

Review

Symmetries in Classical and Quantum Treatment of Einstein's Cosmological Equations and Mini-Superspace Actions

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Abstract: The use of automorphisms of the various Bianchi-type Lie algebras as Lie-point symmetries of the corresponding Einstein field equations entails a reduction of their order and ultimately leads to the entire solution space. When a valid reduced action principle exists, the symmetries of the configuration mini-supermetric space can also be used, in conjunction with the constraints, to provide local or non-local constants of motion. At the classical level, depending on their number, these integrals can even secure the acquisition of the entire solution space without any further solving of the dynamical equations. At the quantum level, their operator analogues can be used, along with the Wheeler–DeWitt equation, to define unique wave functions that exhibit singularity-free behavior at a semi-classical level.

Keywords: Symmetries of ODE's; Einstein cosmological actions; minisuperspace

1. Introduction

For pure gravity without any symmetries, Einstein's field equations consist of an infinite number of equations because such systems have infinite degrees of freedom. On the other hand, spatial homogeneity reduces the general case to a system of ten coupled O.D.E.'s (ordinary differential equations) with respect to time [1,2]: one equation is quadratic in the velocities $\dot{\gamma}_{\alpha\beta}$ and algebraic in N^2 ($G_{00} = 0$); three are linear in velocities and also algebraic in N^α ($G_{0i} = 0$); and the remaining six spatial equations ($G_{ij} = 0$) are linear in $\dot{\gamma}_{\alpha\beta}$ and depending on $N, \dot{N}, N^\alpha, \dot{N}^\alpha, \gamma_{\alpha\beta}, \dot{\gamma}_{\alpha\beta}$.

The natural way to proceed with this set of equations is to solve the quadratic constraint for N^2 and the linear constraint equations for as many N^α 's as possible. Then substitute these results into the remaining spatial equations. Subsequently, the spatial equations can be solved for only $6 - 4 = 2$ independent accelerations $\ddot{\gamma}_{\alpha\beta}$. The only exceptions occur for Bianchi Type II and III, where we can solve for $6 - 3 = 3$ accelerations; thus, only two of the three linear constraints are independent. However, a linear combination of the N^α 's remains arbitrary and in equilibrium with the existence of the extra independent acceleration. Therefore, in every Bianchi type, the general solution consists of four arbitrary functions of time, whose specification corresponds to a choice of time and space coordinates.

The usual way of approaching this problem in the literature is by gauge fixing the lapse function N and shift vector N^a before doing any calculations. Specifically, the most common choices for the lapse function are to consider N depending explicitly on time or being a constant value. The situation for the spatial coordinates is more vague. Most of the time, N^α 's are set to zero, and some other times,

N^α 's are retained. However, such choices are not necessarily equivalent. The spatial equations can be solved for all six independent $\dot{\gamma}_{\alpha\beta}(t)$ after this gauge fixing. Thus, the constraint equations reduce to algebraic restrictions between the integration constants.

Therefore, setting $N^\alpha(t) = 0$ leads to $\gamma_{\alpha\beta} = \text{diag}(a^2(t), b^2(t), c^2(t))$ for Class A types, etc. However, for the Bianchi Types VIII and IX, the hypothesis $N^\alpha(t) = 0$ and $\gamma_{\alpha\beta} = \text{diag}(a^2(t), b^2(t), c^2(t))$ is known to be linked to the kinematics and/or the dynamics although in a somewhat vague way; see, e.g., [3] and Ryan in [1]. In all other cases, this or any other simplifying hypothesis used is interpreted only as an ansatz to be tested at the end, i.e., after having solved all the (further simplified) equations. For example, to take an extreme case, the diagonality of $\gamma_{\alpha\beta}(t)$ together with the vanishing of the shift vector is known to lead to the incompatibility of Bianchi Types IV and VII (Class B) [4,5], as well as for the biaxial Type VIII cases $(a^2, a^2, c^2), (a^2, b^2, a^2)$, [5]. The diversity of the various ansatz appearing in the literature causes a considerable degree of fragmentation.

Automorphisms has been long suspected and/or known to play an important role in a unified treatment of Bianchi types. This was firstly mentioned in [6]. Time-dependent automorphism matrices have recently been used by Jantzen [7,8] as a convenient parametrization of a general positive definite 3×3 scale factor matrix $\gamma_{\alpha\beta}(t)$, in terms of (showed for) a diagonal matrix. Samuel and Ashtekar have seen automorphisms as a result of general coordinate transformations [9], linking them to topological considerations.

In the present work, we also adopt a space-time point of view and try to avoid the above-mentioned fragmentation. This is achieved by revealing those G.C.T.'s (general coordinate transformations) that permit us to simplify the line-element without losing manifest spatial homogeneity. Thus, we are able to uncover special automorphic transformations of $\gamma_{\alpha\beta}(t)$, along with corresponding changes of N, N^α , which allow us to set $N^\alpha = 0$ and bring $\gamma_{\alpha\beta}(t)$ to some irreducible, simple form.

The structure of the work is as follows: In Section 2, we briefly review the classical dynamics in the scene of [10,11]. Section 3 gives an example of the way in which we exploit the freedom to set the shift to zero; the solution space of Bianchi I is presented. In Section 4, we present an example of using the time-dependent automorphism in order to simplify the scale factor matrix $\gamma_{\alpha\beta}$. In Section 5, the reduced Lagrangian dynamics is reviewed, and the various integrals of motion are presented, along with a quantization scheme. In Section 6, we give an example by treating classically and quantum mechanically an FLRW (Friedmann–Lemaître–Robertson–Walker). Finally, some concluding remarks are also given.

2. Classical Kinematics

In this section, we review the time-dependent automorphism inducing the diffeomorphisms used in [10,11] to describe the dynamics of Bianchi-type spatially-homogeneous space-times. Our starting point is the general line element, written in the adopted coordinates (t, x^i) [12]:

$$ds^2 = (N^\alpha(t)N_\alpha(t) - N^2(t))dt^2 + 2N_\alpha(t)\sigma_i^\alpha dx^i dt + \gamma_{\alpha\beta}(t)\sigma_i^\alpha \sigma_j^\beta dx^i dx^j, \tag{1}$$

where σ^α are the invariant basis one forms, defined through:

$$d\sigma^\alpha = C_{\beta\gamma}^\alpha \sigma^\beta \wedge \sigma^\gamma \Leftrightarrow \sigma_{i,j}^\alpha - \sigma_{j,i}^\alpha = C_{\beta\gamma}^\alpha \sigma_j^\beta \sigma_i^\gamma, \tag{2}$$

with $C_{\beta\gamma}^\alpha$ being the structure constants characterizing the Lie algebra of the corresponding Bianchi group. If we use the above line element, Einstein's field equations assume, for each Bianchi model, the form:

$$E_0 = K^{\alpha\beta}K_{\alpha\beta} - K^2 - R = 0 \tag{3a}$$

$$E_\alpha = K_\alpha^\mu C_{\mu\gamma}^\gamma - K_\gamma^\mu C_{\alpha\mu}^\gamma = 0 \tag{3b}$$

$$E_{\alpha\beta} = \dot{K}_{\alpha\beta} + (2K_\alpha^\tau K_{\tau\beta} - K K_{\alpha\beta}) + 2N^\rho (K_{\alpha\nu} C_{\beta\rho}^\nu + K_{\beta\nu} C_{\alpha\rho}^\nu) - N R_{\alpha\beta} = 0, \tag{3c}$$

