symmetry

MDPI

Article

β-Differential of a Graph

Ludwin A. Basilio [1], Sergio Bermudo [2], Jesús Leaños [1] and José M. Sigarreta [3,*]

[1] Academic Unit of Mathematics, Autonomous University of Zacatecas, Paseo la Bufa, int. Calzada Solidaridad, 98060 Zacatecas, Mexico; ludwin.ali@gmail.com (L.A.B.); jleanos@mate.reduaz.mx (J.L.)
[2] Department of Economics, Quantitative Methods and Economic History, Pablo de Olavide University, Carretera de Utrera Km. 1, 41013 Sevilla, Spain; sbernav@upo.es
[3] Faculty of Mathematics, Autonomous University of Guerrero, Carlos E. Adame 5, Col. La Garita, 39350 Acapulco, Guerrero, Mexico
* Correspondence: josemariasigarretaalmira@yahoo.es; Tel.: +52-744-481-0216

Received: 12 September 2017; Accepted: 26 September 2017; Published: 30 September 2017

Abstract: Let $G = (V, E)$ be a simple graph with vertex set V and edge set E. Let D be a subset of V, and let $B(D)$ be the set of neighbours of D in $V \setminus D$. The *differential* $\partial(D)$ of D is defined as $|B(D)| - |D|$. The maximum value of $\partial(D)$ taken over all subsets $D \subseteq V$ is the *differential* $\partial(G)$ of G. For $\beta \in (-1, \Delta)$, the *β-differential* $\partial_\beta(G)$ of G is the maximum value of $\{|B(D)| - \beta|D| : D \subseteq V\}$. Motivated by an influential maximization problem, in this paper we study the β-differential of G.

Keywords: differential of a graph; domination number

1. Introduction

Social networks, such as Facebook or Twitter, have served as an important medium for communication and information disseminating. As a result of their massive popularity, social networks now have a wide variety of applications in the viral marketing of products and political campaigns. Motivated by its numerous applications, some authors [1–3] have proposed several influential maximization problems, which share a fundamental algorithmic problem for information diffusion in social networks: the problem of determining the best group of nodes to influence the rest. As it was shown in [4], the study of the differential of a graph G, could be motivated from such scenarios. In this work we generalize the notion of differential of a graph and provide new applications. Let us first give some basic notation and then we motivate such a generalization.

Throughout this paper, $G = (V, E)$ is a simple graph of order $n \geq 3$ with vertex set V and edge set E. Let u, v be distinct vertices of V, and let S be a subset of V. We will write $u \sim v$ whenever u and v are adjacent in G. If S is nonempty, then $N_S(v)$ denotes the set of neighbors that v has in S, i.e., $N_S(v) := \{u \in S : u \sim v\}$; the degree of v in S is denoted by $\delta_S(v) := |N_S(v)|$. As usual, $N(v)$ is the set of neighbors that v has in V, i.e., $N(v) := \{u \in V : u \sim v\}$; and $N[v]$ is the closed neighborhood of v, i.e., $N[v] := N(v) \cup \{v\}$. We denote by $\delta(v) := |N(v)|$ the degree of v in G, and by $\delta(G)$ and $\Delta(G)$ the minimum and the maximum degree of G, respectively. The subgraph of G induced by S will be denoted by $G[S]$, and the complement of S in V by $\overline{S}$. Then $N_{\overline{S}}(v)$ is the set of neighbors that v has in $\overline{S} = V \setminus S$. We let $N(S) := \bigcup_{v \in S} N(v)$ and $N[S] := N(S) \cup S$. Finally, we will use $B(S)$ to denote the set of vertices in $\overline{S}$ that have a neighbour in S, and $C(S)$ to denote $\overline{S \cup B(S)}$. Then $\{S, B(S), C(S)\}$ is a partition of V. An *external private neighbor* of $v \in S$ (with respect to S) is a vertex $w \in N(v) \cap \overline{S}$ such that $w \notin N(u)$ for every $u \in S \setminus \{v\}$. The set of all external private neighbors of v is denoted by $\operatorname{epn}[v, S]$.

To motivate the notion of β-differential of a graph, assume for a moment that our graph $G = (V, E)$ represents a map of a country, where V is the set of cities of G and E is the set of roads between cities of G. To avoid weights, we could assume that all the cities of G have the same population and have the same importance, and also that all roads have the same length. A supermarket chain wants to build some supermarkets in that country and they are studying which are the best places to do it. For that, they might consider that every supermarket will give service to the own city and the neighboring cities. Moreover, according to some previous studies, the cost of building a new supermarket is $\alpha > 0$ times the benefit that can be obtained by each city in an specific number of years. In consequence, if we consider the unit as the amount of money that we can obtain from a city in that amount of years and we build a supermarket in each vertex of a set $D \subseteq V$, we have that the benefit that we obtain is $|B(D)| + |D| - \alpha|D| = |B(D)| - (\alpha - 1)|D|$, or equivalently, $|B(D)| - \beta|D|$ for $\beta = \alpha - 1$. Such a value is denoted by $\partial_\beta(D)$ and it is called the *β-differential* of D. We are naturally interested in determining the following value:

$$\partial_\beta(G) := \max\{\partial_\beta(D) : D \subseteq V\} = \max\{|B(D)| - \beta|D| : D \subseteq V\}.$$

The number $\partial_\beta(G)$ is the *β-differential* of G. Let Δ be the maximum degree of G. Note that if $v \in V$ has degree Δ, then $\partial_\beta(G) \geq \partial_\beta(\{v\}) = \Delta - \beta$. Thus, if $\beta < \Delta$, there will always be a choice of places giving benefits. On the other hand, if $\beta \geq \Delta$ and $D \subseteq V$, then $\partial_\beta(D) = |B(D)| - \beta|D| \leq \Delta|D| - \beta|D| = (\Delta - \beta)|D| \leq 0$, and hence no set of locations will make benefits. For these reasons, we restrict our study of $\partial_\beta(G)$ to the values of β belonging to $(-1, \Delta)$.

The particular case in which $\beta = 1$ is called the differential of G, and it is usually denoted by $\partial(G)$. The study of $\partial(G)$ together with a variety of other kinds of differentials of a set, started in [5]. In particular, several bounds for $\partial(G)$ were given. The differential of a graph has also been investigated in [4,6–15], and it was proved in [11] that $\partial(G) + \gamma_R(G) = n$, where n is the order of the graph G and $\gamma_R(G)$ is the Roman domination number of G, so every bound for the differential of a graph can be used to get a bound for the Roman domination number. The differential of a set D was also considered in [16], where it was denoted by $\eta(D)$, and the minimum differential of an independent set was considered in [17]. The case of the B-differential of a graph or enclaveless number, defined as $\psi(G) := \max\{|B(D)| : D \subseteq V\}$, was studied in [5,18].

Notice that if G is disconnected, and $G_1, \ldots, G_k$ are its connected components, then $\partial_\beta(G) = \partial_\beta(G_1) + \cdots + \partial_\beta(G_k)$. In view of this, from now on we only consider connected graphs.

Other graph parameters that we will use in this paper are the dominating number and the packing number of G. We recall that a set $S \subseteq V$ is a *dominating set* if every vertex $v \in \overline{S}$ is adjacent to a vertex in S. The *domination number* $\gamma(G)$ is the minimum cardinality among all dominating sets. A *packing* of a graph G is a set of vertices in G that are pairwise at distance more than two. The *packing number* $\rho(G)$ of G is the size of a largest packing in G. An *open packing* of G is a set $S \subseteq V$ such that $N(u) \cap N(v) = \emptyset$ for every two different vertices $u, v \in S$. The *open packing number* $\rho^o(G)$ of G is the size of a largest open packing in G.

