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Nonoscillatory Solutions to Second-Order Neutral Difference Equations

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Abstract: We study asymptotic behavior of nonoscillatory solutions to second-order neutral difference equation of the form: $\Delta(r_n\Delta(x_n + p_n x_{n-\tau})) = a_n f(n, x_n) + b_n$. The obtained results are based on the discrete Bihari type lemma and a Stolz type lemma.

Keywords: second-order difference equation; asymptotic behavior; nonoscillatory solution; quasi-difference

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1. Introduction

We are concerned with the following nonlinear second-order difference equations

$$\Delta(r_n\Delta(x_n + p_n x_{n-\tau})) = a_n f(n, x_n) + b_n, \tag{1}$$

where

$$\tau \in \mathbb{N},\ r_n, a_n, b_n, p_n \in \mathbb{R}, \quad f : \mathbb{N} \times \mathbb{R} \to \mathbb{R}, \quad r_n > 0, \quad p_n \to \lambda \in \mathbb{R}.$$

Here $\mathbb{N}$, $\mathbb{R}$ denote the set of nonnegative integers and all real numbers, respectively. By a *solution* of Equation (1), we mean a sequence x which satisfies Equation (1) for all large n. A solution x is said to be *nonoscillatory* if it is eventually positive or eventually negative; otherwise, it is called oscillatory.

In the sequel, we will use the following notation:

$$r_n^* = \sum_{i=1}^{n-1} \frac{1}{r_n}, \tag{2}$$

by convention $r_1^* = 0$.

The second-order difference equations have been a subject of numerous studies. In particular, investigation of neutral difference equations is important since such equations have applications in various problems of physics, biology, and economics. Recently, there have been many papers devoted to the oscillation of solutions to equations of the type defined by Equation (1) (see, for example, [1–8] and the references cited therein). In comparison with oscillation, there are not as many results on the nonoscillation of these equations.

The asymptotic behavior of solutions of Equation (1) in the case $p_n \equiv 0$ has been studied for several decades by many authors ([5,9–14]), while some generalizations on time-scale variants of the equation have been studied in [15–17]. However, there are relatively few works devoted to the study of the asymptotic behavior of nonoscillatory solutions expressed by Equation (1) when $p_n \not\equiv 0$. In 2003, using the Leray–Schauder theorem, Agarwal et al. [18] obtained sufficient conditions for the existence of nonoscillatory solutions for the discrete equation

$$\Delta(r_n\Delta(x_n + px_{n-k})) + F(n+1, x_{n+1-\sigma}) = 0.$$

Liu et al. in [19], proved the existence of uncountably many bounded nonoscillatory solutions to the problem

$$\Delta(r_n\Delta(x_n + px_{n-k})) + f(n, x_{n-d_1 n}, \dots, x_{n-d_{kn}}) = c_n,$$

using Banach's fixed point theorem, under the Lipschitz continuity condition. Galewski et al. [20] studied the existence of a bounded solution to the more general equation

$$\Delta(r_n\,(\Delta(x_n + p_n x_{n-k}))^{\gamma}) + q_n x_n^{\alpha} + a_n f(n, x_{n+1}) = 0,$$

using the techniques connected with the measure of noncompactness. Some sufficient conditions for the existence of a nonoscillatory solution to the equation

$$\Delta(r_n\Delta(x_n + px_{n-\tau})) + a_n f(x_{n-k}) - b_n x_{n-l} = 0,$$

for $p \neq -1$ were obtained by Tian et al. in [21]. Moreover, for classification of nonoscillatory solutions to equations of the type defined by Equation (1), see [22–26].

In [27], the following equation was considered:

$$\Delta^2(x_n + px_{n-\tau}) = a_n f(n, x_n) + b_n.$$

The results obtained in [27] were extended to higher-order equations in [28]. In this paper, we present generalizations in a different direction, namely to difference equations with quasi-difference of the type defined by Equation (1). In Theorem 1, using the discrete Bihari type lemma and discrete L'Hospital's type lemma, we obtain sufficient conditions, under which all nonoscillatory solutions of Equation (1) have the property

$$x_n = cr_n^* + \mathrm{o}(r_n^*).$$

Moreover, in Theorem 2, we show that, under some additional conditions, all nonoscillatory solutions of Equation (1) have the property

$$x_n = cr_n^* + d + \mathrm{o}(1).$$

The results are new even for linear equations of the type defined by Equation (1) and when $p_n = 0$. We also present applications of the obtained results to some special cases of Equation (1).

2. Main Results

For the proof of the main results, we will need some auxiliary lemmas.