where $K_{\alpha\beta}$ is the extrinsic curvature tensor:

$$K_{\alpha\beta} = -\frac{1}{2N}(\dot{\gamma}_{\alpha\beta} + 2\gamma_{\alpha\nu}C_{\beta\rho}^{\nu}N^{\rho} + 2\gamma_{\beta\nu}C_{\alpha\rho}^{\nu}N^{\rho}), \tag{4}$$

and $R_{\alpha\beta}$ are the triad components of the Ricci tensor $R_{ij} \equiv R_{\alpha\beta}\sigma_i^{\alpha}\sigma_j^{\beta}$, with:

$$R_{\alpha\beta} = C_{\sigma\tau}^{\kappa}C_{\mu\nu}^{\lambda}\gamma_{\alpha\kappa}\gamma_{\beta\lambda}\gamma^{\sigma\nu}\gamma^{\tau\mu} + 2C_{\beta\lambda}^{\kappa}C_{\alpha\kappa}^{\lambda} + 2C_{\alpha\kappa}^{\mu}C_{\beta\lambda}^{\nu}\gamma_{\mu\nu}\gamma^{\kappa\lambda} + 2C_{\alpha\kappa}^{\lambda}C_{\mu\nu}^{\mu}\gamma_{\beta\lambda}\gamma^{\kappa\nu}. \tag{5}$$

The above equations form what is known as a perfect mathematical ideal: the time derivative of Equation (3a) and (3b) becomes an identity when $\dot{K}_{\alpha\beta}$ is substituted from Equation (3c).

If one aims at the general solution of the above system, one must seek transformations of the coordinates, which retain the general functional form of the line element (1). To this end, let us consider the transformation $x^i = f^i(\tilde{x}^j, t)$ and compute its action on the line element.

$$\begin{aligned} ds^2 &= [(N^{\alpha}(t)N_{\alpha}(t) - N^2(t)) + \frac{\partial f^i}{\partial t} \frac{\partial f^j}{\partial t} \sigma_i^{\alpha}(f)\sigma_j^{\beta}(f)\gamma_{\alpha\beta}(t) \\ &+ 2\sigma_i^{\alpha}(f) \frac{\partial f^i}{\partial t} N_{\alpha}(t)]dt^2 \\ &+ 2\sigma_i^{\alpha}(f) \frac{\partial f^i}{\partial \tilde{x}^m} [N_{\alpha}(t) + \sigma_j^{\beta}(f) \frac{\partial f^j}{\partial t} \gamma_{\alpha\beta}(t)]d\tilde{x}^m dt \\ &+ \sigma_i^{\alpha}(f)\sigma_j^{\beta}(f)\gamma_{\alpha\beta}(t) \frac{\partial f^i}{\partial \tilde{x}^m} \frac{\partial f^j}{\partial \tilde{x}^n} d\tilde{x}^m d\tilde{x}^n. \end{aligned} \tag{6}$$

We can always find a non-singular matrix $\Lambda_{\mu}^{\alpha}(\tilde{x}, t)$ and a triplet $P^{\alpha}(\tilde{x}, t)$, such that:

$$\begin{aligned} \sigma_i^{\alpha}(f) \frac{\partial f^i}{\partial \tilde{x}^m} &= \Lambda_{\mu}^{\alpha}(\tilde{x}, t)\sigma_m^{\mu}(\tilde{x}) \\ \sigma_i^{\alpha}(f) \frac{\partial f^i}{\partial t} &= P^{\alpha}(\tilde{x}, t). \end{aligned} \tag{7}$$

If these transformations are to preserve manifest spatial homogeneity, Λ_{μ}^{α} and P^{α} should not depend on the spatial point \tilde{x} , i.e., $\Lambda_{\mu}^{\alpha}(\tilde{x}, t) = \Lambda_{\mu}^{\alpha}(t)$ and $P^{\alpha}(\tilde{x}, t) = P^{\alpha}(t)$. This requirement alters the nature of Equation (7) from mere allocations to a system of first order, highly non-linear PDE's for the unknown functions $f^i(\tilde{x}, t)$. For these particular f^i 's, one can write down the line element (1) in the form:

$$ds^2 = (\tilde{N}^{\alpha}(t)\tilde{N}_{\alpha}(t) - \tilde{N}^2(t))dt^2 + 2\tilde{N}_{\alpha}(t)\sigma_i^{\alpha}(\tilde{x})d\tilde{x}^i dt + \tilde{\gamma}_{\alpha\beta}(t)\sigma_i^{\alpha}(\tilde{x})\sigma_j^{\beta}(\tilde{x})d\tilde{x}^i d\tilde{x}^j. \tag{8}$$

With the allocations:

$$\tilde{N} = N \quad \tilde{N}_{\alpha} = \Lambda_{\alpha}^{\rho}(N_{\rho} + \gamma_{\rho\sigma}P^{\sigma}) \quad \tilde{\gamma}_{\mu\nu} = \Lambda_{\mu}^{\alpha}\Lambda_{\nu}^{\beta}\gamma_{\alpha\beta}. \tag{9}$$

The Frobenius theorem guarantees that there exist local solutions to the system (7), as long as the integrability conditions $\frac{\partial^2 x^i}{\partial \tilde{x}^m \partial \tilde{x}^n} = \frac{\partial^2 x^i}{\partial \tilde{x}^n \partial \tilde{x}^m}$ and $\frac{\partial^2 x^i}{\partial \tilde{x}^m \partial t} = \frac{\partial^2 x^i}{\partial t \partial \tilde{x}^m}$ hold true. These conditions, by repeated use of Equation (7), reduce to the form:

$$\Lambda_{\rho}^{\alpha}C_{\beta\gamma}^{\rho} = C_{\mu\nu}^{\alpha}\Lambda_{\beta}^{\mu}\Lambda_{\gamma}^{\nu} \Leftrightarrow C_{\beta\gamma}^{\alpha} = S_{\mu}^{\alpha}\Lambda_{\beta}^{\mu}\Lambda_{\gamma}^{\nu}C_{\mu\nu}^{\alpha} \tag{10}$$

$$2P^{\mu}C_{\mu\nu}^{\alpha}\Lambda_{\beta}^{\nu} = \dot{\Lambda}_{\beta}^{\alpha}, \tag{11}$$

with S being the matrix inverse to Λ . The first of the above equations signifies that Λ belongs to the automorphisms of the corresponding Lie algebra. The solution to the above equations can easily be found for all Bianchi types. The general result is that in all cases, three arbitrary functions of time,

along with a number of constants, enter the form of Λ_μ^α and P^ρ . There is also a fourth arbitrary function of time due to the usual freedom to change the time coordinate $t = g(\tilde{t})$. All this freedom is at our disposal in order to simplify the line element and thus the ensuing Equation (3a)–(3c). There are essentially two ways to use the three arbitrary functions of time:

1. The first is to make the shift \tilde{N}^α zero. Then, the residual rigid “gauge” freedom described by $\Lambda_\beta^\alpha = \text{constant}$, $P^\rho = 0$ provides us with Lie point symmetries, which can be used to reduce the order of the equations and ultimately acquire the entire solution space. In such a way, the general solution of Bianchi Types I–VII has been uncovered.
2. The second is to simplify the scale factor matrix and then proceed to solve the reduced form of the equations. This option is more suitable for the case of Bianchi Types VIII and IX, since in this case, the time-dependent Λ suffices to diagonalize $\gamma_{\alpha\beta}$, and then, Equation (3b) enforces $N^\alpha = 0$.