2. The Function $f_G(\beta) = \partial_\beta(G)$

Throughout this section Δ denotes the maximum degree of $G = (V, E)$ and $\beta \in (-1, \Delta)$. Clearly, the values of $\partial_\beta(G)$ can be considered as a function $f_G : (-1, \Delta) \to \mathbb{R}$, which is defined as $f_G(\beta) := \partial_\beta(G)$. A subset $D \subseteq V$ satisfying $\partial_\beta(D) = \partial_\beta(G)$ is called a *β-∂-set* or a *β-differential set*. If D has minimum (maximum) cardinality among all β-differential sets, then D is a *minimum* (*maximum*) *β-differential set*. We will write D^m_β (respectively, D^M_β) to indicate that D is a minimum (respectively, maximum) β-differential set. Since the value of $\partial_\beta(G)$ can be achieved by several subsets of V, a natural problem is to determine the properties of such β-differential sets. Our goal in this section is to establish several properties of these sets. In particular, as a consequence of some of them we will show in Theorem 1 that f_G is a continuous function. We have seen before that, if $v \in V$ has maximum degree, then $\partial_\beta(\{v\}) > 0$ for any admissible β. In our next results we continue our study in this direction and we show that the positive value of the β-differential of a subset D of vertices of G will depend on the values of $|D|$ and β.

Proposition 1. *Let $G = (V, E)$ be a graph. If $\beta \in (-1, 1]$ and $k \in \mathbb{N}$ such that $k < \dfrac{n}{\beta+1}$, then there exists a subset $D \subseteq V$ such that $|D| = k$ and $\partial_\beta(D) > 0$.*

Proof 1. If G contains a dominating set $D \subseteq V$ of cardinality k, then $\partial_\beta(D) = |B(D)| - \beta|D| = n - k(\beta+1) > n - \dfrac{n}{\beta+1}(\beta+1) = 0$, as desired. Now we suppose $\gamma(G) > k$, and consider a maximum matching of G, say $M = \{u_1v_1, \ldots, u_mv_m\}$. It is known that $m = |M| \geq \gamma(G) > k$. Let $D = \{u_1, \ldots, u_k\}$. If u_m or v_m is adjacent to a vertex in D, then $\partial_\beta(D) = |B(D)| - \beta|D| \geq k + 1 - k = 1 > 0$. Thus we can assume that neither u_m nor v_m is adjacent to any vertex in D. Since G is connected, then at least one of u_m or v_m is adjacent to a vertex of $V \setminus (D \cup \{u_m, v_m\})$. Without loss of generality, we suppose that u_m is adjacent to a vertex $x \in V \setminus (D \cup \{u_m, v_m\})$. If $x \notin \{v_1, \ldots, v_k\}$, then $D' = \{u_1, \ldots, u_{k-1}, u_m\}$ satisfies $\partial_\beta(D') = |B(D')| - \beta|D'| \geq k + 1 - k = 1 > 0$. If $x = v_j$ for some $j \in \{1, \ldots, k\}$, then $D' = \{u_1, \ldots, u_{j-1}, v_j, u_{j+1}, \ldots, u_k\}$ satisfies $\partial_\beta(D') = |B(D')| - \beta|D'| \geq k + 1 - k = 1 > 0$. □

Taking into account that $\frac{n}{\beta+1} \leq \frac{n-\gamma(G)}{\beta}$ when $\beta \in (-1, 1]$, the next proposition shows that the upper bound on the size of D in Proposition 1 cannot be relaxed.

Proposition 2. *Let $G = (V, E)$ be a graph. Every set $D \subseteq V$ such that $|D| \geq \min\left\{\frac{n}{\beta+1}, \frac{n-\gamma(G)}{\beta}\right\}$ satisfies $\partial_\beta(D) \leq 0$.*

Proof 2. Firstly, if $D \subseteq V$ is a set such that $|D| \geq n/(\beta+1)$, then $\beta|D| \geq n - |D| \geq |B(D)|$, consequently, $\partial_\beta(D) \leq 0$. Secondly, if $|D| \geq (n - \gamma(G))/\beta$, then $\beta|D| \geq n - \gamma(G) \geq |B(D)|$, which again implies $\partial_\beta(D) \leq 0$. □

Lemma 1. *If $G = (V, E)$ is a graph and $\beta_1 < \beta_2$, then $\partial_{\beta_1}(G) > \partial_{\beta_2}(G)$.*

Proof 3. If $\beta_1 < \beta_2$, then $|B(D)| - \beta_1|D| > |B(D)| - \beta_2|D|$ for every $D \subseteq V$. Since the number of subsets of V is finite, we conclude that $\partial_{\beta_1}(G) > \partial_{\beta_2}(G)$. □

Proposition 3. *Let $G = (V, E)$ be a graph. If $\beta \notin \mathbb{N}$, then every minimum β-differential set is a maximal β-differential set and every maximum β-differential set is a minimal β-differential set.*

Proof 4. Let D^m_β be a minimum β-differential set and let D be a β-differential set such that $D^m_\beta \subset D$. In such a case, $D \setminus D^m_\beta = \{u_1, ..., u_r\} \subseteq B(D^m_\beta) \cup C(D^m_\beta)$ and $\{u_i; z_{i_1}, ..., z_{i_{k_i}}\}$ are disjoint stars (not necessarily induced stars) with centers u_i for every $i \in \{1, \ldots, r\}$, such that $z_s \in C(D^m_\beta)$ for every $s \in \{i_j : i \in \{1, \ldots, r\}, j \in \{1, \ldots, k_i\}\}$ and $\sum_{u_i \in B(D^m_\beta)}(-1 + k_i - \beta) + \sum_{u_i \in C(D^m_\beta)}(k_i - \beta) = 0$. Since $\beta \notin \mathbb{N}$, there exists $i \in \{1, \ldots, r\}$ such that $u_i \in B(D^m_\beta)$ and $-1 + k_i - \beta > 0$, or $u_i \in C(D^m_\beta)$ and $k_i - \beta > 0$, thus $\partial_\beta(D^m_\beta \cup \{u_i\}) > \partial_\beta(D^m_\beta)$, a contradiction. Analogously, we can prove that every maximum β-differential set is minimal. □

Now we establish a couple of relationships between dominating sets and β-differential sets of G.

Lemma 2. *Let $G = (V, E)$ be a graph and let A be a dominating set in G. If $D \subseteq V$ with $|D| > |A|$, then $\partial_\beta(D) < \partial_\beta(A)$. In particular,*

$$\partial_\beta(G) = \max\{\partial_\beta(D) : D \subseteq V, |D| \leq \gamma(G)\}.$$

Proof 5. Let $D \subseteq V$ with $|D| > |A|$. Then

$$\partial_\beta(A) = |B(A)| - \beta|A| = |V| - |A| - \beta|A| > |V| - |D| - \beta|D| \geq |B(D)| - \beta|D| = \partial_\beta(D).$$

□

Proposition 4. *Let $G = (V, E)$ be a graph of order n. If $\beta \in [-1, 0]$, then $\partial_\beta(G) = n - (1+\beta)\gamma(G)$. That is, every β-differential set is a minimum dominating set.*

Proof 6. Let $A, D \subseteq V$ such that A is a dominating set with $\gamma(G) = |A|$ and $\partial_\beta(G) = \partial_\beta(D)$. It is known that $|B(A)| = \max\{|B(S)| : S \subseteq V\}$. If $\beta = 0$ we have that $\partial_0(A) = |B(A)| \geq |B(D)| = \partial_0(D)$, and so $|B(A)| = |B(D)|$. Then $|B(D)| + |A| = n$, or equivalently, $\partial_0(D) = n - \gamma(G)$, as required. Now we suppose that $\beta < 0$. By Lemma 2 we know that $|D| \leq |A|$. If $|D| < |A|$, then

$$\partial_\beta(A) = |B(A)| - \beta|A| \geq |B(D)| - \beta|A| > |B(D)| - \beta|D| = \partial_\beta(D),$$

a contradiction. Finally, since $|D| = |A|$ and $\partial_\beta(D) = |B(D)| - \beta|D| = |B(D)| - \beta|A| \leq |B(A)| - \beta|A| = \partial_\beta(A)$, we have that $|B(D)| = |B(A)|$ and, consequently, that D is a minimum dominating set. □

In view of Proposition 4, unless otherwise stated, from now on we will only consider $\beta > 0$. Note that the trees shown in Figures 1–3, and the 1- and 2-differential sets marked (in black) suggest that if $\beta_1 < \beta_2$ then $|D^M_{\beta_2}| \leq |D^m_{\beta_1}|$. This question will be answered in Lemma 3.