Lemma 1. *Assume x, p, z are real sequences, x is bounded, $\tau \in \mathbb{N}$,*

$$z_n = x_n + p_n x_{n-\tau},$$

for $n \geq \tau$, $p_n \to \lambda \in \mathbb{R}$, $|\lambda| \neq 1$, and $z_n \to \alpha \in \mathbb{R}$. Then x is convergent and

$$\lim_{n\to\infty} x_n = \frac{\alpha}{1+\lambda}.$$

Proof. The assertion is a consequence of ([27], Lemma 1). □

Remark 1. *Lemma 1 was essentially proved in Lemma 1 in [29], where the case of complex sequences was studied in detail for the case of constant sequence p_n. For the case of sequences in Banach spaces, see Lemma 1 in [30].*

The following lemma is a discrete version of Bihari type lemma.

Lemma 2. *Assume a, w are real sequences, $n_0 \in \mathbb{N}$, $g : [0, \infty) \to [0, \infty)$, $\lambda \in [0, \infty)$,*

$$\sum_{n=1}^{\infty} |a_k| < \infty, \quad g(\lambda) > 0, \quad \int_{\lambda}^{\infty} \frac{ds}{g(s)} = \infty, \quad |w_n| \le \lambda + \sum_{k=n_0}^{n-1} |a_k| g(|w_k|),$$

for $n \ge n_0$, and g is nondecreasing. Then the sequence w is bounded.

Proof. The assertion is a consequence of ([28], Lemma 4.1). □

In the proof of Theorem 1, we will use the following Stolz-type lemma, which should be a folklore one, but it is difficult to find a specific reference in the literature. Because of this, for the completeness and benefit of the reader, we will provide a proof of the lemma.

Lemma 3. *Assume x, y are real sequences, y is bounded and eventually strictly monotonic, and the sequence $(\Delta x_n / \Delta y_n)$ is convergent. Then the sequence x is convergent. Moreover, if $\lim_{n\to\infty} y_n \neq 0$, then the sequence (x_n / y_n) is convergent.*

Proof. First assume that the sequence y is eventually increasing. Let $\varepsilon > 0$ and

$$L = \lim_{n\to\infty} \frac{\Delta x_n}{\Delta y_n}.$$

Choose an index k such that

$$L - \varepsilon \le \frac{\Delta x_n}{\Delta y_n} \le L + \varepsilon \quad \text{and} \quad \Delta y_n > 0,$$

for $n \ge k$. Then

$$(L - \varepsilon)\Delta y_n \le \Delta x_n \le (L + \varepsilon)\Delta y_n,$$

for $n \ge k$. Summing from k to $n - 1$, we obtain

$$(L - \varepsilon)(y_n - y_k) \le x_n - x_k \le (L + \varepsilon)(y_n - y_k).$$

Since y is bounded, there exists a positive constant S such that $|y_n - y_m| \le S$ for any n, m. Therefore, we have

$$Ly_n - \varepsilon S - Ly_k + x_k \le x_n \le Ly_n + \varepsilon S - Ly_k + x_k, \tag{3}$$

for any $n \ge k$. Let

$$U = L \lim_{n\to\infty} y_n. \tag{4}$$

Choose an index $q \ge k$ such that $U - \varepsilon \le Ly_n \le U + \varepsilon$ for $n \ge q$. Let $T = x_k - Ly_k$. Then, using Equations (3) and (4), we have

$$U - \varepsilon - \varepsilon S + T \le x_n \le U + \varepsilon + \varepsilon S + T,$$

for any $n \ge q$. Hence, $|x_n - x_m| \le 2\varepsilon(S + 1)$ for any $n, m \ge q$. Therefore, the sequence x is convergent. If y is eventually decreasing, then the proof of convergence of x is analogous. The last part of the lemma is now obvious. □

Remark 2. *The following simple example shows that, in Lemma 3, the limit of x_n/y_n can be different than the limit of $\Delta x_n/\Delta y_n$. Let*

$$x_n = 2 - \frac{1}{n}, \quad y_n = 1 - \frac{1}{n},$$

then the sequence y is bounded, increasing and

$$\lim_{n\to\infty} \frac{x_n}{y_n} = 2 \neq \lim_{n\to\infty} \frac{\Delta x_n}{\Delta y_n} = 1.$$

The next lemma will be used in the proof of Corollary 1. This lemma is probably known, but for the convenience of the reader, we give a proof.