In the spirit of the first way, we present the complete solution space for Bianchi Type I. The rest of the above-mentioned types afford the same method: one brings in normal form the generator of the maximal Abelian subgroup, thus reducing the order of the system of equations for the corresponding dependent functions. At this point, since the initial automorphism group is solvable, we are assured that the remaining generators are still symmetries of the reduced equations. By repeating the same procedure, we end up with one differential equation in terms of one dependent variable. This equation through a finite Lie–Backlund transformation is cast into the Painlevé VI transcendental. This holds true for all Types II–VII; see [13–16].

3. Bianchi Type I

In this model, all structure constants are zero. The general solution of Equations (10) and (11) is:

$$\Lambda_\beta^\alpha = \text{constant} \quad (12)$$

$$P^\alpha = (P^1(t), P^2(t), P^3(t)). \quad (13)$$

Therefore, with the use of three arbitrary functions of P^α , it is possible to set the shift vector equal to zero, $\tilde{N}_\alpha = 0$, by using (9) and selecting:

$$P^\rho = -\gamma^{\rho\sigma} N_\sigma. \quad (14)$$

It is also useful to choose the time gauge $N = \sqrt{\gamma}$. Under these conditions, Equation (3b) is identically satisfied, while (3a) and (3c) become:

$$E_0 \equiv \text{Tr}[(\gamma^{-1}\dot{\gamma})^2] - (\text{Tr}(\gamma^{-1}\dot{\gamma}))^2 = 0 \quad (15)$$

$$E_\beta^\alpha \equiv \frac{d}{dt}(\gamma^{-1}\dot{\gamma}) = 0, \quad (16)$$

where γ is a general symmetric matrix with all its components depending on time. The second equation is integrated and gives:

$$\gamma^{-1}\dot{\gamma} = \Theta \Rightarrow \dot{\gamma} = \gamma\Theta, \quad (17)$$

where Θ is a general constant 3×3 matrix. Because there is additional freedom to use the constant automorphisms Λ_β^α (see Equation (9) with $P^\alpha = 0$), the matrix Θ can be simplified:

$$\gamma = \Lambda^T \tilde{\gamma} \Lambda \Rightarrow \Lambda^T \dot{\tilde{\gamma}} \Lambda = \Lambda^T \tilde{\gamma} \Lambda \Theta \Rightarrow \dot{\tilde{\gamma}} = \tilde{\gamma} \Lambda \Theta \Lambda^{-1} \Rightarrow \tilde{\Theta} = \Lambda \Theta \Lambda^{-1}. \quad (18)$$

Thus, the solution space is divided into different parts according to the nature of the eigenvalues of Θ .

3.1. Three Unequal Real Eigenvalues

If one assumes the following decomposition of a general matrix Λ as $\Lambda_1\Lambda_2\Lambda_3$:

$$\Lambda_1 = \begin{bmatrix} \sqrt{\lambda_1} & 0 & 0 \\ 0 & \sqrt{\lambda_2} & 0 \\ 0 & 0 & \sqrt{\lambda_3} \end{bmatrix}, \quad \Lambda_2 = \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}, \quad \Lambda_3 = O_3, \quad O_3^T O_3 = I. \quad (19)$$

Λ_3 can be used to bring the matrix Θ into upper-triangular form:

$$\Theta = \begin{bmatrix} \theta_1 & \theta_{12} & \theta_{13} \\ 0 & \theta_2 & \theta_{23} \\ 0 & 0 & \theta_3 \end{bmatrix} \quad (20)$$

The next step is to use Λ_2 to diagonalize Θ . Therefore, without loss of generality, we have:

$$\dot{\gamma} = \gamma\Theta \quad (21)$$

where $\Theta = \text{diag}(\kappa, \lambda, \mu)$ on which the matrix Λ_1 does not have impact (because both of these are diagonal and the transformation is a similarity transformation). Since γ is a symmetric matrix, it follows that $\dot{\gamma}$ is symmetric, as well. Therefore, there are additional conditions for the product $\gamma\Theta$, that is, it should hold:

$$\gamma\Theta = (\gamma\Theta)^T \Rightarrow \gamma\Theta - \Theta\gamma = 0. \quad (22)$$

These relations, for $\theta_1, \theta_2, \theta_3$, being different and non-zero, lead to the following restrictions:

$$\gamma_{12} = \gamma_{13} = \gamma_{23} = 0. \quad (23)$$

Therefore, we have the equations:

$$\kappa^2 + \lambda^2 + \mu^2 - (\kappa + \lambda + \mu)^2 = 0 \Rightarrow \kappa\lambda + \kappa\mu + \lambda\mu = 0 \quad (24)$$

and:

$$\dot{\gamma}_{11} = \kappa\gamma_{11}, \quad \dot{\gamma}_{22} = \lambda\gamma_{22}, \quad \dot{\gamma}_{33} = \mu\gamma_{33} \quad (25)$$

This system is simple and can be straightforwardly integrated; however, it is useful to deal with it more formally as a consequence of the existence of the unused symmetries corresponding to Λ_1 . The transformations in the space of the dependent variables $(\gamma_{11}, \gamma_{22}, \gamma_{33})$:

$$\tilde{\gamma}_{11} = \lambda_1\gamma_{11}, \quad \tilde{\gamma}_{22} = \lambda_2\gamma_{22}, \quad \tilde{\gamma}_{33} = \lambda_3\gamma_{33}, \quad (26)$$

constitute the symmetries of the system of the above equations. The three symmetry generators are:

$$X_1 = \gamma_{11} \frac{\partial}{\partial \gamma_{11}}, \quad X_2 = \gamma_{22} \frac{\partial}{\partial \gamma_{22}}, \quad X_3 = \gamma_{33} \frac{\partial}{\partial \gamma_{33}}. \quad (27)$$

The three corresponding generators commute; there is thus a transformation bringing them in canonical form, which is:

$$\gamma_{11} = e^a, \quad \gamma_{22} = e^b, \quad \gamma_{33} = e^c. \quad (28)$$

In the new dependent variables, the generators become:

$$X_1 = \frac{\partial}{\partial a}, \quad X_2 = \frac{\partial}{\partial b}, \quad X_3 = \frac{\partial}{\partial c}. \quad (29)$$

showing that the system will not depend on (a, b, c) , but only on their derivatives. Therefore, we have:

$$\dot{a} = \kappa, \quad \dot{b} = \lambda, \quad \dot{c} = \mu \Rightarrow \tag{30}$$

$$a = \kappa t + d_1, \quad b = \lambda t + d_2, \quad c = \mu t + d_3, \tag{31}$$

with $\kappa\lambda + \lambda\mu + \kappa\mu = 0$. The constants d_1, d_2, d_3 are absorbed with a scaling of x, y, z , and the line element is:

$$ds^2 = -e^{x+\lambda+\mu}tdt^2 + e^{\kappa t}dx^2 + e^{\lambda t}dy^2 + e^{\mu t}dz^2, \tag{32}$$

which is the Kasner solution [17].