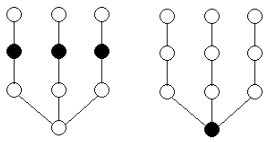

Figure 1. $|D^m_1| = 3$ and $|D^M_2| = 1$.

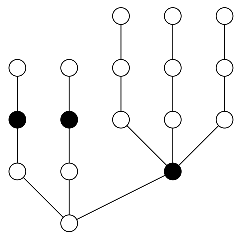

Figure 2. $|D^m_1| = |D^M_2| = 3$.

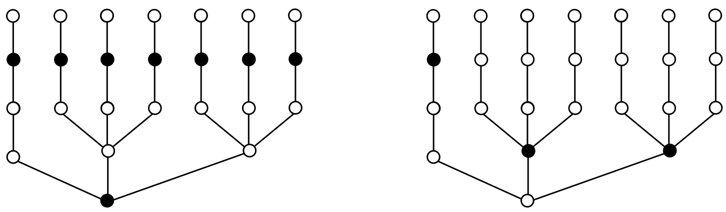

Figure 3. $|D^m_1| = 8$ and $|D^M_2| = 3$.

Lemma 3. *Let $G = (V, E)$ be a graph. If $\beta_1 < \beta_2$, and there is no $D \subseteq V$ such that $\partial_{\beta_1}(G) = \partial_{\beta_1}(D)$ and $\partial_{\beta_2}(G) = \partial_{\beta_2}(D)$, then for every $D_1, D_2 \subseteq V$ such that $\partial_{\beta_1}(G) = \partial_{\beta_1}(D_1)$ and $\partial_{\beta_2}(G) = \partial_{\beta_2}(D_2)$, we have $|D_2| \leq |D_1| - 1$.*

Proof 7. Let D_1 and D_2 be such that $\partial_{\beta_1}(G) = \partial_{\beta_1}(D_1)$ and $\partial_{\beta_2}(G) = \partial_{\beta_2}(D_2)$. By hypothesis $\partial_{\beta_2}(G) \neq \partial_{\beta_2}(D_1)$, so $|B(D_2)| - \beta_2|D_2| > |B(D_1)| - \beta_2|D_1|$. Hence

$$\begin{aligned} |B(D_2)| - \beta_1|D_2| &= |B(D_2)| - |B(D_1)| + |B(D_1)| - \beta_1|D_2| + \beta_1|D_1| - \beta_1|D_1| \\ &> \beta_2|D_2| - \beta_2|D_1| + \beta_1|D_1| - \beta_1|D_2| + \partial_{\beta_1}(G) \\ &= (\beta_2 - \beta_1)(|D_2| - |D_1|) + \partial_{\beta_1}(G). \end{aligned}$$

Therefore, if $|D_2| - |D_1| \geq 0$, then $|B(D_2)| - \beta_1|D_2| > \partial_{\beta_1}(G)$, a contradiction. □

Lemma 4. *Let $G = (V, E)$ be a graph. If $\beta_1 < \beta_2$, then for every β_1-differential set D_1 and every β_2-differential set D_2 it is satisfied $|D_2| \leq |D_1|$ and $|B(D_2)| \leq |B(D_1)|$.*

Proof 8. By absurdum we suppose that $|D_2| > |D_1|$. Since $\partial_{\beta_2}(D_1) \leq \partial_{\beta_2}(D_2)$, then

$$|B(D_1)| - \beta_2|D_1| \leq |B(D_2)| - \beta_2|D_2| = |B(D_2)| - \beta_1|D_2| - |D_2|(\beta_2 - \beta_1).$$

Therefore,

$$|B(D_1)| - \beta_2|D_1| + |D_2|(\beta_2 - \beta_1) = |B(D_1)| - \beta_1|D_1| + (|D_2| - |D_1|)(\beta_2 - \beta_1) \leq \partial_{\beta_1}(D_2).$$

Since $(|D_2| - |D_1|)(\beta_2 - \beta_1) > 0$, we have $\partial_{\beta_1}(D_1) = |B(D_1)| - \beta_1|D_1| < \partial_{\beta_1}(D_2)$, a contradiction.

Finally, since $|B(D_2)| - \beta_1|D_2| \leq \partial_{\beta_1}(G) = |B(D_1)| - \beta_1|D_1|$, we have

$$|B(D_2)| - |B(D_1)| \leq \beta_1(|D_2| - |D_1|) \leq 0.$$

□

Looking also Figures 1–3, it might be thought that every minimum β-differential set is included in a maximum β-differential set, but this is not true, as we can see in Figure 4, where black vertices sets are the minimum 1-differential set and $\frac{1}{2}$-differential set, respectively, and grey vertices set are the maximum 1-differential set and $\frac{1}{2}$-differential set, respectively.

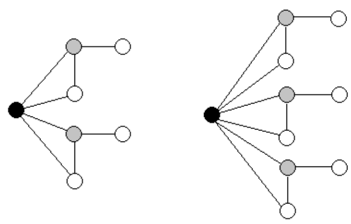

Figure 4. On the left $|D_1^m| = 1$ and $|D_1^M| = 2$, and on the right $|D_{\frac{1}{2}}^m| = 1$ and $|D_{\frac{1}{2}}^M| = 3$.

If β is an irrational number, then $|D_\beta^m| = |D_\beta^M|$, because $|B(D_\beta^M)| - \beta|D_\beta^M| = |B(D_\beta^m)| - \beta|D_\beta^m|$ implies that $\beta(|D_\beta^M| - |D_\beta^m|) = |B(D_\beta^M)| - |B(D_\beta^m)|$.

Proposition 5. *Let $G = (V, E)$ be a graph. If $\beta_1 < \beta_2$ and there exists $D \subseteq V$ such that $\partial_{\beta_1}(G) = \partial_{\beta_1}(D)$ and $\partial_{\beta_2}(G) = \partial_{\beta_2}(D)$, then for every $\beta \in (\beta_1, \beta_2)$ it holds $\partial_\beta(G) = \partial_\beta(D)$ and $|D| = |D_{\beta_2}^M| = |D_{\beta_1}^m|$.*

Proof 9. Let $\beta \in (\beta_1, \beta_2)$ and let $D' \subseteq V$ be a β-differential set, by Lemma 4 we have

$$|D| \leq |D'| \leq |D| \quad \text{and} \quad |B(D)| \leq |B(D')| \leq |B(D)|,$$

thus, $\partial_\beta(D) = |B(D)| - \beta|D| = |B(D')| - \beta|D'| = \partial_\beta(G)$. Finally, since D is a β_2-differential set, by Lemma 4, we have $|D| \le |D^m_{\beta_1}|$. Using now that D is also a β_1-differential set, we have $|D| \ge |D^m_{\beta_1}|$. The equality $|D| = |D^M_{\beta_2}|$ can be obtained analogously. □

Theorem 1. *Let $G = (V, E)$ be a graph, then the function $f_G(\beta) = \partial_\beta(G)$ is continuous for every $\beta \in (-1, \Delta)$.*

Proof 10. It follows from Lemma 3 and Proposition 5 that the graphic representation of the function $f_G(\beta)$ is formed by pieces of straight lines with negative slope. That is, there exists a partition $0 < \beta_1 < \beta_2 < \cdots < \beta_{r-1} < \beta_r = \Delta$ of $[0, \Delta]$ such that

$$f_G(\beta) = \begin{cases} n - (1+\beta)\gamma(G) & -1 < \beta \le 0 \\ a_1 - b_1\beta & 0 < \beta \le \beta_1 \\ a_2 - b_2\beta & \beta_1 < \beta \le \beta_2 \\ \vdots & \vdots \\ a_r - b_r\beta & \beta_{r-1} < \beta \le \beta_r \end{cases}$$

where $a_i, b_i \in \mathbb{N}$. Moreover, $b_i \le b_{i-1} - 1$ and $r \le \gamma(G)$. Observe that $f_G(\beta)$ is a continuous function because, if $a_i - b_i\beta_i > a_{i+1} - b_{i+1}\beta_i$, then there exist $\delta > 0$ and $\beta' \in (\beta_i, \beta_i + \delta)$ such that $a_i - b_i\beta' > a_{i+1} - b_{i+1}\beta'$, so $f_G(\beta')$, since it is a maximum, should be equal to $a_i - b_i\beta'$, a contradiction. □

For instance, in the graphs shown in Figure 5 we have $f(\beta) = 4 - 3\beta$ if $\beta \in \left(0, \frac{1}{2}\right]$ and $f(\beta) = 3 - \beta$ if $\beta \in \left(\frac{1}{2}, 3\right]$.