Lemma 4. *Assume $\sigma \in (0,\infty)$ and $r_n = n^{1-\sigma}$. Then*

$$r_n^* = \sigma^{-1} n^{\sigma} + \mathrm{o}(n^{\sigma}). \tag{5}$$

Proof. By Theorem 2.2 in [31], we have

$$\Delta n^{\sigma} = \sigma n^{\sigma-1} + \mathrm{o}(n^{\sigma-1}).$$

Since $\Delta r_n^* = r_n^{-1} = n^{\sigma-1}$, we have

$$\frac{\Delta r_n^*}{\Delta n^{\sigma}} = \frac{n^{\sigma-1}}{\sigma n^{\sigma-1} + \mathrm{o}(n^{\sigma-1})} = \frac{1}{\sigma + \mathrm{o}(1)} \to \frac{1}{\sigma}.$$

By the Stolz–Cesaro theorem,

$$\frac{r_n^*}{n^{\sigma}} \to \frac{1}{\sigma} \quad \Rightarrow \quad \frac{r_n^*}{n^{\sigma}} = \sigma^{-1} + \mathrm{o}(1),$$

and we obtain Equation (5). □

Theorem 1. *Assume $g : [0,\infty) \to [0,\infty)$ is nondecreasing, $\alpha \in (0,\infty)$, $g(\alpha) > 0$,*

$$\sum_{n=1}^{\infty} |a_n| < \infty, \quad \sum_{n=1}^{\infty} |b_n| < \infty \quad , \tag{6}$$

$$p_n \geq 0, \quad p_n \to \lambda \in \mathbb{R}, \quad \frac{r_n^*}{r_{n+1}^*} \to \mu \in \mathbb{R}, \quad \lambda \neq 1 \neq \lambda\mu^{\tau}, \tag{7}$$

$$\int_{\alpha}^{\infty} \frac{ds}{g(s)} = \infty, \quad |f(n,u)| \leq g\left(\frac{|u|}{r_n^*}\right) \quad \text{for } (n,u) \in \mathbb{N}\times\mathbb{R}. \tag{8}$$

Then every nonoscillatory solution x of Equation (1) has the property

$$x_n = c r_n^* + \mathrm{o}(r_n^*),$$

where c is a real constant.

Proof. Let x be a nonoscillatory solution of Equation (1). Then there is an index n_0, such that $x_n > 0$ for any $n \geq n_0$ or $x_n < 0$ for any $n \geq n_0$. Set

$$z_n = x_n + p_n x_{n-\tau}. \tag{9}$$

Then

$$|x_n| < |z_n|, \tag{10}$$

for $n \geq n_1 = n_0 + \tau$, and Equation (1) takes the form

$$\Delta(r_n \Delta z_n) = a_n f(n, x_n) + b_n.$$

Let us denote $z_{n_1} = c_0$ and $r_{n_1}\Delta z_{n_1} = c_1$. Summing the above equation from n_1 to $n-1$, we obtain

$$r_n \Delta z_n = c_1 + \sum_{j=n_1}^{n-1} a_j f(j, x_j) + \sum_{j=n_1}^{n-1} b_j. \tag{11}$$

Dividing both sides of Equation (11) by r_n and summing again, we have

$$z_n = c_0 + c_1 \sum_{i=n_1}^{n-1} \frac{1}{r_i} + \sum_{i=n_1}^{n-1} \frac{1}{r_i} \sum_{j=n_1}^{i-1} a_j f(j, x_j) + \sum_{i=n_1}^{n-1} \frac{1}{r_i} \sum_{j=n_1}^{i-1} b_j.$$

Hence, using Equation (2), we have

$$|z_n| \leq |c_0| + |c_1| r_n^* + \sum_{i=n_1}^{n-1} \frac{1}{r_i} \sum_{j=n_1}^{i-1} |a_j||f(j, x_j)| + \sum_{i=n_1}^{n-1} \frac{1}{r_i} \sum_{j=n_1}^{i-1} |b_j|.$$

Changing the order of summation, we obtain

$$\begin{aligned} |z_n| &\leq |c_0| + |c_1| r_n^* + \sum_{i=n_1}^{n-1} |a_i||f(i, x_i)| \sum_{j=i}^{n-1} \frac{1}{r_j} + \sum_{i=n_1}^{n-1} |b_j| \sum_{j=i}^{n-1} \frac{1}{r_i} \\ &\leq |c_0| + |c_1| r_n^* + r_n^* \sum_{i=n_1}^{n-1} |a_i||f(i, x_i)| + r_n^* \sum_{i=n_1}^{n-1} |b_j|. \end{aligned}$$