3.2. Three Real with Two Equal Eigenvalues

In this case, the Θ matrix, using the restrictions from Equation (18), can be brought in the irreducible form:

$$\Theta = \begin{pmatrix} \theta_1 & 1 & 0 \\ 0 & \theta_1 & 0 \\ 0 & 0 & \theta_2 \end{pmatrix}, \tag{33}$$

where θ_1 and θ_2 are real valued. Extra restrictions on the θ 's are produced by Equation (15), resulting in:

$$-2\theta_1(\theta_1 + 2\theta_2) = 0. \tag{34}$$

The later equation implies that $\theta_2 \neq 0$ in order to have a non-trivial solution. Thus, by dividing it with θ_2^2 :

$$-2\theta_1(\theta_1 + 2\theta_2) = 0 \Rightarrow \lambda(\lambda + 2) = 0, \tag{35}$$

where $\lambda = \theta_1/\theta_2$. This equation has two solutions, $\lambda = 0$ and $\lambda = -2$. Therefore, using Equation (21) and then setting $\tau = \theta_2 t$, we evaluate the elements of γ :

$$\gamma_{11} = \gamma_{13} = \gamma_{23} = 0 \tag{36}$$

$$\gamma_{12} = c_1 e^{\lambda\tau} \quad \gamma_{22} = e^{\lambda\tau}(c_1 + c_2 t) \quad \gamma_{33} = c_3 e^{\lambda\tau}. \tag{37}$$

Finally, the line element without the non-essential constants is written:

$$ds^2 = -e^{(2\lambda+1)\tau}d\tau^2 + e^{\tau}dx^2 + \tau e^{\lambda\tau}dy^2 + 2e^{\lambda\tau}dydz. \tag{38}$$

This metric belongs to the Harrison [18] family of metrics and has a homothety produced by:

$$H = 2\partial_\tau + 2x\lambda\partial_x + (\lambda + 1)y\partial_y + (z(\lambda + 1) - y)\partial_z. \tag{39}$$

Furthermore, for the special case of $\lambda = 0$, this metric is a pp-wave.

3.3. Case with One Real and Two Complex Conjugate Eigenvalues

In this case, the corresponding Θ matrix is reduced to:

$$\Theta = \begin{pmatrix} \theta_1 & 0 & 0 \\ 0 & \theta_2 & \theta_3 \\ 0 & -\theta_3 & \theta_2 \end{pmatrix}, \tag{40}$$

where the eigenvalues are θ_1 , $\theta_2 + i\theta_3$ and $\theta_2 - i\theta_3$. If we follow the same procedure as previously, the matrix form for $\gamma_{\alpha\beta}$ is written as:

$$\gamma_{\alpha\beta} = \begin{pmatrix} c_1 e^{\theta_1 t} & 0 & 0 \\ 0 & e^{\theta_2 t} [c_2 \cos(\theta_3 t) - c_3 \sin(\theta_3 t)] & e^{\theta_2 t} [c_2 \cos(\theta_3 t) + c_3 \sin(\theta_3 t)] \\ 0 & e^{\theta_2 t} [c_2 \cos(\theta_3 t) + c_3 \sin(\theta_3 t)] & e^{\theta_2 t} [-c_2 \cos(\theta_3 t) + c_3 \sin(\theta_3 t)] \end{pmatrix}, \quad (41)$$

where c_1 , c_2 and c_3 are real constants. The first condition for these constants is produced by Equation (15) reading:

$$-2(2\theta_1\theta_2 + \theta_2^2 + \theta_3^2) = 0. \quad (42)$$

The constants θ_1 and θ_3 are non-zero, so it is possible to divide the former equation by $\theta_1^2\theta_3$, and then, redefine the constants as:

$$2\lambda + \beta\lambda^2 + \beta = 0 \rightarrow \beta = -\frac{2\lambda}{\lambda^2 + 1}, \quad \beta = \frac{\theta_3}{\theta_1}, \quad \lambda = \frac{\theta_2}{\theta_3}. \quad (43)$$

Therefore, by setting $\tau = \beta\theta_1 t$ and by doing proper transformations in order to absorb the non-essential constants, we produce the final metric:

$$ds^2 = -e^{(2\lambda+\beta^{-1})\tau} d\tau^2 + e^{\tau/\beta} dx^2 - e^{\lambda\tau} \sin \tau dy^2 + 2e^{\lambda\tau} \cos \tau dy dz + e^{\lambda\tau} \sin \tau dz^2, \quad (44)$$

which has a homothety generated by the following vector field:

$$H = -4\lambda\partial_\tau - 4\lambda^2 x \partial_x + (y(-\lambda+1) + 2\lambda z) \partial_y - (z(\lambda^2 - 1) + 2\lambda y) \partial_z. \quad (45)$$

The same metric has been produced for the first time by Harrison [18], who however used a different approach.

3.4. Case with Three Equal Eigenvalues

In this case, the Θ matrix is:

$$\Theta = \begin{pmatrix} \theta_1 & 1 & 0 \\ 0 & \theta_1 & 1 \\ 0 & 0 & \theta_1 \end{pmatrix}. \quad (46)$$

If we follow the same procedure as previously, the matrix form for $\gamma_{\alpha\beta}$ is written:

$$\gamma_{\alpha\beta} = e^{\theta_1 t} \begin{pmatrix} 2c_1 & -2c_2 t & 2c_3 \\ -2c_2 t & 1 & 0 \\ 2c_3 & 0 & 0 \end{pmatrix}, \quad (47)$$

where c_1 , c_2 and c_3 are real constants. From Equation (15), we have that $\theta_1 = 0$.

Thus, by doing proper transformations in order to absorb the non-essential constants, we produce the final metric:

$$ds^2 = dt^2 + 2t^2 dx^2 + dy^2 - 4t dx dy + 4t dx dz, \quad (48)$$

which has homothety described by the following vector field:

$$H = t\partial_t + y\partial_y + 2z\partial_z. \quad (49)$$

Furthermore, this metric is a pp-wave because for the Killing field $u = \xi_3 = \partial_z$, we have $u^\alpha_{;\beta} = 0$ and $u^\alpha u_\alpha = 0$.

4. Diagonalizability of $\gamma_{\alpha\beta}$ for Types VIII–IX

Let us begin by recalling that, in three dimensions, the structure constants tensor $C_{\beta\gamma}^\alpha$ can be represented as [19]:

$$C_{\beta\gamma}^\alpha = m^{\alpha\delta} \epsilon_{\delta\beta\gamma} + v_\beta \delta_\gamma^\alpha - v_\gamma \delta_\beta^\alpha, \tag{50}$$

with $m^{\alpha\beta}$ a symmetric tensor density of weight -1 and $v_\beta = \frac{1}{2} C_{\beta\alpha}^\alpha$. The condition that Λ is an automorphism of the Lie algebra translates into the following requirements:

$$m^{\alpha\beta} = |S|^{-1} S_\gamma^\alpha S_\delta^\beta m^{\gamma\delta} \tag{51}$$

$$v_\alpha = \Lambda_\alpha^\beta v_\beta. \tag{52}$$

The symmetry groups for Bianchi VIII and IX $m^{\alpha\beta}$ are characterized by:

$$m^{\alpha\beta} = \text{diag}(\epsilon, 1, 1) \text{ and } v_\beta = 0, \tag{53}$$

where $\epsilon = -1$ for Bianchi VIII and $\epsilon = 1$ for Bianchi IX.