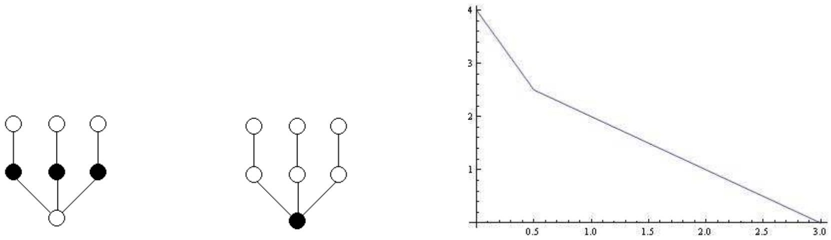

Figure 5. β-differential set when $\beta \in \left(0, \frac{1}{2}\right]$ (on the left) and β-differential set when $\beta \in \left(\frac{1}{2}, 3\right]$ (on the right).

Notice that from the point of view explained in the introduction, if the cost of building a supermarket is $\alpha = \frac{7}{5}$ (that is $\beta = \frac{2}{5}$), it is more profitable to build three supermarket giving service to all the towns. However, if the cost of building a supermarkets is $\alpha = \frac{8}{5}$ (that is $\beta = \frac{3}{5}$), it is more profitable to build only one supermarket, leaving without service to three towns.

Note that there exist another generalizations in graphs using continuous parameters, for instance, α-domination number in [19], where the resulting function is not continuous.

It might be also thought that the intervals where the function is a straight line are big, but there are graphs where these intervals are really small. For instance, if we consider the graph G_r with $3r + 1$ vertices shown in Figure 6, the β-differential set is an unitary set containing the black vertex when $\beta > \frac{1}{r-1}$, and the set containing the grey vertices when $\beta \le \frac{1}{r-1}$.

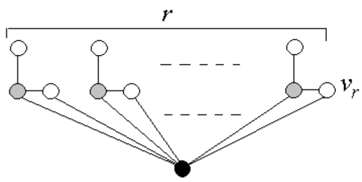

Figure 6. An example of a graph G_r such that $\partial_\beta(G_r) = 1 + 2r - \beta r$ if $\beta \leq \frac{1}{r-1}$, and $\partial_\beta(G_r) = 2r - \beta$ if $\beta > \frac{1}{r-1}$.

As $v_r \in B(D_\beta)$ for every β-differential set D_β in G_r, we can consider a graph G whose vertices are $V(G) = \bigcup_{i=0}^{j} V(G_{r+i})$ and edges $E = \bigcup_{i=0}^{j} E(G_{r+i}) \cup \{v_s v_{s+1} : s \in \{r, \ldots, r+j-1\}$. In such a case, the partition of the interval $(0, \Delta)$ for the definition of the piecewise function $f_G(\beta)$ is $0 < \frac{1}{r+j-1} < \frac{1}{r+j-2} < \cdots < \frac{1}{r-1} < \Delta$.

3. Bounds on the β-Differential of a Graph

As we have mentioned in the introduction, $\partial_\beta(G)$ will be the maximum benefit we could obtain if the cost of placing the considered service is $\alpha = \beta + 1$, so it will be interesting to get lower and upper bounds for this benefit.

Proposition 6. *Let $G = (V, E)$ be a graph with order n and maximum degree Δ. Then $\Delta - \beta \leq \partial_\beta(G) \leq n - (1 + \beta)$.*

Proof 11. Let $v \in V$ such that $\delta(v) = \Delta$. Then $\partial(\{v\}) = \Delta - \beta \leq \partial_\beta(G)$. Now, for any β-differential set D we have that

$$\partial_\beta(G) = |B(D)| - \beta|D| \leq n - 1 - \beta|D| \leq n - 1 - \beta.$$

□

Proposition 7. *Let $G = (V, E)$ be a graph with order n and maximum degree Δ. The following properties hold.*

(a) $\partial_\beta(G) = n - (1 + \beta)$ *if and only if* $\Delta = n - 1$.
(b) $\partial_\beta(G) = n - (2 + \beta)$ *if and only if* $\Delta = n - 2$.
(c) *If* $\beta > 1$, *then* $\partial_\beta(G) = n - (3 + \beta)$ *if and only if* $\Delta = n - 3$.

Proof 12. *(a)* If $\Delta = n - 1$, by Proposition 6 we have $n - 1 - \beta \leq \partial_\beta(G) \leq n - 1 - \beta$, then $\partial_\beta(G) = n - 1 - \beta$. If $\partial_\beta(G) = n - 1 - \beta$ and D is a β-differential set, then we have $n - 1 - \beta = |B(D)| - \beta|D| \leq n - 1 - \beta|D|$. Therefore, $|D| \leq 1$, that is, $|D| = 1$ and $|B(D)| = n - 1$, which means that $\Delta = n - 1$.

(b) If $\Delta = n - 2$, by Proposition 6 and (a) we have $n - 2 - \beta \leq \partial_\beta(G) < n - 1 - \beta$. If D is a β-differential set such that $|D| \geq 2$, then $n - 2 - \beta \leq |B(D)| - \beta|D| \leq |B(D)| - 2\beta$, consequently, $n - 2 - \beta \leq |B(D)|$. Since $\beta > 0$, we have $n - 1 \leq |B(D)|$, which is a contradiction. If D is a β-differential set such that $|D| = 1$, by $n - 2 - \beta \leq |B(D)| - \beta|D|$ and (a) we obtain $|B(D)| = n - 2$, so $\partial_\beta(G) = n - (2 + \beta)$. Now, if $\partial_\beta(G) = n - 2 - \beta$, there exists a β-differential set D such that $n - 2 - \beta = |B(D)| - \beta|D| \leq |B(D)| - \beta$, therefore $|B(D)| \geq n - 2$. By (a) we know that $|B(D)| \neq n - 1$, then $|B(D)| = n - 2$ and, using again that $|B(D)| - \beta|D| = n - 2 - \beta$, we conclude that $|D| = 1$, which means that $\Delta = n - 2$.

(c) If $\partial_\beta(G) = n - (3 + \beta)$, there exists a β-differential set D such that $n - (3 + \beta) = |B(D)| - \beta|D| \leq |B(D)| - \beta$, then $|B(D)| \geq n - 3$. By (a) we know that $|B(D)| \neq n - 1$. If $|B(D)| = n - 2$, then we have $|D| \leq 2$ and $|D| = 1 + \frac{1}{\beta}$, which is a contradiction with the fact that $\beta > 1$. Therefore, $|B(D)| = n - 3$ and, consequently, $|D| = 1$, which means that $\Delta = n - 3$. Finally, if $\Delta = n - 3$ and D is a β-differential set, by Proposition 6 we have $n - 3 - \beta \leq |B(D)| - \beta|D| \leq |B(D)| - \beta$,

then $|B(D)| = n-2$ or $|B(D)| = n-3$. If $|B(D)| = n-2$, since $\Delta = n-3$, we have $|D| = 2$ and $-1-\beta \le -\beta|D| = -2\beta$, a contradiction. If $|B(D)| = n-3$, since $n-3-\beta \le |B(D)| - \beta|D| = n-3-\beta|D|$, we have $|D| = 1$ and $\partial_\beta(G) = n-3-\beta$.
□

Let us note that, if we consider the path P_5 with five vertices and $\beta < 1$, then we have $\Delta = n-3$ but $\partial_\beta(P_5) = n-2-2\beta \neq n-(3+\beta)$.

To characterize the graphs G such that $\partial_\beta(G) = \Delta - \beta$ is much more difficult. A characterization of these graphs when $\beta = 1$ was given in [5], but only for trees. Next we will give some properties that the graphs verifying that equality must satisfy.

