v_2^2 + \frac{105}{2v^2}v_1^2v_2 - \frac{315}{8v^3}v_1^4. \end{cases}$$

The solutions of this system are given by

$$\begin{cases} u(x,t) = \left[C_1^{(1)}(t)x^3 + C_2^{(1)}(t)x^2 + C_3^{(1)}(t)x + C_4^{(1)}(t)\right]^{-\frac{2}{3}}, \\ v(x,t) = \left[C_1^{(2)}(t)x^3 + C_2^{(2)}(t)x^2 + C_3^{(2)}(t)x + C_4^{(2)}(t)\right]^{-\frac{2}{3}}, \end{cases}$$

where $C_1^{(1)}(t), C_2^{(1)}(t), C_3^{(1)}(t), C_4^{(1)}(t), C_1^{(2)}(t), C_2^{(2)}(t), C_3^{(2)}(t)$ *and* $C_4^{(2)}(t)$ *satisfy the eight-dimensional dynamical system*

$$\begin{cases} C_1^{(1)'} = -\frac{3}{2}a_1C_1^{(1)} - \frac{3}{2}c_1C_1^{(2)}, \\ C_2^{(1)'} = -\frac{2}{3}{C_2^{(1)}}^2 + 2C_1^{(1)}C_3^{(1)} - \frac{3}{2}a_1C_2^{(1)} - \frac{3}{2}c_1C_2^{(2)}, \\ C_3^{(1)'} = -\frac{2}{3}C_2^{(1)}C_3^{(1)} + 6C_1^{(1)}C_4^{(1)} - \frac{3}{2}a_1C_3^{(1)} - \frac{3}{2}c_1C_3^{(2)}, \\ C_4^{(1)'} = -\frac{2}{3}{C_3^{(1)}}^2 + 2C_2^{(1)}C_4^{(1)} - \frac{3}{2}a_1C_4^{(1)} - \frac{3}{2}c_1C_4^{(2)} - \frac{3}{2}b_1, \\ C_1^{(2)'} = -\frac{3}{2}a_2C_1^{(2)} - \frac{3}{2}c_2C_1^{(1)}, \\ C_2^{(2)'} = -\frac{2}{3}{C_2^{(2)}}^2 + 2C_1^{(2)}C_3^{(2)} - \frac{3}{2}a_2C_2^{(2)} - \frac{3}{2}c_2C_2^{(1)}, \\ C_3^{(2)'} = -\frac{2}{3}C_2^{(2)}C_3^{(2)} + 6C_1^{(2)}C_4^{(2)} - \frac{3}{2}a_2C_3^{(2)} - \frac{3}{2}c_2C_3^{(1)}, \\ C_4^{(2)'} = -\frac{2}{3}{C_3^{(2)}}^2 + 2C_2^{(2)}C_4^{(2)} - \frac{3}{2}a_2C_4^{(2)} - \frac{3}{2}c_2C_4^{(1)} - \frac{3}{2}b_2. \end{cases}$$

Example 6. *RD system*

$$\begin{cases} u_t = \left(u^{-\frac{4}{3}}u_x\right)_x + a_1u + b_1u^{\frac{7}{3}} - \frac{3s}{20}u^{-\frac{1}{3}} + c_1u^{\frac{7}{3}}v^{-\frac{4}{3}}, \\ v_t = \left(v^{-\frac{4}{3}}v_x\right)_x + a_2v + b_2v^{\frac{7}{3}} - \frac{3s}{20}v^{-\frac{1}{3}} + c_2v^{\frac{7}{3}}u^{-\frac{4}{3}} \end{cases}$$

admits CLBS

$$\begin{cases} \eta_1 = u_5 - \frac{35}{3u}u_1u_4 + \left(-\frac{70}{3u}u_2 + \frac{700}{9u^2}u_1^2 + s\right)u_3 + \frac{350}{3u^2}u_1u_2^2 \\ \qquad - \left(\frac{7s}{u}u_1 + \frac{9100}{27u^3}u_1^3\right)u_2 + \frac{14560}{81u^4}u_1^5 + \frac{70s}{9u^2}u_1^3 + \frac{4s^2}{25}u_1, \\ \eta_2 = v_5 - \frac{35}{3v}v_1v_4 + \left(-\frac{70}{3v}v_2 + \frac{700}{9v^2}v_1^2 + s\right)v_3 + \frac{350}{3v^2}v_1v_2^2 \\ \qquad - \left(\frac{7s}{v}v_1 + \frac{9100}{27v^3}v_1^3\right)v_2 + \frac{14560}{81v^4}v_1^5 + \frac{70s}{9v^2}v_1^3 + \frac{4s^2}{25}v_1. \end{cases}$$

The solutions of this system are given as below.