From Equations (51) and (52), we can deduce that $\Lambda_\beta^\alpha(t)$ is an element of the three-dimensional proper Lorentz group or the group of rotations for VIII and IX respectively. Since there are no other restrictions on $\Lambda_\beta^\alpha(t)$, $P^\alpha(t)$ takes the form:

$$P^\alpha = \frac{1}{4\epsilon} \epsilon_{\beta\tau\kappa} m^{\alpha\beta} \Lambda_\gamma^\tau \Lambda_\delta^\kappa m^{\gamma\delta}. \tag{54}$$

Due to the fact that i) $\Lambda_\beta^{\alpha'}$ s are the isometries of the Minkowski metric in the first case and of the Euclidean in the second one and ii) $\gamma_{\alpha\beta}$ is positive definite and thus always diagonalizable, we arrive at the reduced form $\gamma_{\alpha\beta} = \text{diag}(a^2(t), b^2(t), c^2(t))$.

At this point, having spent our freedom in three arbitrary functions of time, we are left with the unknown lapse and shift in addition to $a(t)$, $b(t)$, $c(t)$. Invoking the linear constraint Equation (3b), we get the relations:

$$N^1 (b(t)^2 - c(t)^2) = 0 \quad N^2 (a(t)^2 - c(t)^2) = 0 \quad N^3 (a(t)^2 - b(t)^2) = 0, \tag{55}$$

which in the generic case ($a(t) \neq b(t) \neq c(t) \neq a(t)$) imply that the shift vector is zero. The particular cases of axisymmetry are analyzed as follows:

Case IX Suppose $a(t) = b(t)$. Then, Equation (55) implies that $N^1 = N^2 = 0$ and $N^3(t)$ is unrestricted.

However, there is an extra rotation in the plane (1–2), which has no effect on the form of $\gamma_{\alpha\beta}$. This particular matrix, being an automorphism, can be used to absorb the N^3 ; see Equations (9) and (54). The situation with $a(t) = c(t)$ or $b(t) = c(t)$ is exactly the same.

Case VIII For the case $b(t) = c(t)$, we follow exactly the same reasoning and arrive at zero shift, as well. Of particular interest and less known is the fact that the case $b(t) = a(t)$ or $c(t) = a(t)$ leads to the incompatibility of the resulting Einstein Equation (3c) (for the first time reported in [5]); they require $-\frac{2n(t)^2}{a(t)} = 0$. This fact can, in view of the transformations (9), be understood as follows: while the transformation matrix that leaves the form of $\gamma_{\alpha\beta}$ invariant is still a rotation, the corresponding allowed one is a boost; thus, the form of $\gamma_{\alpha\beta}$ becomes block diagonal when the shift is zero, showing the incompatibility.

5. Reduced Dynamics

Our starting point is the action of Einstein’s gravity plus matter:

$$S = \frac{c^4}{16\pi G} \int \sqrt{-g} R d^4x + S_m, \tag{56}$$

where g is the determinant of the space-time metric $g_{\mu\nu}$, R the Ricci scalar and S_m the action of the matter content. In the case that the manifold has a specific group of isometries, such as spatial homogeneity, the line element can be decomposed in the following form:

$$ds^2 = -N(t)^2 dt^2 + \gamma_{\kappa\lambda}(t) \sigma_i^\kappa(x) \sigma_j^\lambda(x) dx^i dx^j, \quad (57)$$

where N is the lapse function. Note that the more general line element should also have a shift term ($2N_\alpha \sigma_i^\alpha(x) dx^i dt$), which can always be made to vanish through a particular coordinate transformation [10] (however, one has to keep in mind that in several cases, it may so happen that this absorption can be performed only locally, since it can result in altering the topology of the time axis). If one inserts this line element into Einstein's equations (obtained by varying the action with respect to $g_{\mu\nu}$):

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}, \quad (58)$$

with $T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}}$, one arrives at a set of coupled ordinary differential equations with t as the independent dynamical variable. On the other hand, if the action (56) is calculated using the decomposed metric (57), integrating out the non-dynamical degrees of freedom, the full gravitational system may be successfully described by the reduced action. If this happens, this action is called valid. This reduced system has finite degrees of freedom, and it is thus more easily quantized.

The Lagrangian of these mini-superspace systems has the general form (for simplicity in what follows, we choose to work in units $c = G = \hbar = 1$, a choice that reflects the typical geometric and quantum mechanical units):

$$L = \frac{1}{2N(t)} \bar{G}_{\alpha\beta}(q) q'^\alpha(t) q'^\beta(t) - N(t) V(q), \quad (59)$$

where $' = \frac{d}{dt}$, while $V(q)$ and $\bar{G}_{\alpha\beta}(q)$ are the mini-superspace potential and metric, respectively. The variables q^α are functions of the matrix scale factors $\gamma_{\mu\nu}$, i.e., $q^\alpha = q^\alpha(\gamma_{\mu\nu})$. This Lagrangian corresponds to a singular system. The associated Hamiltonian is written as:

$$H_T = N \mathcal{H} + u_N p_N,$$

which is produced with the help of the Dirac–Bergmann algorithm for singular systems [20,21]. This Hamiltonian has the following primary and secondary constraints:

$$p_N \approx 0, \quad (60a)$$

$$\mathcal{H} = \frac{1}{2} \bar{G}^{\alpha\beta} p_\alpha p_\beta + V(q) \approx 0. \quad (60b)$$

In what follows, we will use the constant potential parametrization in which all the information about the system is included inside the mini-superspace metric. This parametrization is achieved if we adopt the following transformation $N \mapsto n = NV$; thus, the transformed Lagrangian is:

$$L = \frac{1}{2n(t)} G_{\alpha\beta}(q) q'^\alpha(t) q'^\beta(t) - n(t), \quad (61)$$

with $G_{\alpha\beta} = V \bar{G}_{\alpha\beta}$ being the new mini-superspace metric. The corresponding Hamiltonian constraint becomes:

$$\mathcal{H} = \frac{1}{2} G^{\alpha\beta} p_\alpha p_\beta + 1 \approx 0.$$

It is easily proven that there can be conserved quantities (linear in momenta), modulo the constraint [22], with the general form:

$$Q_I = \xi_I^\alpha(q) p_\alpha + \int n(t) \omega_I(q(t)) dt, \quad (62)$$

with:

$$\mathcal{L}_\xi G_{\alpha\beta} = \omega(q) G_{\alpha\beta},$$

where \mathcal{L}_ξ is the Lie derivative with respect to the configuration space vector ξ . The above conserved charges correspond to conformal ($\omega(q) \neq 0$), Killing ($\omega(q) = 0$) or homothetic ($\omega(q) = \text{constant}$) vector fields of $G_{\alpha\beta}$. The charges that correspond to Killing vector fields strongly commute with the Hamiltonian, a property that is extremely useful in the process of quantization. In the system under consideration, there could arise non-trivial higher order symmetries; see for example [23–26]. These symmetries are generated by second order Killing tensors, defined through:

$$\nabla_\mu K_{\nu\lambda} + \nabla_\lambda K_{\mu\nu} + \nabla_\nu K_{\lambda\mu} = 0, \quad (63)$$

$$K_{\mu\nu} = K_{\nu\mu}. \quad (64)$$

These tensors are separated into two categories: the first includes the trivial tensors constructed by tensor products of Killing vector fields:

$$K_{\mu\nu} = \frac{1}{2} (\xi_\mu \otimes \xi_\nu + \xi_\nu \otimes \xi_\mu), \quad (65)$$

or the metric itself, while the second contains all the rest, which are designated as non-trivial.