Proposition 8. *Let $G = (V, E)$ be a graph with maximum degree Δ. If $\partial_\beta(G) = \Delta - \beta$ and $v \in V$ is such that $\delta(v) = \Delta$, then:*

(a) $\Delta(G[V \setminus N[v]]) \le \beta$.
(b) $\delta_{\overline{N[v]}}(u) \le \beta + 1$ *for every* $u \in N(v)$.
(c) $|N_{\overline{N[v]}}(A)| + |N_{N(v)\setminus A}(A)| \le \Delta - 1 + \beta(|A| - 1)$ *for every* $A \subseteq N(v)$.

Proof 13. We suppose that $\partial_\beta(G) = \Delta - \beta$ and we take any vertex $v \in V$ such that $\delta(v) = \Delta$. If $\Delta(G[V \setminus N[v]]) > \beta$, there exist $\{u, u_1, \ldots, u_j\} \subseteq V \setminus N[v]$ such that $u \sim u_i$ for every $i \in \{1, \ldots, j\}$ and $j > \beta$. In such a case, $\partial_\beta(\{v, u\}) = \Delta + j - 2\beta > \Delta - \beta$, a contradiction. If there exists $u \in N(v)$ such that $\delta_{\overline{N[v]}}(u) > \beta + 1$, then $\partial_\beta(\{v, u\}) > \Delta - 1 + \beta + 1 - 2\beta = \Delta - \beta$, a contradiction. If there exists $A \subseteq N(v)$ such that $|N_{\overline{N[v]}}(A)| + |N_{N(v)\setminus A}(A)| > \Delta - 1 + \beta(|A| - 1)$, then $\partial_\beta(A) = |N_{\overline{N[v]}}(A)| + |N_{N(v)\setminus A}(A)| + 1 - \beta|A| > \Delta - \beta$, a contradiction. □

Note that conditions (a)–(c) in the above Proposition are not enough to guarantee that $\partial_\beta(G) = \Delta - \beta$. The graph G shows in Figure 7 satisfies these three conditions but $\partial_2(G) = 4 > \Delta - 2$.

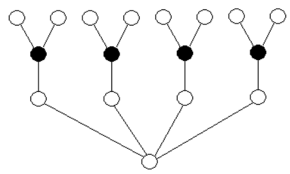

Figure 7. An example with $\beta = 2$ and $\partial_2(G) = 4$.

Proposition 9. *Let $G = (V, E)$ be a graph with order n and maximum degree Δ. If $\beta \in (0, 1)$ and $\partial_\beta(G) = \Delta - \beta$, then $n \le 2\Delta + 1$. Moreover, if $\beta < \frac{1}{\Delta - 1}$, then $n \le 2\Delta$.*

Proof 14. The first statement is directly obtained by Proposition 8. Assume that $\beta < \frac{1}{\Delta-1}$ and $n = 2\Delta + 1$. If $\delta(v) = \Delta$, then $\partial_\beta(N(v)) = \Delta + 1 - \beta\Delta > \Delta - \beta$, a contradiction. □

Another lower and upper bound is shown in the following lemma.

Lemma 5. *Let $G = (V, E)$ be a graph with order n. Then*

$$n - (1+\beta)\gamma(G) \le \partial_\beta(G) \le n - \gamma(G) - \beta.$$

Proof 15. For any set of vertices D it is known that $|B(D)| \le n - \gamma(G)$. Therefore, for any β-differential set D we have

$$\partial_\beta(G) = |B(D)| - \beta|D| \le n - \gamma(G) - \beta|D| \le n - \gamma(G) - \beta.$$

Finally, if D is a minimum dominating set, then $n - (1+\beta)\gamma(G) = \partial_\beta(D) \le \partial_\beta(G)$. □

Next, we will characterize all trees attaining the upper bound given in this lemma. For that, we will need the following result. We recall that a *wounded spider* is a graph that results by subdividing at most $m-1$ edges of the complete bipartite graph $K_{1,m}$.

Lemma 6 ([20]). *If $G=(V,E)$ is a tree, then $\gamma(G)=n-\Delta$ if only if G is a wounded spider.*

Theorem 2. *If G is a tree of order n, then $\partial_\beta(G)=n-\gamma(G)-\beta$ if only if G is a wounded spider.*

Proof 16. Assume that $\partial_\beta(G)=n-\gamma(G)-\beta$, and let D be a β-differential set of G. Since $|B(D)|-\beta|D|=n-\gamma(G)-\beta$ and we know that $|B(D)|\le n-\gamma(G)$, we deduce that $D=\{v\}$ for some $v\in V$, and $\delta(v)=n-\gamma(G)$. Therefore, $\delta(v)=\Delta$ and, by Lemma 6 we have that G is a wounded spider. If G is a wounded spider, again by Lemma 6 we have that $\Delta=n-\gamma(G)$, so $\partial_\beta(G)\ge\Delta-\beta\ge n-\gamma(G)-\beta$. Finally, using Lemma 5 we conclude that $\partial_\beta(G)=n-\gamma(G)-\beta$. □

Proposition 10. *If $G=(V,E)$ is a graph with minimum degree δ. Then,*

(a) *if $\beta\in(0,\delta-1)$, then $\partial_\beta(G)\ge\rho^o(G)(\delta-\beta-1)$,*
(b) *if $\beta\in(0,\delta)$, then $\partial_\beta(G)\ge\rho(G)(\delta-\beta)$.*

Proof 17. (a) Let S be a maximum open packing in G. If $u\in S$ then $\delta_{\overline{[S]}}(u)\ge\delta-1$, and so $\partial_\beta(S)\ge|S|(\delta-1)-\beta|S|=\rho^o(G)(\delta-\beta-1)$. The proof of (b) is analogous. □

Proposition 11. *Let $G=(V,E)$ be a graph with maximum degree Δ. If $\beta\in[1,\Delta)$, then $\partial_{\beta-1}(G)-\gamma(G)\le\partial_\beta(G)\le\partial_{\beta-1}(G)-1$.*

Proof 18. On one hand

$$\begin{aligned}\partial_{\beta-1}(G) &= \max\{|B(D)|-\beta|D|+|D| : D\subseteq V\}\\ &\ge \max\{|B(D)|-\beta|D|+1 : D\subseteq V\}\\ &= \max\{|B(D)|-\beta|D| : D\subseteq V\}+1=\partial_\beta(G)+1.\end{aligned}$$

On the other hand, if D is a $(\beta-1)$-differential set, then $|D|\le\gamma(G)$ and

$$\partial_{\beta-1}(G)=|B(D)|-\beta|D|+|D|\le|B(D)|-\beta|D|+\gamma(G)\le\partial_\beta(G)+\gamma(G).$$

□

Proposition 12. *Let K_n, P_n and C_n be the complete, path and cycle graph of order n and let $S_{n,m}$ and $K_{n,m}$ be the double star and the bipartite complete graph of orders $n+m+2$ and $n+m$ respectively. Then*

$$\partial_\beta(K_n)=\partial_\beta(W_n)=n-1-\beta.$$

$$\partial_\beta(P_n)=\partial_\beta(C_n)=\begin{cases}\left\lfloor\frac{n}{3}\right\rfloor(2-\beta)+1-\beta & \text{if } \beta\in(0,1) \text{ and } n\equiv 2 \pmod 3\\ \left\lfloor\frac{n}{3}\right\rfloor(2-\beta) & \text{otherwise.}\end{cases}$$

If $m\ge n$

$$\partial_\beta(K_{n,m})=\begin{cases} m+n-2(1+\beta) & \text{if } 0<\beta<n-2\\ m-\beta & \text{if } \beta\ge n-2.\end{cases}$$

$$\partial_\beta(S_{n,m})=\begin{cases} m+n-2\beta & \text{if } 0<\beta<n-1\\ m+1-\beta & \text{if } \beta\ge n-1.\end{cases}$$

Proof 19. $\partial_\beta(K_n)=\partial_\beta(W_n)=n-1-\beta$ follows immediately from Proposition 6. Let $V(P_n)=V(C_n)=\{u_1,\dots,u_n\}$ with $n=3k$ or $n=3k+1$. Let $D=\{u_2,u_5,\dots,u_{3\lfloor\frac{n}{3}\rfloor-1}\}$ then

$\partial_\beta(D) = \lfloor \frac{n}{3} \rfloor (2-\beta)$. Since any other set has β-differential less than or equal to $\partial_\beta(D)$, then $\partial_\beta(P_n) = \partial_\beta(C_n) = \lfloor \frac{n}{3} \rfloor (2-\beta)$. Similarly, we can check the other cases. □

Lemma 7. *Let $G = (V, E)$ be a graph. If D is a minimum (respectively, maximum) β-differential set of G, then $|B(D)| \geq (\lfloor \beta \rfloor + 1)\,|D|$ (respectively, $|B(D)| \geq \lceil \beta \rceil |D|$).*

Proof 20. If D is a minimum β-differential set, then for every $v \in D$, the number k of vertices in $B(D)$ which are adjacent to v but not to any $w \in D \setminus \{v\}$, that means that they are private neighbors of v with respect to D, must satisfy $k > \beta$, and so $k \geq \lfloor \beta \rfloor + 1$. If we consider the same situation when D is a maximum β-differential set, it must be satisfied $k \geq \beta$, that is, $k \geq \lceil \beta \rceil$. □

Observe that $\lfloor \beta \rfloor + 1 = \lceil \beta \rceil$ when $\beta \notin \mathbb{N}$.