- *For* $s > 0$,

$$\begin{cases} u(x,t) = \Big[C_1^{(1)}(t) + C_2^{(1)}(t)\sin\left(\frac{\sqrt{5s}}{5}x\right) + C_3^{(1)}(t)\cos\left(\frac{\sqrt{5s}}{5}x\right) \\ \qquad + C_4^{(1)}(t)\sin\left(\frac{2\sqrt{5s}}{5}x\right) + C_5^{(1)}(t)\cos\left(\frac{2\sqrt{5s}}{5}x\right)\Big]^{-\frac{3}{4}}, \\ v(x,t) = \Big[C_1^{(2)}(t) + C_2^{(2)}(t)\sin\left(\frac{\sqrt{5s}}{5}x\right) + C_3^{(2)}(t)\cos\left(\frac{\sqrt{5s}}{5}x\right) \\ \qquad + C_4^{(2)}(t)\sin\left(\frac{2\sqrt{5s}}{5}x\right) + C_5^{(2)}(t)\cos\left(\frac{2\sqrt{5s}}{5}x\right)\Big]^{-\frac{3}{4}}, \end{cases}$$

where $C_1^{(1)}(t), C_2^{(1)}(t), C_3^{(1)}(t), C_4^{(1)}(t), C_5^{(1)}(t), C_1^{(2)}(t), C_2^{(2)}(t), C_3^{(2)}(t), C_4^{(2)}(t)$ *and* $C_5^{(2)}(t)$ *satisfy the ten-dimensional dynamical system*

$$\begin{cases} C_1^{(1)'} = \frac{s}{5}C_1^{(1)^2} - \frac{3s}{40}C_2^{(1)^2} - \frac{3s}{40}C_3^{(1)^2} - \frac{3s}{5}C_4^{(1)^2} - \frac{3s}{5}C_5^{(1)^2} - \frac{4}{3}a_1C_1^{(1)} - \frac{4}{3}c_1C_1^{(2)} - \frac{4}{3}b_1, \\ C_2^{(1)'} = \frac{3s}{5}C_2^{(1)}C_5^{(1)} - \frac{3s}{5}C_3^{(1)}C_4^{(1)} + \frac{s}{5}C_1^{(1)}C_2^{(1)} - \frac{4}{3}a_1C_2^{(1)} - \frac{4}{3}c_1C_2^{(2)}, \\ C_3^{(1)'} = -\frac{3s}{5}C_2^{(1)}C_4^{(1)} - \frac{3s}{5}C_3^{(1)}C_5^{(1)} + \frac{s}{5}C_1^{(1)}C_3^{(1)} - \frac{4}{3}a_1C_3^{(1)} - \frac{4}{3}c_1C_3^{(2)}, \\ C_4^{(1)'} = \frac{3s}{20}C_2^{(1)}C_3^{(1)} - \frac{2s}{5}C_1^{(1)}C_4^{(1)} - \frac{4}{3}a_1C_4^{(1)} - \frac{4}{3}c_1C_4^{(2)}, \\ C_5^{(1)'} = -\frac{3s}{40}C_2^{(1)^2} + \frac{3s}{40}C_3^{(1)^2} - \frac{2s}{5}C_1^{(1)}C_5^{(1)} - \frac{4}{3}a_1C_5^{(1)} - \frac{4}{3}c_1C_5^{(2)}, \\ C_1^{(2)'} = \frac{s}{5}C_1^{(2)^2} - \frac{3s}{40}C_2^{(2)^2} - \frac{3s}{40}C_3^{(2)^2} - \frac{3s}{5}C_4^{(2)^2} - \frac{3s}{5}C_5^{(2)^2} - \frac{4}{3}a_2C_1^{(2)} - \frac{4}{3}c_2C_1^{(1)} - \frac{4}{3}b_2, \\ C_2^{(2)'} = \frac{3s}{5}C_2^{(2)}C_5^{(2)} - \frac{3s}{5}C_3^{(2)}C_4^{(2)} + \frac{s}{5}C_1^{(2)}C_2^{(2)} - \frac{4}{3}a_2C_2^{(2)} - \frac{4}{3}c_2C_2^{(1)}, \\ C_3^{(2)'} = -\frac{3s}{5}C_2^{(2)}C_4^{(2)} - \frac{3s}{5}C_3^{(2)}C_5^{(2)} + \frac{s}{5}C_1^{(2)}C_3^{(2)} - \frac{4}{3}a_2C_3^{(2)} - \frac{4}{3}c_2C_3^{(1)}, \\ C_4^{(2)'} = \frac{3s}{20}C_2^{(2)}C_3^{(2)} - \frac{2s}{5}C_1^{(2)}C_4^{(2)} - \frac{4}{3}a_2C_4^{(2)} - \frac{4}{3}c_2C_4^{(1)}, \\ C_5^{(2)'} = -\frac{3s}{40}C_2^{(2)^2} + \frac{3s}{40}C_3^{(2)^2} - \frac{2s}{5}C_1^{(2)}C_5^{(2)} - \frac{4}{3}a_2C_5^{(2)} - \frac{4}{3}c_2C_5^{(1)}. \end{cases}$$