The corresponding phase space quantities $K = K^{\mu\nu} p_\mu p_\nu$ have vanishing Poisson brackets with the Hamiltonian constraint and therefore constitute constants of motion:

$$K = K^{\mu\nu} p_\mu p_\nu \Rightarrow \{K, \mathcal{H}\} = 0. \quad (66)$$

The next step is to construct a canonical quantization scheme for the Lagrangian (61). We assume that the mini-supermetric $G_{\alpha\beta}$ possesses some Killing vector fields ξ_I and some Killing tensor fields K_J where I, J are indices that label each of them. To each Killing vector and tensor field corresponds an integral of motion. We begin by assigning differential operators to momenta and replace Poisson brackets by commutators:

$$p_n \mapsto \widehat{p}_n = -i \frac{\partial}{\partial n}, \quad p_\alpha \mapsto \widehat{p}_\alpha = -i \frac{\partial}{\partial q^\alpha}, \quad \{, \} \rightarrow -i[,],$$

while the operators corresponding to q^α are considered to act multiplicatively. In order to solve the factor ordering problem of the kinetic term of \mathcal{H} , we choose the conformal Laplacian (or Yamabe operator) (see e.g., [27]),

$$\widehat{\mathcal{H}} = -\frac{1}{2\mu} \partial_\alpha (\mu G^{\alpha\beta} \partial_\beta) + \frac{d-2}{8(d-1)} \mathcal{R} + 1, \quad (67)$$

where $\mu(q) = \sqrt{|\det G_{\alpha\beta}|}$, $\partial_\alpha = \frac{\partial}{\partial q^\alpha}$, \mathcal{R} is the Ricci scalar and d the dimension of the mini-superspace. This choice is essentially unique on account of:

1. The requirement for the operator to be scalar under coordinate transformations of the configuration space variables q^d .
2. The requirement to contain up to second derivatives of $G_{\mu\nu}$ since the classical constraint is quadratic in momenta.

3. The requirement to be covariant under conformal scalings of $G_{\mu\nu}$, since this is also a property of the classical system.

A further property of the definition (67) is that it will be Hermitian in the Hilbert space of the quantum states, if of course appropriate boundary conditions are fulfilled, e.g., square integrability of the derivatives of $\Psi(q)$ [28]. However, in view of the fact that any particular combination of the q^a 's can, at the classical level, be considered as representing the time, such a property is not in general expected to hold. This can be considered as a reflection of the famous problem of time in quantum gravity/cosmology [29,30].

As far as the classical symmetries (62) and (66) are concerned, they can naturally be transferred to operators by formally assigning to Q_I the general expression for linear first order, Hermitian operators [31] and to K_J a pseudo-Laplacian operator [32]; thus, the corresponding forms are respectively:

$$\widehat{Q}_I = -\frac{\hbar}{2\mu} (\mu \zeta_I^\alpha \partial_\alpha + \partial_\alpha (\mu \zeta_I^\alpha)) = -\hbar \zeta_I^\alpha \partial_\alpha, \quad (68)$$

$$\widehat{K}_J = -\frac{1}{\mu} \partial_\alpha [\mu K_J^{\alpha\beta} \partial_\beta]. \quad (69)$$

In (68), the last equality holds due to the ζ_I 's being Killing vector fields. The linear symmetries exactly commute with the Hamiltonian only in the constant potential parametrization [33] i.e., commutators of the following form are zero:

$$[\widehat{Q}_I, \widehat{\mathcal{H}}] = 0. \quad (70)$$

By virtue of the above relations, it is possible to use some of the \widehat{Q}_I 's as quantum observables together with $\widehat{\mathcal{H}}$ as long as they have a common set of eigenfunctions. Therefore, our quantum system will obey the following conditions:

$$\widehat{p}_n \Psi = 0 \quad (71a)$$

$$\widehat{\mathcal{H}} \Psi = 0 \quad (71b)$$

$$\widehat{Q}_I \Psi = \kappa_I \Psi, \quad (71c)$$

where the first is restriction induced by the primary constraint and the second one is the Wheeler–DeWitt equation. The number of \widehat{Q}_I that can be consistently imposed on the wave function is prescribed by the integrability condition [34]:

$$C_{IJ}^M \kappa_M = 0, \quad (72)$$

where κ_M are the eigenvalues and C_{IJ}^M the structure constants of the sub-algebra under consideration.

Similar considerations apply when some of the \widehat{K}_J are also used. In order for these quadratic quantum observables to be consistently imposed, there are geometric conditions involving the metric and the $K_J^{\alpha\beta}$'s that need to be satisfied [32].

6. Massless Field in the FLRW Universe

6.1. Classical Treatment

We now turn to the study of FLRW geometries [35]. These belong to the Bianchi classification; note, however, that they appear as special solutions of several Bianchi types. The spatially closed case belongs to the Bianchi Type IX model, the spatially open to Type V, while the spatially flat to the

Bianchi Type I. We solve in detail the $k \neq 0$ cases and only mention the final result for the spatially flat case. The general form of the line element of an FLRW space-time is:

$$ds^2 = -N^2(t)dt^2 + a^2(t) \left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \right), \quad (73)$$

where $N(t)$ is the lapse function and $a(t)$ is the scale factor. For this space-time, the total Lagrangian for the gravity plus matter system is:

$$L = 6Nka - \frac{6a\dot{a}^2}{N} + \frac{a^3\dot{\phi}^2}{2N}, \quad (74)$$

where a term of a total time derivative has been discarded. This Lagrangian has a singular form, and the procedure of integration of the system via the conditional symmetries applies here. To this end, we have to turn to the constant potential parametrization (For $k = 0$, the Lagrangian is already in the constant potential parametrization. This results in the existence of numerous rheonomic integrals of motion corresponding to the infinite number of conformal Killing fields in two dimensions.) by setting $n = 6kNa$, resulting in:

$$L = n - \frac{36ka^2\dot{a}^2}{n} + \frac{3ka^4\dot{\phi}^2}{n}, \quad (75)$$

with the corresponding Hamiltonian constraint and supermetric respectively:

$$\mathcal{H} = -\frac{p_a^2}{72ka^2} + \frac{p_\phi^2}{12ka^4} - 1 \approx 0, \quad G_{\alpha\beta} = 6ka \begin{pmatrix} -12a & 0 \\ 0 & a^3 \end{pmatrix}. \quad (76)$$

This supermetric represents a flat two-dimensional space, admitting the following three symmetries and the homothety:

$$\xi_1 = \frac{e^{\phi/\sqrt{3}}}{a} \partial_a - \frac{2\sqrt{3}e^{\phi/\sqrt{3}}}{a^2} \partial_\phi, \quad \xi_2 = \frac{e^{-\phi/\sqrt{3}}}{a} \partial_a + \frac{2\sqrt{3}e^{-\phi/\sqrt{3}}}{a^2} \partial_\phi, \quad \xi_3 = \partial_\phi, \quad \xi_h = \frac{a}{4} \partial_a, \quad (77)$$

where the numbered indices denote the Killing fields, while h denotes the homothetic field. These symmetry generators satisfy a Lie bracket algebra with the following non-vanishing elements:

$$[\xi_1, \xi_3] = -\frac{1}{\sqrt{3}} \xi_1, \quad [\xi_2, \xi_3] = \frac{1}{\sqrt{3}} \xi_2, \quad [\xi_1, \xi_h] = \frac{1}{2} \xi_1, \quad [\xi_2, \xi_h] = \frac{1}{2} \xi_2. \quad (78)$$