Proposition 13. *Let $G = (V, E)$ be a graph. If D is a β-differential set of G, then $(\lceil \beta \rceil - \beta)\,|D| \leq \partial_\beta(G)$. Moreover, if D is a minimum β-differential set, then $(\lfloor \beta \rfloor - \beta + 1)\,|D| \leq \partial_\beta(G)$.*

Proof 21. It is enough to prove the first statement for a maximum β-differential set. By Lemma 7 we have$\lceil \beta \rceil |D| \leq |B(D)|$, so $|D|\,(\lceil \beta \rceil - \beta) \leq \partial_\beta(G)$. If D is a minimum β-differential set, Again by Lemma 7 we have $(\lfloor \beta \rfloor + 1)\,|D| \leq |B(D)|$, so $(\lfloor \beta \rfloor - \beta + 1)\,|D| \leq \partial_\beta(G)$. □

Theorem 3. *Let $G = (V, E)$ be a graph with maximum degree Δ.*

(i) If $\beta \in (0, 1]$, then $\frac{(\Delta-\beta)\partial(G)}{\Delta-1} \leq \partial_\beta(G)$.
(ii) If $\beta \in (1, \Delta)$, then $\frac{(\lfloor\beta\rfloor-\beta+1)\partial(G)}{\lfloor\beta\rfloor} \leq \partial_\beta(G)$.

Proof 22. (i) Let D be a 1-differential set of G. Since $\partial(G) \leq (\Delta - 1)|D|$, we have

$$\begin{aligned}\partial_\beta(G) &\geq |B(D)| - \beta|D| = |B(D)| - |D| + (1-\beta)|D| = \partial(G) + (1-\beta)|D| \\ &\geq \partial(G) + \frac{(1-\beta)\partial(G)}{\Delta - 1} = \frac{(\Delta-\beta)\partial(G)}{\Delta-1}.\end{aligned}$$

(ii) Let D be a 1-differential set of G. Since $1 - \beta < 0$, by Proposition 13 we have

$$\begin{aligned}\partial_\beta(G) &\geq |B(D)| - \beta|D| = |B(D)| - |D| + (1-\beta)|D| = \partial(G) + (1-\beta)|D| \\ &\geq \partial(G) + \frac{(1-\beta)\partial_\beta(G)}{(\lfloor\beta\rfloor - \beta + 1)},\end{aligned}$$

then $\left(1 - \frac{(1-\beta)}{(\lfloor\beta\rfloor-\beta+1)}\right)\partial_\beta(G) \geq \partial(G)$ or, equivalently $\partial_\beta(G) \geq \frac{(\lfloor\beta\rfloor-\beta+1)\partial(G)}{\lfloor\beta\rfloor}$. □

Theorem 4. *Let $G = (V, E)$ be a graph of order n and minimum degree δ, and let $\beta \in (0, \delta)$. Then,*

$$\partial_\beta(G) \geq \left(\frac{\lfloor\beta\rfloor - \beta + 1}{\beta\lfloor\beta\rfloor + 2\lfloor\beta\rfloor + 2}\right) n.$$

Proof 23. Let D be a minimum β-differential set. Since $\beta < \delta$ we have that every vertex in $C(D)$ has at least one neighbor in $B(D)$, that is, $B(D)$ is a dominating set. On one hand, since $\partial_\beta(G) \geq$

$|B(B(D))| - \beta|B(D)| = |D| + |C(D)| - \beta|B(D)|$, we have $|C(D)| \le \partial_\beta(G) + \beta|B(D)| - |D| = (1+\beta)\partial_\beta(G) + (\beta^2 - 1)|D|$. Now, using that, by Proposition 13, $(\lfloor\beta\rfloor - \beta + 1)\,|D| \le \partial_\beta(G)$, we have

$$\begin{aligned} n &= |D| + |B(D)| + |C(D)| \le (\beta+1)\partial_\beta(G) + |B(D)| + \beta^2|D| \\ &= (\beta+2)\partial_\beta(G) + (\beta^2+\beta)|D| \le (\beta+2)\partial_\beta(G) + \left(\frac{\beta^2+\beta}{\lfloor\beta\rfloor - \beta + 1}\right)\partial_\beta(G) \\ &= \left(\frac{(\beta+2)\,(\lfloor\beta\rfloor - \beta+1) + \beta^2+\beta}{\lfloor\beta\rfloor - \beta+1}\right)\partial_\beta(G) = \left(\frac{\beta\lfloor\beta\rfloor + 2\lfloor\beta\rfloor + 2}{\lfloor\beta\rfloor - \beta + 1}\right)\partial_\beta(G). \end{aligned}$$

□

Note that (i) in Theorem 3 is attained for any graph with order n and maximum degree $\Delta = n-1$. On the other hand, (ii) is attained in any double star, like the one shown in Figure 8, when $\beta \in \mathbb{N}$ and $r = s = 1 + \beta$.

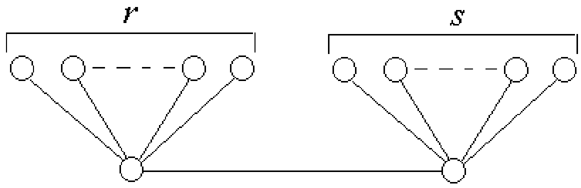

Figure 8. This graph show that the bound (ii) in Theorem 3 is attained when $\beta \in \mathbb{N}$ and $r = s = 1 + \beta$.

On the other hand, notice that, if $\beta \ge \delta$ it is not possible to give a bound similar to the one given in Theorem 4. For instance, it fails for the graph shown in Figure 9 with $\delta = 2, \beta = 3$ and $k \ge 29$, where C_k represents a cycle of k vertices, we have $n = 6 + k$ and $\partial_3(G) = 2$.

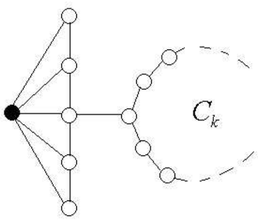

Figure 9. This graph show that Theorem 4 can fail when $\beta > \delta$.

Theorem 5. *Let $G = (V, E)$ be a graph of order n, size m and maximum degree Δ. Then*

$$\partial_\beta(G) \ge \frac{(2m - n\lfloor\beta\rfloor)(\lfloor\beta\rfloor - \beta + 1)}{\Delta(\lfloor\beta\rfloor + 2) + 1}.$$

Proof 24. We note that if D is a minimum β-differential set of G, then the following properties hold:

(1) $|D| \le \dfrac{\partial_\beta(G)}{\lfloor\beta\rfloor - \beta + 1}$.
(2) If $v \in B(D)$, then $\delta_{C(D)}(v) \le \lfloor\beta\rfloor + 1$.
(3) If $v \in C(D)$, then $\delta_{C(D)}(v) \le \lfloor\beta\rfloor$.