- *For* $s = 0$,

$$\begin{cases} u(x,t) = \left[C_1^{(1)}(t)x^4 + C_2^{(1)}(t)x^3 + C_3^{(1)}(t)x^2 + C_4^{(1)}(t)x + C_5^{(1)}(t)\right]^{-\frac{3}{4}}, \\ v(x,t) = \left[C_1^{(2)}(t)x^4 + C_2^{(2)}(t)x^3 + C_3^{(2)}(t)x^2 + C_4^{(2)}(t)x + C_5^{(2)}(t)\right]^{-\frac{3}{4}}, \end{cases}$$

where $C_1^{(1)}(t), C_2^{(1)}(t), C_3^{(1)}(t), C_4^{(1)}(t), C_5^{(1)}(t), C_1^{(2)}(t), C_2^{(2)}(t), C_3^{(2)}(t), C_4^{(2)}(t)$ *and* $C_5^{(2)}(t)$ *satisfy the ten-dimensional dynamical system*

$$\begin{cases} C_1^{(1)'} = 2C_1^{(1)}C_3^{(1)} - \frac{3}{4}C_2^{(1)^2} - \frac{4}{3}a_1C_1^{(1)} - \frac{4}{3}c_1C_1^{(2)}, \\ C_2^{(1)'} = -C_2^{(1)}C_3^{(1)} + 6C_1^{(1)}C_4^{(1)} - \frac{4}{3}a_1C_2^{(1)} - \frac{4}{3}c_1C_2^{(2)}, \\ C_3^{(1)'} = \frac{3}{2}C_2^{(1)}C_4^{(1)} + 12C_1^{(1)}C_5^{(1)} - C_3^{(1)^2} - \frac{4}{3}a_1C_3^{(1)} - \frac{4}{3}c_1C_3^{(2)}, \\ C_4^{(1)'} = 6C_2^{(1)}C_5^{(1)} - C_3^{(1)}C_4^{(1)} - \frac{4}{3}a_1C_4^{(1)} - \frac{4}{3}c_1C_4^{(2)}, \\ C_5^{(1)'} = 2C_3^{(1)}C_5^{(1)} - \frac{3}{4}C_4^{(1)^2} - \frac{4}{3}a_1C_5^{(1)} - \frac{4}{3}c_1C_5^{(2)} - \frac{4}{3}b_1, \\ C_1^{(2)'} = 2C_1^{(2)}C_3^{(2)} - \frac{3}{4}C_2^{(2)^2} - \frac{4}{3}a_2C_1^{(2)} - \frac{4}{3}c_2C_1^{(1)}, \\ C_2^{(2)'} = -C_2^{(2)}C_3^{(2)} + 6C_1^{(2)}C_4^{(2)} - \frac{4}{3}a_2C_2^{(2)} - \frac{4}{3}c_2C_2^{(1)}, \\ C_3^{(2)'} = \frac{3}{2}C_2^{(2)}C_4^{(2)} + 12C_1^{(2)}C_5^{(2)} - C_3^{(2)^2} - \frac{4}{3}a_2C_3^{(2)} - \frac{4}{3}c_2C_3^{(1)}, \\ C_4^{(2)'} = 6C_2^{(2)}C_5^{(2)} - C_3^{(2)}C_4^{(2)} - \frac{4}{3}a_2C_4^{(2)} - \frac{4}{3}c_2C_4^{(1)}, \\ C_5^{(2)'} = 2C_3^{(2)}C_5^{(2)} - \frac{3}{4}C_4^{(2)^2} - \frac{4}{3}a_2C_5^{(2)} - \frac{4}{3}c_2C_5^{(1)} - \frac{4}{3}b_2, \end{cases}$$