For the case $k = 0$, the corresponding algebra is the same, but with structure constant coefficients being $C_{31}^1 = C_{23}^2 = \frac{\sqrt{3}}{4}$, $C_{1h}^1 = C_{2h}^2 = \frac{1}{2}$. The corresponding integrals of motion in the configuration space are:

$$Q_1 = -\frac{12e^{\frac{\phi}{\sqrt{3}}}ka \left(6\dot{a} + \sqrt{3}a\dot{\phi} \right)}{n}, \quad Q_2 = \frac{12e^{-\frac{\phi}{\sqrt{3}}}ka \left(-6\dot{a} + \sqrt{3}a\dot{\phi} \right)}{n}, \quad (79)$$

$$Q_3 = \frac{6ka^4\dot{\phi}}{n}, \quad Q_h = -\frac{18ka^3\dot{a}}{n} + \int dt n(t). \quad (80)$$

In order to determine the line element, the system $Q_i = \kappa_i$ for $i = 1, 2, 3, h$ with κ_i constants is algebraically solved. The solution is:

$$a = \frac{2 \times 3^{1/4} \sqrt{\kappa_3} e^{\frac{\phi}{2\sqrt{3}}}}{\sqrt{-\kappa_1 + \kappa_2 e^{\frac{2\phi}{\sqrt{3}}}}}, \quad (81)$$

$$\dot{a} = -\frac{\sqrt{\kappa_3} e^{\frac{\phi}{2\sqrt{3}}} (\kappa_1 + \kappa_2 e^{\frac{2\phi}{\sqrt{3}}}) \dot{\phi}}{3^{1/4} (-\kappa_1 + \kappa_2 e^{\frac{2\phi}{\sqrt{3}}})^{3/2}}, \quad (82)$$

$$n = \frac{288 \kappa_3 e^{\frac{2\phi}{\sqrt{3}}} k \dot{\phi}}{(\kappa_1 - \kappa_2 e^{\frac{2\phi}{\sqrt{3}}})^2}, \quad (83)$$

$$\int dt n(t) = -\kappa_h - \frac{\sqrt{3} \kappa_3 (\kappa_1 + \kappa_2 e^{\frac{2\phi}{\sqrt{3}}})}{2(\kappa_1 - \kappa_2 e^{\frac{2\phi}{\sqrt{3}}})}, \quad (84)$$

while a relation for the constants appearing in the system is also found by the only non-trivial consistency condition $\frac{\partial}{\partial t} \int dt n(t) = n$,

$$\kappa_1 \kappa_2 + 144k = 0. \quad (85)$$

This relation is the constraint equation of the theory, and it is interesting to note that the Casimir invariant of the algebra is the term $Q_1 Q_2$, which is also the kinetic part of the Hamiltonian.

For the case of $k = 0$, the system of equations is $Q_i = \kappa_i$, where $i = 1, 2, 3, h$ must be solved by setting one of the constants κ_1, κ_2 equal to zero. This is necessary because of the vanishing of the constraint and the fact that its kinetic part when expressed with respect to Q_i 's equals $Q_1 Q_2$. As before, it is also here true that this product is the Casimir invariant of the algebra.

At this stage, the solution will still contain an arbitrary function of time, representing the time reparametrization invariance (since no gauge fixing has been so far assumed in the derivation). Choosing the gauge $\phi = \ln t$, we find that the solution is:

$$a = \frac{2 \times 3^{1/4} \sqrt{\kappa_3} t^{\frac{1}{2\sqrt{3}}}}{\sqrt{-\kappa_1 + \kappa_2 t^{2/\sqrt{3}}}}, \quad N = \frac{8 \times 3^{3/4} \sqrt{\kappa_3} t^{-1 + \frac{\sqrt{3}}{2}}}{(-\kappa_1 + \kappa_2 t^{2/\sqrt{3}})}. \quad (86)$$

By inserting it in the line element (73) and performing proper coordinate transformations in order to absorb the redundant constants, the final line element of the space-time is:

$$ds^2 = -\frac{\lambda}{4\sqrt{T}(1+Tk)^3} dT^2 + \frac{\lambda\sqrt{T}}{(1+Tk)} \left(\frac{dr^2}{1-r^2k} + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \right). \quad (87)$$

This geometry has one essential constant, as can be shown by using the methodology of [36,37]. The Ricci scalar is:

$$R = -\frac{3(Tk+1)^3}{2T^{3/2}\lambda}, \quad (88)$$

where we have set $\lambda = -\frac{\kappa_3}{\sqrt{3}k^{3/2}}$ rendering the metric element singular for both $T \rightarrow 0$ and $T \rightarrow \infty$. The same steps for the spatially flat case lead us to the solution:

$$ds^2 = -dT^2 + T^{2/3} dr^2 + T^{2/3} r^2 d\theta^2 + T^{2/3} r^2 \sin^2 \theta d\varphi^2. \quad (89)$$

This space-time metric is conformally flat with Ricci scalar:

$$R = -\frac{2}{3T^2}. \quad (90)$$

It does not contain any essential constants characterizing the geometry and the matter content of the space-time, and as can be seen from the form of the Ricci scalar, a singularity for $T \rightarrow 0$ appears.

6.2. Canonical Quantization and Semiclassical Analysis

We next canonically quantize the classical system. Promoting the constraints, as well as the first integrals Q_i to operators and imposing them on the wave function gives the following quantum equations:

$$\hat{Q}_1 \Psi = -\frac{ie^{\phi/\sqrt{3}}(-6\partial_\phi \Psi + \sqrt{3}a\partial_a \Psi)}{\sqrt{3}a^2} = \kappa_1 \Psi, \quad (91)$$

$$\hat{Q}_2 \Psi = -\frac{ie^{-\phi/\sqrt{3}}(6\partial_\phi \Psi + \sqrt{3}a\partial_a \Psi)}{\sqrt{3}a^2} = \kappa_2 \Psi, \quad (92)$$

$$\hat{Q}_3 \Psi = -i\partial_\phi \Psi = \kappa_3 \Psi, \quad (93)$$

$$\hat{\mathcal{H}} \Psi = \frac{-144ka^4 \Psi - 12\partial_{\phi\phi} \Psi + a(\partial_a \Psi + a\partial_{aa} \Psi)}{144ka^4} = 0, \quad (94)$$

where the measure is $\mu(a, \phi) = 6\sqrt{3}a^3k$. The quantum equations $\hat{Q}_i \Psi = \kappa_i \Psi$ that can be imposed simultaneously according to the condition (72) are the two-dimensional (\hat{Q}_1, \hat{Q}_2) and the one-dimensional $\hat{Q}_1, \hat{Q}_2, \hat{Q}_3$. The one-dimensional sub-algebras spanned by the operators \hat{Q}_1, \hat{Q}_2 give solutions that are special cases of the two-dimensional case [38].