Let r be the number of edges from $B(D)$ to $C(D)$. Then from (3) and (2) we have

$$\left(\sum_{u \in C(D)} \delta(u)\right) - |C(D)|\lfloor\beta\rfloor \le r \le |B(D)|(\lfloor\beta\rfloor + 1).$$

Therefore,

$$\begin{aligned} 2m &\leq |D|\Delta + |B(D)|\Delta + \sum_{u \in C(D)} \delta(u) \leq |D|\Delta + |B(D)|\Delta + |B(D)|(\lfloor\beta\rfloor + 1) + |C(D)|\lfloor\beta\rfloor \\ &= |D|\Delta + |B(D)|(\Delta+1) + (n - |D|)\lfloor\beta\rfloor \\ &= |D|(\Delta - \lfloor\beta\rfloor) + \partial_\beta(G)(\Delta+1) + \beta|D|(\Delta+1) + n\lfloor\beta\rfloor \\ &= (\Delta - \lfloor\beta\rfloor + \beta(\Delta+1))|D| + \partial_\beta(G)(\Delta+1) + n\lfloor\beta\rfloor \\ &\leq \left(\frac{\Delta - \lfloor\beta\rfloor + \beta(\Delta+1)}{\lfloor\beta\rfloor - \beta + 1}\right)\partial_\beta(G) + \partial_\beta(G)(\Delta+1) + n\lfloor\beta\rfloor \\ &= \left(\frac{\Delta - \lfloor\beta\rfloor + \beta(\Delta+1) + (\Delta+1)(\lfloor\beta\rfloor - \beta + 1)}{\lfloor\beta\rfloor - \beta + 1}\right)\partial_\beta(G) + n\lfloor\beta\rfloor \\ &= \left(\frac{\Delta(\lfloor\beta\rfloor + 2) + 1}{\lfloor\beta\rfloor - \beta + 1}\right)\partial_\beta(G) + n\lfloor\beta\rfloor. \end{aligned}$$

In consequence, $\partial_\beta(G) \geq \frac{(2m - n\lfloor\beta\rfloor)(\lfloor\beta\rfloor - \beta + 1)}{\Delta(\lfloor\beta\rfloor + 2) + 1}$. □

We present now a technical lemma which will be used in the proof of Theorem 6.

Lemma 8. *If* $\beta \in (0, \delta)$, *then*

$$\left(\frac{\lceil\beta\rceil(1 + \lfloor\beta\rfloor)}{\delta - \lfloor\beta\rfloor} + 2 + \lfloor\beta\rfloor\right)\left(\frac{\Delta - \beta}{1 + \lfloor\beta\rfloor - \beta} + 1\right) > 2 + \Delta(2 + \lfloor\beta\rfloor).$$

Proof 25. We write $\beta = k + \frac{\alpha}{10}$, where $k \in \mathbb{N}$ and $\alpha \in [0, 10)$, so the inequality is

$$\left(\frac{(k+1)^2}{\delta - k} + 2 + k\right)\left(\frac{\Delta - k - \frac{\alpha}{10}}{1 - \frac{\alpha}{10}} + 1\right) > 2 + \Delta(2 + k)$$

or, equivalently,

$$\left(\frac{(k+1)^2 + (2+k)(\delta - k)}{\delta - k}\right)\left(\frac{10\Delta - 10k - 2\alpha + 10}{10 - \alpha}\right) > 2 + \Delta(2 + k).$$

Since $h_1(\delta) := \frac{(k+1)^2 + (2+k)(\delta-k)}{\delta - k}$ is decreasing in δ and $h_2(\alpha) = \frac{10\Delta - 10k - 2\alpha + 10}{10 - \alpha}$ is increasing in α, we have

$$\begin{aligned} &\left(\frac{(k+1)^2 + (2+k)(\delta - k)}{\delta - k}\right)\left(\frac{10\Delta - 10k - 2\alpha + 10}{10 - \alpha}\right) \\ &\qquad \geq \left(\frac{(k+1)^2 + (2+k)(\Delta - k)}{\Delta - k}\right)\left(\frac{10\Delta - 10k + 10}{10}\right) \\ &\qquad = \left(\frac{(k+1)^2 + (2+k)(\Delta - k)}{\Delta - k}\right)(\Delta - k + 1) \\ &\qquad = (k+1)^2 + (2+k)(\Delta - k) + \frac{(k+1)^2}{\Delta - k} + (2+k) \\ &\qquad = k(k+2) + 1 + (2+k)\Delta - (2+k)k + \frac{(k+1)^2}{\Delta - k} + (2+k) \\ &\qquad = 1 + \Delta(2+k) + \frac{(k+1)^2}{\Delta - k} + (2+k) > 2 + \Delta(2+k). \end{aligned}$$

□

Theorem 6. *Let $G = (V, E)$ be a graph of order n, minimum degree δ and maximum degree Δ. Let $\beta < \delta$ and $h(k) := \left(\frac{\lceil\beta\rceil(1+\lfloor\beta\rfloor)}{\delta-\lfloor\beta\rfloor} + 2 + \lfloor\beta\rfloor\right)\left(\frac{\Delta-\beta}{1+\lfloor\beta\rfloor-\beta} + k + 1\right)$, where $k \in \mathbb{N}$. If $n \geq h(k)$, then*

$$\partial_\beta(G) \geq \Delta - \beta + (k+1)(\lfloor\beta\rfloor - \beta + 1).$$

Proof 26. We proceed by induction on k. For $k = 0$ we suppose that $n \geq h(0)$ and take $v \in V$ such that $\delta(v) = \Delta$. If there exists $u \in B(\{v\})$ with $\delta_{C(\{v\})}(u) \geq \lfloor\beta\rfloor + 2$, then for $D = \{v, u\}$ we obtain $\partial_\beta(D) \geq \Delta - 1 + \lfloor\beta\rfloor + 2 - 2\beta = \Delta - \beta + (\lfloor\beta\rfloor - \beta + 1)$. Therefore, we can assume $\delta_{C(\{v\})}(u) \leq \lfloor\beta\rfloor + 1$ for every $u \in B(\{v\})$. Note that if there exists a $w \in C(\{v\})$ such that $N(w) \cap B(\{v\}) = \emptyset$, then

$$\partial(\{v, w\}) \geq \Delta + \delta - 2\beta = \Delta - \beta + \delta - \beta \geq \Delta - \beta + (\lfloor\beta\rfloor - \beta + 1),$$

because $\delta \geq \lfloor\beta\rfloor + 1$. If we assume that $N(w) \cap B(\{v\}) \neq \emptyset$ for every $w \in C(\{v\})$, and hence $|C(\{v\})| \leq \Delta(1 + \lfloor\beta\rfloor)$. From Lemma 8 it follows

$$n = 1 + \Delta + |C(\{v\})| \leq 1 + \Delta(2 + \lfloor\beta\rfloor) < h(0),$$

contradicting the hypothesis. Now, we suppose that the theorem is true for k and $n \geq h(k+1)$. Let $\mathcal{M}$ be the collection of all β-differential sets of G such that every $D \in \mathcal{M}$ satisfies that every vertex $v \in D$ has at least $\lfloor\beta\rfloor + 1$ external private neighbors with respect to D. That is, $|\text{epn}[v, D]| \geq \lfloor\beta\rfloor + 1$. Let $D' \in \mathcal{M}$ with maximum cardinality. Since $n \geq h(k+1) \geq k$, by induction hypothesis we know that $\partial_\beta(D') \geq \Delta - \beta + (k+1)(\lfloor\beta\rfloor - \beta + 1)$. Moreover, as $|B(D')| \geq (\lfloor\beta\rfloor + 1)|D'|$, we also have $\partial_\beta(G) \geq (\lfloor\beta\rfloor - \beta + 1)|D'|$.

If there exists $w \in C(D')$ such that $\delta_{C(D')}(w) > \lfloor\beta\rfloor$, then we have

$$\partial(D' \cup \{w\}) \geq \Delta - \beta + (k+1)(\lfloor\beta\rfloor - \beta + 1) + \lfloor\beta\rfloor - \beta + 1 = \Delta - \beta + (k+2)(\lfloor\beta\rfloor - \beta + 1)$$

and we are done. Therefore, we suppose that for every $w \in C(D')$ it is satisfied $\delta_{C(D')}(w) \leq \lfloor\beta\rfloor$. If m' is the number of edges in $G[C(D')]$, then

$$m' \leq \frac{(n - |D'| - |B(D')|)\lfloor\beta\rfloor}{2}.$$

We suppose that there exists $v \in D'$ and $u \in B(\{v\})$ such that $\delta_{C(D')}(u) \geq 1 + \beta$. If $|\text{epn}[v, D']| = \lfloor\beta\rfloor + 1$, then $D'' = (D' \setminus \{v\}) \cup \{u\}$ gives a β-differential bigger than $\partial_\beta(D')$, which is impossible. If $|\text{epn}[v, D']| > \lfloor\beta\rfloor + 1$, then $D'' = D' \cup \{u\} \in \mathcal{M}$ contradicting the choice of D'. Thus, we can assume that $\delta_{C(D')}(u) < 1 + \beta$ for any $u \in B(\{v\})$ and $v \in D'$, that is, $\delta_{C(D')}(u) \leq \lceil\beta\rceil$ for any $u \in B(\{v\})$ and $v \in D'$.