Hence, by Equation (8),

$$\begin{aligned} \frac{|z_n|}{r_n^*} &\leq \frac{|c_0|}{r_{n_0}^*} + |c_1| + \sum_{i=n_1}^{n-1} |a_i||f(i, x_i)| + \sum_{i=n_1}^{n-1} |b_j| \\ &\leq d_1 + \sum_{i=n_1}^{n-1} |a_i| \, g\left(\frac{|x_i|}{r_i^*}\right) \leq d_1 + \sum_{i=n_1}^{n-1} |a_i| \, g\left(\frac{|z_i|}{r_i^*}\right), \end{aligned}$$

where d_1 is an appropriate constant. Therefore, by Lemma 2, there exists a constant K such that

$$\frac{|z_n|}{r_n^*} \leq K, \tag{12}$$

for any $n \geq n_1 + \tau$. On the other hand, we have

$$\sum_{i=n_1}^{n-1} |a_i||f(i, x_i)| \leq \sum_{i=n_1}^{n-1} |a_i| g\left(\frac{|x_i|}{r_i^*}\right) \leq \sum_{i=n_1}^{n-1} |a_i| g\left(\frac{|z_i|}{r_i^*}\right) \leq g(K) \sum_{i=n_1}^{n-1} |a_i|.$$

Therefore, the series $\sum_{i=n_1}^{\infty} a_i f(i, x_i)$ is absolutely convergent. Thus, by Equations (11) and (6), we see that the sequence $(r_n \Delta z_n)$ is convergent. Note that $\Delta r_n^* = r_n^{-1}$. Hence,

$$\frac{\Delta z_n}{\Delta r_n^*} = r_n \Delta z_n.$$

If the sequence (r_n^*) is unbounded, then by the Stolz–Cesaro Theorem we have

$$\lim_{n\to\infty} \frac{z_n}{r_n^*} = \lim_{n\to\infty} \frac{\Delta z_n}{\Delta r_n^*} = \lim_{n\to\infty} r_n \Delta z_n.$$

If the sequence (r_n^*) is bounded, then by Lemma 3 the sequence (z_n/r_n^*) is convergent. Now,

$$w_n = \frac{z_n}{r_n^*}, \quad y_n = \frac{x_n}{r_n^*}, \quad u_n = \frac{p_n r_{n-\tau}^*}{r_n^*}. \tag{13}$$

Then, Equation (9) implies

$$w_n = y_n + u_n y_{n-\tau}.$$

Using Equations (10) and (12), we have

$$|y_n| = \frac{|x_n|}{r_n^*} \le \frac{|z_n|}{r_n^*} \le K.$$

It is easy to see that the assumption

$$\frac{r_n^*}{r_{n+1}^*} \to \mu$$

implies

$$\frac{r_{n-\tau}^*}{r_n^*} \to \mu^\tau.$$

Hence, by Equation (13), $u_n \to \lambda\mu^\tau \neq 1$. By Lemma 1, we have

$$\lim_{n\to\infty} \frac{x_n}{r_n^*} = \lim_{n\to\infty} y_n = \frac{\lim_{n\to\infty} w_n}{1+\lambda\mu^\tau} = c.$$

Therefore,

$$\frac{x_n}{r_n^*} = c + \mathrm{o}(1) \Rightarrow x_n = c r_n^* + \mathrm{o}(r_n^*).$$

□

Theorem 1 extends Theorem 1 in [27].

Note that checking the assumption $\frac{r_n^*}{r_{n+1}^*} \to \mu \in \mathbb{R}$ of Theorem 1 may be difficult, so the following result can be useful.

Lemma 5. *Assume at least one of the following conditions holds*

$$(a)\ \ r_n^* = \mathrm{O}(1), \quad (b)\ \ r_n^{-1} = \mathrm{O}(1), \quad (c)\ \ \frac{r_{n+1}}{r_n} \to 1.$$

Then

$$\frac{r_n^*}{r_{n+1}^*} \to 1. \tag{14}$$

Proof. (a) Assume $r_n^* = \mathrm{O}(1)$. Since the sequence (r_n^*) is positive and increasing, there exists a limit $\lim_{n\to\infty} r_n^* = \omega \in (0,\infty)$. Then $\lim_{n\to\infty} r_{n+1}^* = \omega$ and we have Equation (14).

Now, assume that the sequence r^* is unbounded. Then $r_n^* \to \infty$.

(b) If the sequence (r_n^{-1}) is bounded, then

$$\frac{r_n^*}{r_{n+1}^*} = \frac{r_n^*}{r_n^* + \frac{1}{r_n}} = \frac{1}{1+\frac{1}{r_n r_n^*}} \to \frac{1}{1+0}.$$

(c) Note that

$$\frac{\Delta r_n^*}{\Delta r_{n+1}^*} = \frac{\frac{1}{r_n}}{\frac{1}{r_{n+1}}} = \frac{r_{n+1}}{r_n}.$$

Hence, by the Stolz–Cesaro theorem, (c) implies Equation (14). □

Note that, if r is a potential sequence, i.e., $r_n = n^\omega$, where ω is a fixed real number, then $r_{n+1}/r_n \to 1$. In this case, from Theorem 1, we have the following corollary.