- *For* $s < 0$,

$$\begin{cases} u(x,t) = \Big[C_1^{(1)}(t) + C_2^{(1)}(t)\sinh\left(\frac{\sqrt{-5s}}{5}x\right) + C_3^{(1)}(t)\cosh\left(\frac{\sqrt{-5s}}{5}x\right) \\ \quad +C_4^{(1)}(t)\sinh\left(\frac{2\sqrt{-5s}}{5}x\right) + C_5^{(1)}(t)\cosh\left(\frac{2\sqrt{-5s}}{5}x\right)\Big]^{-\frac{3}{4}}, \\ v(x,t) = \Big[C_1^{(2)}(t) + C_2^{(2)}(t)\sinh\left(\frac{\sqrt{-5s}}{5}x\right) + C_3^{(2)}(t)\cosh\left(\frac{\sqrt{-5s}}{5}x\right) \\ \quad +C_4^{(2)}(t)\sinh\left(\frac{2\sqrt{-5s}}{5}x\right) + C_5^{(2)}(t)\cosh\left(\frac{2\sqrt{-5s}}{5}x\right)\Big]^{-\frac{3}{4}}, \end{cases}$$

where $C_1^{(1)}(t), C_2^{(1)}(t), C_3^{(1)}(t), C_4^{(1)}(t), C_5^{(1)}(t), C_1^{(2)}(t), C_2^{(2)}(t), C_3^{(2)}(t), C_4^{(2)}(t)$ *and* $C_5^{(2)}(t)$ *satisfy the ten-dimensional dynamical system*