6.2.1. Subalgebra (\hat{Q}_1, \hat{Q}_2)

For the case of the two-dimensional sub-algebra, we solve Equations (91), (92) and (94). The solution for the wave function is:

$$\Psi = A \exp\left(i\frac{a^2}{4}(\kappa_1 e^{-\frac{\phi}{\sqrt{3}}} + \kappa_2 e^{\frac{\phi}{\sqrt{3}}})\right). \quad (95)$$

The semiclassical analysis is next performed following Bohm as explained in the Introduction. This wave function is written in polar form, and we can see that the amplitude Ω is constant. Therefore, the quantum potential will vanish, rendering the solution for this case the same as the classical metric. Indeed, if we solve the semiclassical solutions:

$$\frac{1}{2}a \left(-e^{-\phi/\sqrt{3}}(\kappa_1 + e^{2\phi/\sqrt{3}}\kappa_2)\right) = \frac{144ka\dot{a}}{n}, \quad (96)$$

$$\frac{a^2}{12} \left(\sqrt{3}e^{-\phi/\sqrt{3}}(\kappa_1 - e^{2\phi/\sqrt{3}}\kappa_2)\right) = -\frac{72ka^2\dot{\phi}}{n}, \quad (97)$$

with phase function $S = \frac{1}{4}a^2 e^{-\frac{\phi}{\sqrt{3}}}(\kappa_1 + \kappa_2 e^{\frac{2\phi}{\sqrt{3}}})$, we indeed find the same line element as in the classical case. The same conclusion also holds for $k = 0$.

6.2.2. Subalgebra \hat{Q}_3

In the case of the one-dimensional algebra, the system of equations is formed by Equations (93) and (94). The wave function is:

$$\Psi_{cl}(a, \phi) = e^{i\phi\kappa_3} (A_1 I_{-i\sqrt{3}\kappa_3}(6a^2) + B_1 I_{i\sqrt{3}\kappa_3}(6a^2)), \tag{98}$$

$$\Psi_{op}(a, \phi) = e^{i\phi\kappa_3} (A_2 J_{-i\sqrt{3}\kappa_3}(6a^2) + B_2 J_{i\sqrt{3}\kappa_3}(6a^2)), \tag{99}$$

for the closed and open case respectively. In order to write the wave function in polar form, for the sake of the semiclassical analysis, approximation limits are taken for small and large arguments of the Bessel functions. The use of the simplifying assumptions $A_1 = B_1$, $A_2 = B_2$ renders the, common for the two cases, wave function:

$$\Psi_{sm} \approx c_1 e^{i\kappa_3\phi} \cos \ln a. \tag{100}$$

Similarly, for the large values assuming again $A_1 = B_1$, $A_2 = B_2$, the wave function becomes:

$$\Psi_{la}^{cl} \approx \frac{e^{a^2}}{a} e^{i\kappa_3\phi}, \quad \Psi_{la}^{op} \approx \frac{\sin(6a^2)}{a} e^{i\kappa_3\phi}. \tag{101}$$

The quantum potential for small values does not vanish $\mathcal{Q}_{sm} = \frac{1}{144ka^4}$, while $\mathcal{Q}_{la}^{cl} = -\frac{1+4a^4}{144a^4k}$ and $\mathcal{Q}_{la}^{op} = \frac{144ka^4-1}{144ka^4}$. The phase function is $S = \kappa_3\phi$. The solution of the semiclassical equations with respect to (a, n) is:

$$a = c, \quad n = \frac{6ka^4}{\kappa_3} \dot{\phi}, \tag{102}$$

and has a remaining freedom for the scalar field, which we select to be such that the lapse function $N(t)$ of the semiclassical element is the same as for the classical, that is:

$$\phi(t) = - \frac{8 \times 3^{3/4} t^{\sqrt{3}/2} \sqrt{-\frac{48kt^2/\sqrt{3}}{\kappa_1} - \frac{\kappa_1}{3} \kappa_3^{3/2}} \left(-3 + \sqrt{1 + \frac{144kt^2/\sqrt{3}}{\kappa_1^2}} {}_2F_1 \left(\frac{1}{2}, \frac{3}{4}; \frac{7}{4}; -\frac{144kt^2/\sqrt{3}}{\kappa_1^2} \right) \right)}{c^3 (144kt^2/\sqrt{3}\kappa_1 + \kappa_1^3)}, \tag{103}$$

where ${}_2F_1(a, b; c; d)$ is the Gauss hypergeometric function. Inserting the solution in the four-dimensional element and after proper coordinate transformations, the space-time metric is written:

$$ds^2 = -\frac{\lambda}{4\sqrt{T}(1+T\epsilon)^3} dt^2 + \frac{1}{1-\epsilon r^2} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2, \tag{104}$$

where the sign (+) accounts for the closed case and (−) for the open case while the identification $c^2 = \frac{\lambda^2}{16}$ has been considered in order for the constant λ to coincide with that one in the classical metric. This space-time has the interesting property of having a constant Ricci scalar $R = 6k$, all higher derivatives of its Riemann tensor zero and all curvature scalars constructed from its Riemann tensor constant. Hence, there is no curvature and/or a higher derivative curvature singularity.

Following the same procedure for the spatially-flat case, we find that the wave function is of the form:

$$\Psi(a, \phi) = e^{i\kappa_3\phi} \left(A_3 \cos(2\sqrt{3}\kappa_3 \ln a) + B_3 \sin(2\sqrt{3}\kappa_3 \ln a) \right).$$

Note that in this case, there is no need to make an approximation in order to write it in polar form as it is already in this form with $\Omega = A_3 \cos(2\sqrt{3}\kappa_3 \ln a) + B_3 \sin(2\sqrt{3}\kappa_3 \ln a)$ and $S = \kappa_3\phi$. The line element we obtain is:

$$ds^2 = -dT^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2. \tag{105}$$

This space-time is the Minkowski space-time and does not contain any essential constants. Therefore, there is no singularity for the range of all times, since no approximation limits have been considered here.

7. Resume

We have seen how the automorphic motions of the scale factor matrix, the lapse and the shift arise from particular space-time coordinate transformation. These motions can be used to either set the shift vector to zero or simplify the scale factor matrix. In the former case, as we have seen in the example of Bianchi Type I (Section 3), the remaining rigid symmetry of constant automorphisms provides Lie point symmetries of the equations of motion. In the latter case, once the time-dependent automorphic matrices have been used to simplify $\gamma_{\alpha\beta}(t)$, the linear constraints provide some extra information about the shift. Thus, in Section 4, we have given the examples of Bianchi Types VIII and IX, showing how the linear constraints dictate the vanishing of the shift vector when $\gamma_{\alpha\beta}$ is diagonalized.

When the equations admit a reduced Lagrangian, further symmetries of the configuration space metric can be present, which result in linear local and non-local integrals of motion, as well as quadratic (higher order symmetries produced by Killing tensors). A general discussion about these subjects is given in Section 5, at the classical level. Moreover, at the quantum level, the WDW (Wheeler–De Witt) equation can be supplemented by the quantum analog of the integrals of motion; thus, unique wave functions can be obtained, which exhibit a non-singular semi-classical behavior. An example of such a type of treatment is given in Section 6, where a massless scalar field coupled to an FLRW geometry is presented.

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