Corollary 1. *Assume $\sigma \in (0,\infty)$, $r_n = n^{1-\sigma}$, $h : [0,\infty) \to [0,\infty)$ is nondecreasing, $\alpha \in (0,\infty)$, $h(\alpha) > 0$,*

$$p_n \geq 0, \quad p_n \to \lambda \in \mathbb{R}, \quad \lambda \neq 1, \quad \sum_{n=1}^{\infty} |a_n| < \infty, \quad \sum_{n=1}^{\infty} |b_n| < \infty,$$

$$\int_\alpha^\infty \frac{ds}{h(s)} = \infty, \quad |f(n,u)| \leq h\left(\frac{|u|}{n^\sigma}\right) \quad \text{for } (n,u) \in \mathbb{N} \times \mathbb{R}.$$

Then every nonoscillatory solution x of Equation (1) has the property

$$x_n = cn^\sigma + \mathrm{o}(n^\sigma),$$

where c is a real constant.

Proof. By Lemma 5(c) we have

$$\frac{r_n^*}{r_{n+1}^*} \to 1.$$

Let $\alpha = \sigma^{-1}$. Then $\alpha > 0$ and, by Lemma 4,

$$r_n^* = \alpha n^\sigma + \mathrm{o}(n^\sigma) = n^\sigma(\alpha + \mathrm{o}(1)) = n^\sigma \mathrm{O}(1) = \mathrm{O}(n^\sigma).$$

Choose a positive constant L such that for any n we have

$$r_n^* \leq Ln^\sigma.$$

Define a function $g : [0,\infty) \to [0,\infty)$ by $g(s) = h(Ls)$. Then g is nondecreasing and

$$\int_{\frac{\alpha}{L}}^\infty \frac{ds}{g(s)} = \infty.$$

Moreover, for any $(n,u) \in \mathbb{N} \times \mathbb{R}$, we have

$$|f(n,u)| \leq h\left(\frac{|u|}{n^\sigma}\right) \leq h\left(\frac{L|u|}{r_n^*}\right) = g\left(\frac{|u|}{r_n^*}\right).$$

Let x be a nonoscillatory solution of Equation (1). By Theorem 1, there exists a constant c' such that $x_n = c' r_n^* + \mathrm{o}(r_n^*)$. Hence,

$$x_n = c'\alpha n^\sigma + \mathrm{o}(\mathrm{O}(n^\sigma)) = cn^\sigma + \mathrm{o}(n^\sigma).$$

□

Theorem 1, applied to the linear equation

$$\Delta(r_n \Delta(x_n + p_n x_{n-\tau})) = q_n x_n, \tag{15}$$

leads to the following corollary.

Corollary 2. *Assume that* $p_n \geq 0, \quad p_n \to \lambda \in \mathbb{R}, \quad \frac{r_n^*}{r_{n+1}^*} \to \mu \in \mathbb{R}, \quad \lambda \neq 1 \neq \lambda\mu^\tau$, *and*

$$\sum_{n=1}^{\infty} r_n^* |q_n| < \infty.$$

Then every nonoscillatory solution (x_n) *of Equation (15) has the asymptotic property*

$$x_n = cr_n^* + \mathrm{o}(r_n^*),$$

where c is a real constant.

Proof. We get the conclusion of Corollary 2 by applying Theorem 1 with

$$a_n = r_n^* q_n, \quad f(n,u) = \frac{u}{r_n^*} \quad \text{and} \quad g(u) = u.$$

□

Applying Theorem 1 to nonlinear difference equation of the form

$$\Delta(r_n \Delta(x_n + p_n x_{n-\tau})) = q_n x_n^\alpha, \quad 0 < \alpha < 1, \tag{16}$$

where (p_n), (q_n) are sequences of real numbers and τ is a nonnegative integer, we have the following corollary.

Corollary 3. *Assume that* $p_n \geq 0, \quad p_n \to \lambda \in \mathbb{R}, \quad \frac{r_n^*}{r_{n+1}^*} \to \mu \in \mathbb{R}, \quad \lambda \neq 1 \neq \lambda\mu^\tau$, *and*

$$\sum_{n=1}^{\infty} (r_n^*)^\alpha |q_n| < \infty.$$

Then every nonoscillatory solution (x_n) *of Equation (16) has the property* $x_n = cr_n^* + \mathrm{o}(r_n^*)$, *where c is a real number.*

Proof. The conclusion follows from Theorem 1 with $a_n = r_n^{*\alpha} q_n$, $f(n,u) = \frac{u^\alpha}{r_n^{*\alpha}}$ and $g(u) = u^\alpha$.
□