$$
\begin{cases}
C_1^{(1)\prime} = \frac{s}{5}C_1^{(1)2} + \frac{3s}{40}C_2^{(1)2} - \frac{3s}{40}C_3^{(1)2} + \frac{3s}{5}C_4^{(1)2} - \frac{3s}{5}C_5^{(1)2} - \frac{4}{3}a_1C_1^{(1)} - \frac{4}{3}c_1C_1^{(2)} - \frac{4}{3}b_1, \\
C_2^{(1)\prime} = \frac{3s}{5}C_2^{(1)}C_5^{(1)} - \frac{3s}{5}C_3^{(1)}C_4^{(1)} + \frac{s}{5}C_1^{(1)}C_2^{(1)} - \frac{4}{3}a_1C_2^{(1)} - \frac{4}{3}c_1C_2^{(2)}, \\
C_3^{(1)\prime} = \frac{3s}{5}C_2^{(1)}C_4^{(1)} - \frac{3s}{5}C_3^{(1)}C_5^{(1)} + \frac{s}{5}C_1^{(1)}C_3^{(1)} - \frac{4}{3}a_1C_3^{(1)} - \frac{4}{3}c_1C_3^{(2)}, \\
C_4^{(1)\prime} = \frac{3s}{20}C_2^{(1)}C_3^{(1)} - \frac{2s}{5}C_1^{(1)}C_4^{(1)} - \frac{4}{3}a_1C_4^{(1)} - \frac{4}{3}c_1C_4^{(2)}, \\
C_5^{(1)\prime} = \frac{3s}{40}C_2^{(1)2} + \frac{3s}{40}C_3^{(1)2} - \frac{2s}{5}C_1^{(1)}C_5^{(1)} - \frac{4}{3}a_1C_5^{(1)} - \frac{4}{3}c_1C_5^{(2)}, \\
C_1^{(2)\prime} = \frac{s}{5}C_1^{(2)2} + \frac{3s}{40}C_2^{(2)2} - \frac{3s}{40}C_3^{(2)2} + \frac{3s}{5}C_4^{(2)2} - \frac{3s}{5}C_5^{(2)2} - \frac{4}{3}a_2C_1^{(2)} - \frac{4}{3}c_2C_1^{(1)} - \frac{4}{3}b_2, \\
C_2^{(2)\prime} = \frac{3s}{5}C_2^{(2)}C_5^{(2)} - \frac{3s}{5}C_3^{(2)}C_4^{(2)} + \frac{s}{5}C_1^{(2)}C_2^{(2)} - \frac{4}{3}a_2C_2^{(2)} - \frac{4}{3}c_2C_2^{(1)}, \\
C_3^{(2)\prime} = \frac{3s}{5}C_2^{(2)}C_4^{(2)} - \frac{3s}{5}C_3^{(2)}C_5^{(2)} + \frac{s}{5}C_1^{(2)}C_3^{(2)} - \frac{4}{3}a_2C_3^{(2)} - \frac{4}{3}c_2C_3^{(1)}, \\
C_4^{(2)\prime} = \frac{3s}{20}C_2^{(2)}C_3^{(2)} - \frac{2s}{5}C_1^{(2)}C_4^{(2)} - \frac{4}{3}a_2C_4^{(2)} - \frac{4}{3}c_2C_4^{(1)}, \\
C_5^{(2)\prime} = \frac{3s}{40}C_2^{(2)2} + \frac{3s}{40}C_3^{(2)2} - \frac{2s}{5}C_1^{(2)}C_5^{(2)} - \frac{4}{3}a_2C_5^{(2)} - \frac{4}{3}c_2C_5^{(1)}.
\end{cases}
$$

6. Conclusions

The method of linear determining equations to construct DC (3) and CLBS (5) of two-component second-order evolution system (4) is provided. The linear determining equations (23) and (24) generalize the classical determining equations within the framework of Lie's operator. The general form of CLBS (5) and DC (3) admitted by the system (4) can be identified by solving the resulting linear determining equations.

As an application of this approach, the general form of DC (3) and CLBS (5) with $n = 3, 4, 5$ of RD system (9) is established in this paper. The reductions of the resulting equations are also constructed due to the compatibility of the admitted DC (3) and the governing system (9). These reductions cannot be obtained within the framework of Lie's classical symmetry method and conditional symmetry method.

All examples except Example 4 in Section 5 involve the power diffusivities with the exponent either $-4/3$ or $-2/3$. Exact solutions of the nonlinear diffusion equations $u_t = (u^{-4/3}u_x)_x$ and $u_t = (u^{-2/3}u_x)_x$ are firstly studied by using local and non-local symmetries by King [48]. The polynomial solutions like the ones in the examples of Section 5 for scalar nonlinear diffusion equations are also constructed by King [49,50]. Moreover, a range of more complicated exact solutions for scalar nonlinear diffusion equations are derived by Cherniha [15] due to the method of the additional generating condition. In addition, the results of Examples 4, 5 and the case of $s = 0$ for Example 6 in Section 5 have been given by Cherniha and King [16] by using the method of the additional generating condition. All of the reductions of the obtained RD system (9) constructed in Section 5, involving either a polynomial, trigonometric or hyperbolic function, are used in [14] for the first time within the framework of the method of the additional generating condition.

The method of linear determining equations can be extended to consider DCs and CLBSs of other types of evolution systems, including a multi-component diffusion system and a high-order evolution system. The discussion about the linear determining equation for evolution system (4) to identify CLBS and DC with η_1 and η_2 possessing different orders is another interesting problem. All of these problems will be involved in our future research.

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