Let r be the number of edges between $B(D')$ and $C(D')$. Then $r \leq \lceil\beta\rceil |B(D')| = \lceil\beta\rceil(\partial_\beta(G) + \beta|D'|)$. Hence,

$$m' \quad \geq \quad \frac{(n - |D'| - |B(D')|)\delta - r}{2} \geq \frac{(n - |D'| - |B(D')|)\delta - \lceil\beta\rceil(\partial_\beta(G) + \beta|D'|)}{2},$$

consequently,

$$\frac{(n - |D'| - |B(D')|)\delta - \lceil\beta\rceil(\partial_\beta(G) + \beta|D'|)}{2} \leq \frac{(n - |D'| - |B(D')|)\lfloor\beta\rfloor}{2}$$

or, equivalently,

$$n \leq \frac{\lceil\beta\rceil}{\delta - \lfloor\beta\rfloor}(\partial_\beta(G) + \beta|D'|) + \partial_\beta(G) + (\beta + 1)|D'|.$$

Finally, using that $|D'| \leq \frac{\partial_\beta(G)}{\lfloor\beta\rfloor-\beta+1}$, we obtain

$$\begin{aligned} n &\leq \frac{\lceil\beta\rceil}{\delta-\lfloor\beta\rfloor}(\partial_\beta(G)+\beta|D'|)+\partial_\beta(G)+(\beta+1)|D'| \\ &\leq \frac{\lceil\beta\rceil}{\delta-\lfloor\beta\rfloor}\left(\partial_\beta(G)+\frac{\beta\partial_\beta(G)}{\lfloor\beta\rfloor-\beta+1}\right)+\partial_\beta(G)+\frac{(\beta+1)\partial_\beta(G)}{\lfloor\beta\rfloor-\beta+1} \\ &= \frac{\lceil\beta\rceil}{\delta-\lfloor\beta\rfloor}\left(\frac{(1+\lfloor\beta\rfloor)\partial_\beta(G)}{\lfloor\beta\rfloor-\beta+1}\right)+\frac{(2+\lfloor\beta\rfloor)\partial_\beta(G)}{\lfloor\beta\rfloor-\beta+1} \\ &= \left(\frac{\lceil\beta\rceil(1+\lfloor\beta\rfloor)}{\delta-\lfloor\beta\rfloor}+2+\lfloor\beta\rfloor\right)\frac{\partial_\beta(G)}{\lfloor\beta\rfloor-\beta+1} \end{aligned}$$

and, as

$$\left(\frac{\lceil\beta\rceil(1+\lfloor\beta\rfloor)}{\delta-\lfloor\beta\rfloor}+2+\lfloor\beta\rfloor\right)\left(\frac{\Delta-\beta+(k+2)(\lfloor\beta\rfloor-\beta+1)}{\lfloor\beta\rfloor-\beta+1}\right)=h(k+1)\leq n,$$

we conclude that $\partial_\beta(G) \geq \Delta-\beta+(k+2)(\lfloor\beta\rfloor-\beta+1)$. □

Acknowledgments: We thank both referees for their valuable and constructive comments. This paper was supported in part by two grants from AEI and FEDER (MTM2016-78227-C2-1-P and MTM2015-69323-REDT), and a grant from CONACYT (FOMIX-CONACyT-UAGro 249818), Mexico. First author's work was also partially supported by Plan Nacional I+D+I Grant MTM2015-70531 and Junta de Andalucía FQM-260.

Author Contributions: These authors contributed to equally to this work.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Kempe, D.; Kleinberg, J.; Tardos, E. Maximizing the spread of influence through a social network. In Proceedings of the Ninth ACM SIGKDD International Conference on Knowledge Discovery and Data Mining, New York, NY, USA, 24–27 August 2003; pp. 137–146, doi:10.1145/956750.956769.
2. Kempe, D.; Kleinberg, J.; Tardos, E. Influential nodes in a diffusion model for social networks. In Proceedings of the 32nd international conference on Automata, Languages and Programming, Lisbon, Portugal, 11–15 July 2005; pp. 1127–1138.
3. Haynes, T.W.; Hedetniemi, S.; Slater, P.J. *Domination in Graphs: Advanced Topics*; Taylor and Francis: Didcot, UK, 1998.
4. Bermudo, S.; Fernau, H. Lower bound on the differential of a graph. *Discret. Math.* **2012**, *312*, 3236–3250, doi:10.1016/j.disc.2012.07.021.
5. Mashburn, J.L.; Haynes, T.W.; Hedetniemi, S.M.; Hedetniemi, S.T.; Slater, P.J. Differentials in graphs. *Util. Math.* **2006**, *69*, 43–54.
6. Basilio, L.A.; Bermudo, S.; Sigarreta, J.M. Bounds on the differential of a graph. *Util. Math.* **2015**, in press.
7. Bermudo, S. On the Differential and Roman domination number of a graph with minimum degree two. *Discret. Appl. Math.* doi:10.1016/j.dam.2017.08.005.
8. Bermudo, S.; De la Torre, L.; Martín-Caraballo, A.M.; Sigarreta, J.M. The differential of the strong product graphs. *Int. J. Comput. Math.* **2015**, *92*, 1124–1134, doi:10.1080/00207160.2014.941359.
9. Bermudo, S.; Fernau, H. Computing the differential of a graph: Hardness, approximability and exact algorithms. *Discret. Appl. Math.* **2014**, *165*, 69–82, doi:10.1016/j.dam.2012.11.013.
10. Bermudo, S.; Fernau, H. Combinatorics for smaller kernels: The differential of a graph. *Theor. Comput. Sci.* **2015**, *562*, 330–345, doi:10.1016/j.tcs.2014.10.007.
11. Bermudo, S.; Fernau, H.; Sigarreta, J.M. The differential and the Roman domination number of a graph. *Appl. Anal. Discret. Math.* **2014**, *8*, 155–171, doi:10.2298/AADM140210003B.
12. Bermudo, S.; Rodríguez, J.M.; Sigarreta, J.M. On the differential in graphs. *Util. Math.* **2015**, *97*, 257–270.
13. Pushpam, P.R.L.; Yokesh, D. Differential in certain classes of graphs. *Tamkang J. Math.* **2010**, *41*, 129–138, doi:10.5556/j.tkjm.41.2010.664.
14. Sigarreta, J.M. Differential in cartesian product graphs. *Ars Comb.* **2016**, *126*, 259–267.

15. Hernández-Gómez, J.C. Differential and operations on graphs. *Int. J. Math. Anal.* **2015**, *9*, 341–349, doi:10.12988/ijma.2015.411344.
16. Goddard, W.; Henning, M.A. Generalised domination and independence in graphs. *Congr. Numer.* **1997**, *123*, 161–171.
17. Zhang, C.Q. Finding critical independent sets and critical vertex subsets are polynomial problems. *SIAM J. Discret. Math.* **1990**, *3*, 431–438, doi:10.1137/0403037.
18. Slater, P.J. Enclaveless sets and MK-systems. *J. Res. Nat. Bur. Stand.* **1977**, *82*, 197–202.
19. Dahme, F.; Rautenbach, D.; Volkmann, L. Some remarks on α-domination. *Discuss. Math. Gr. Theory* **2004**, *24*, 423–430, doi:10.7151/dmgt.1241.
20. Domke, G.S.; Dumbar, J.E.; Markus, L.R. Gallai-type theorems and domination parameters. *Discret. Math.* **1997**, *167–168*, 237–248, doi:10.1016/S0012-365X(97)00231-8.