Example 1. *Consider the difference equation*

$$\Delta\left(n(n+1)\Delta\left(x_n + \frac{2n+1}{n} x_{n-1}\right)\right) = -\frac{12}{(n-1)(n-2)}\, x_n. \tag{17}$$

Here, $r_n = n(n+1)$, $p_n = \frac{2n+1}{n}$, $\tau = 1$, *and* $q_n = -\frac{12}{(n-1)(n-2)}$. *Hence,*

$$r_n^* = 1 - \frac{1}{n}, \quad \frac{r_n^*}{r_{n+1}^*} \to 1, \quad \sum_{n=1}^{\infty} r_n^* |q_n| < \infty.$$

Therefore, all assumptions of Corollary 2 are satisfied. It is not difficult to check that the sequence $x_n = 1 - \frac{2}{n}$ *is a solution of Equation (17) with the property*

$$x_n = 1 - \frac{1}{n} - \frac{1}{n} = r_n^* + \mathrm{o}(r_n^*).$$

Next, we give sufficient conditions under which all nonoscillatory solutions of Equation (1) have the property $x_n = cr_n^* + d + \mathrm{o}(1)$.

Theorem 2. *Assume $g : [0, \infty) \to [0, \infty)$ is nondecreasing,*

$$\sum_{n=1}^{\infty} n|a_n| < \infty, \qquad \sum_{n=1}^{\infty} n|b_n| < \infty, \tag{18}$$

$$p_n \geq 0, \quad p_n \to \lambda \in \mathbb{R}, \quad \lambda \neq 1, \quad p_n - \lambda = \mathrm{o}\left(\frac{1}{r_n^*}\right), \quad r_n^{-1} \to \rho \in \mathbb{R},$$

$$\int_0^{\infty} \frac{ds}{g(s)} = \infty, \quad |f(n,u)| \leq g\left(\frac{|u|}{r_n^*}\right) \quad \text{for } (n,u) \in \mathbb{N} \times \mathbb{R}.$$

Then every nonoscillatory solution x of Equation (1) has the property

$$x_n = cr_n^* + d + \mathrm{o}(1),$$

where c, d are real constants.

Proof. Note that all assumptions of Theorem 1 are satisfied. Let x be a nonoscillatory solution of Equation (1) and let z be defined by Equation (9). As in the proof of Theorem 1, there exists a constant K such that

$$\frac{|x_n|}{r_n^*} \leq \frac{|z_n|}{r_n^*} \leq K,$$

for $n \geq n_1 = n_0 + \tau$. Hence,

$$|f(n, x_n)| \leq g\left(\frac{|x_n|}{r_n^*}\right) \leq g(K),$$

for any $n \geq n_1$. Therefore, by Equations (1) and (18), the series

$$\sum_{n=1}^{\infty} n|\Delta(r_n \Delta z_n)|,$$

is convergent. Choose a constant L such that $r_n^{-1} \leq L$ for any n. Then

$$\sum_{n=1}^{\infty} \frac{1}{r_n} \sum_{j=n}^{\infty} |\Delta(r_j \Delta z_j)| \leq L \sum_{n=1}^{\infty} \sum_{j=n}^{\infty} |\Delta(r_j \Delta z_j)| = L \sum_{n=1}^{\infty} n|\Delta(r_n \Delta z_n)| < \infty.$$

Define a sequence U by

$$U_n = \sum_{j=n}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} \Delta(r_i \Delta z_i).$$

Then

$$U_n = \mathrm{o}(1), \tag{19}$$

and

$$\Delta(r_n \Delta U_n) = -\Delta\left(r_n \frac{1}{r_n} \sum_{i=n}^{\infty} \Delta(r_i \Delta z_i)\right) = -\Delta \sum_{i=n}^{\infty} \Delta(r_i \Delta z_i) = \Delta(r_n \Delta z_n). \tag{20}$$

Define a sequence W by

$$W_n = z_n - U_n. \tag{21}$$

Using Equation (20), we have

$$\Delta(r_n \Delta W_n) = \Delta(r_n \Delta z_n) - \Delta(r_n \Delta U_n) = \Delta(r_n \Delta z_n) - \Delta(r_n \Delta z_n) = 0.$$

Hence, there exists a constant P such that $r_n \Delta W_n = P$ for any n. Summing the equality

$$\Delta W_n = \frac{P}{r_n},$$

from 1 to $n-1$, we obtain

$$W_n = Q + P\sum_{k=1}^{n-1}\frac{1}{r_k} = Pr^*_n + Q, \tag{22}$$

where $Q = W_1$. Using Equations (21), (22), and (19), we obtain

$$z_n = W_n + U_n = Pr^*_n + Q + \mathrm{o}(1). \tag{23}$$

Let z' be a sequence defined by

$$z'_n = x_n + \lambda x_{n-\tau},$$

for $n > \tau$. Then

$$z'_n = z_n + (\lambda - p_n)x_{n-\tau}. \tag{24}$$

By Theorem 1, we have

$$\lim_{n\to\infty}\frac{x_{n-\tau}}{r^*_{n-\tau}} = \lim_{n\to\infty}\frac{x_n}{r^*_n} \in \mathbb{R}.$$

Moreover, since the sequence r^{-1} is convergent, the sequence

$$\frac{r^*_n}{r^*_{n+1}} = \frac{r^*_n}{r^*_n + \frac{1}{r_n}} = \frac{1}{1 + \frac{1}{r_n r^*_n}}$$

is convergent, too. Hence, the sequence

$$\frac{r^*_{n-\tau}}{r^*_n}$$

is convergent. Therefore,

$$(\lambda - p_n)x_{n-\tau} = \mathrm{o}\left(\frac{1}{r^*_n}\right)x_{n-\tau} = \frac{\mathrm{o}(1)}{r^*_n}x_{n-\tau} = \mathrm{o}(1)\frac{x_{n-\tau}}{r^*_{n-\tau}}\frac{r^*_{n-\tau}}{r^*_n} = \mathrm{o}(1).$$

Thus, by Equations (22) and (24),

$$z'_n = z_n + \mathrm{o}(1) = Pr^*_n + Q + \mathrm{o}(1). \tag{25}$$

Let

$$u_n = x_n - \frac{P}{1+\lambda}r^*_n. \tag{26}$$

Then

$$\begin{aligned}
u_n + \lambda u_{n-\tau} &= x_n - \frac{P}{1+\lambda}r^*_n + \lambda x_{n-\tau} - \frac{P\lambda}{1+\lambda}r^*_{n-\tau} \\
&= x_n - \frac{P}{1+\lambda}r^*_n + \lambda x_{n-\tau} - \frac{P\lambda}{1+\lambda}r^*_n + \frac{P\lambda}{1+\lambda}(r^*_n - r^*_{n-\tau}) \\
&= x_n + \lambda x_{n-\tau} - P\left(\frac{1}{1+\lambda} + \frac{\lambda}{1+\lambda}\right)r^*_n + \frac{P\lambda}{1+\lambda}(r^*_n - r^*_{n-\tau}) \\
&= z'_n - Pr^*_n + \frac{P\lambda}{1+\lambda}(r^*_n - r^*_{n-\tau}).
\end{aligned}$$

Hence, by Equation (25), we have

$$u_n + \lambda u_{n-\tau} = Q + \mathrm{o}(1) + \frac{P\lambda}{1+\lambda}(r_n^* - r_{n-\tau}^*).$$

Note that $r_{n+1}^* - r_n^* = r_n^{-1} \to \rho$. Similarly, $r_{n+2}^* - r_{n+1}^* \to \rho$. Hence,

$$r_{n+2}^* - r_n^* = r_{n+2}^* - r_{n+1}^* + r_{n+1}^* - r_n^* \to 2\rho.$$

Analogously, $r_n^* - r_{n-\tau}^* \to \tau\rho$. Hence, the sequence $(u_n + \lambda u_{n-\tau})$ is convergent and, by Lemma 1, the sequence u is convergent, too. Therefore, by Equation (26),

$$x_n = \frac{P}{1+\lambda} r_n^* + u_n = c r_n^* + d + \mathrm{o}(1)$$

where

$$c = \frac{P}{1+\lambda}, \quad d = \lim_{n\to\infty} u_n.$$

□

Remark 3. *Observe that, if the sequence* (r_n^*) *is bounded, then the conclusion of Theorem 2 follows directly from Theorem 1. Indeed, in this case, we have* $r_n^* = \alpha + \mathrm{o}(1)$*. Hence,* $\mathrm{o}(r_n^*) = \mathrm{o}(1)$*, and we have*

$$x_n = c r_n^* + \mathrm{o}(r_n^*) = c r_n^* + \mathrm{o}(1) = c r_n^* + 0 + \mathrm{o}(1).$$

Applying Theorem 2 to a linear equation expressed by (15), we have the following result.

Corollary 4. *Assume that* $p_n \geq 0$, $p_n \to \lambda \in \mathbb{R}$, $\frac{r_n^*}{r_{n+1}^*} \to \mu \in \mathbb{R}$, $\lambda \neq 1 \neq \lambda\mu^\tau$, $p_n - \lambda = \mathrm{o}\left(\frac{1}{r_n^*}\right)$, $\lim_{n\to\infty} r_n^{-1} = \rho \in \mathbb{R}$*, and*

$$\sum_{n=1}^{\infty} n r_n^* |q_n| < \infty.$$

Then every nonoscillatory solution (x_n) *of Equation (15) has the asymptotic property*

$$x_n = c r_n^* + d + \mathrm{o}(1)$$

where c, d *are real constants.*

Example 2. *Consider the difference equation*

$$\Delta\left(\frac{1}{n}\Delta\left(x_n + 2x_{n-1}\right)\right) = \frac{6}{n^2(n+1)(n-1)} x_n. \tag{27}$$

Here, $r_n = \frac{1}{n}$, $p_n = 2$, $\tau = 1$*, and* $q_n = \frac{6}{n^2(n+1)(n-1)}$*. Then*

$$r_n^* = \frac{n(n-1)}{2}, \quad \frac{r_n^*}{r_{n+1}^*} \to 1, \quad \sum_{n=1}^{\infty} r_n^* |q_n| < \infty.$$

Note that all assumptions of Corollary 2 are satisfied. One can see that the sequence $x_n = n^2 - 2n$ *is a solution of Equation (27) with the property*

$$x_n = 2 r_n^* + \mathrm{o}(r_n^*).$$

Note also that the assumption $\sum_{n=1}^{\infty} n r_n^* |q_n| < \infty$ *of Corollary 4 is not satisfied, and the sequence x does not have the property* $x_n = c r_n^* + d + \mathrm{o}(1)$.

Applying Theorem 2 to a nonlinear Equation (16), we have the corollary.

Corollary 5. *Assume that* $p_n \geq 0$, $p_n \to \lambda \in \mathbb{R}$, $\frac{r_n^*}{r_{n+1}^*} \to \mu \in \mathbb{R}$, $\lambda \neq 1 \neq \lambda\mu^\tau$, $p_n - \lambda = \mathrm{o}\left(\frac{1}{r_n^*}\right)$, $\lim_{n\to\infty} r_n^{-1} = \rho \in \mathbb{R}$, *and*

$$\sum_{n=1}^{\infty} n (r_n^*)^\alpha |q_n| < \infty.$$

Then every nonoscillatory solution (x_n) *of Equation (16) has the property*

$$x_n = c r_n^* + d + \mathrm{o}(1)$$

where c, d *are real constants.*

Example 3. *Let* $r_n = 2^n$, $p_n = \frac{1}{2^n}$, $\tau = 1$, $\alpha = \frac{1}{2}$, *and* $q_n = \frac{1}{2^{n+1}\sqrt{2+2^{-n}}}$. *Then Equation (16) takes the form*

$$\Delta\left(2^n \Delta\left(x_n + \frac{1}{2^n} x_{n-1}\right)\right) = \frac{1}{2^{n+1}\sqrt{2+2^{-n}}} \sqrt{x_n}. \tag{28}$$

For this equation, we have

$$r_n^* = 1 - \left(\frac{1}{2}\right)^{n-1}, \quad \mu = \frac{r_n^*}{r_{n+1}^*} \to 1.$$

Then

$$\sum_{n=1}^{\infty} n r_n^* |q_n| < \sum_{n=1}^{\infty} \frac{n}{2^{n+1}} < \infty.$$

Therefore, since all assumptions of Corollary 5 are satisfied, every nonoscillatory solution (x_n) *of Equation (28) has the property*

$$x_n = c r_n^* + d + \mathrm{o}(1)$$

where c, d *are real constants. The sequence* $x_n = 1 - \frac{1}{2^{n-1}} + 1 + \frac{1}{2^n} = 2 + \frac{1}{2^n}$ *is one of such solutions.*

Remark 4. *This paper is devoted to nonoscillatory solutions. But, in the case* $p_n \equiv 0$, *our results are true for all solutions. This follows from the proofs of Theorems 1 and 2, respectively.*

3. Conclusions

In this paper, we have presented sufficient conditions, under which all nonoscillatory solutions of Equation (1) have the property $x_n = c r_n^* + \mathrm{o}(r_n^*)$ or the property $x_n = c r_n^* + d + \mathrm{o}(1)$. The presented results are new even for linear equations of the type defined by Equation (1), and in the case when $p_n \equiv 0$. The first part of the proof of Theorem 1, based on the summation method and the use of discrete Bihari type lemma, is in principle standard (see [27,28,32,33]). The second part of the proof required a new approach with the use of Lemma 3. The difficulty was choosing appropriate conditions for the sequences p and r. In Theorem 2, this problem was even greater. Our results can be generalized in two directions. First, one can try to get a more accurate approximation of solutions, e.g., with an accuracy of $\mathrm{o}(n^s)$, where s is a nonpositive real number. Secondly, one can try to obtain similar results for higher-order equations. This problem is not easy to solve. A comparison between [27] and [28] illustrates the scale of this difficulty.

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