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Computing the Metric Dimension of Gear Graphs

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Abstract: Let $G = (V, E)$ be a connected graph and $d(u, v)$ denote the distance between the vertices u and v in G. A set of vertices W resolves a graph G if every vertex is uniquely determined by its vector of distances to the vertices in W. A metric dimension of G is the minimum cardinality of a resolving set of G and is denoted by $dim(G)$. Let $J_{2n,m}$ be a m-level gear graph obtained by m-level wheel graph $W_{2n,m} \cong mC_{2n} + k_1$ by alternatively deleting n spokes of each copy of C_{2n} and J_{3n} be a generalized gear graph obtained by alternately deleting $2n$ spokes of the wheel graph W_{3n}. In this paper, the metric dimension of certain gear graphs $J_{2n,m}$ and J_{3n} generated by wheel has been computed. Also this study extends the previous result given by Tomescu et al. in 2007.

Keywords: Metric dimension; basis; resolving set; gear graph; generalized gear graph

MSC: 05C12; 05C90; 05C15; 05C62

1. Introduction and Preliminary Results

In a connected graph $G(V, E)$, where V is the set of vertices and E is the set of edges. The distance $d(u, v)$ between two vertices $u, v \in V$ is the length of the shortest path between them and the diameter of G denoted by $diam(G)$ is the maximum distance between pairs of vertices $u, v \in V(G)$. Let $W = \{v_1, v_2, \ldots, v_k\}$ be an order set of vertices of G and u be a vertex of G. The representation $r(u|W)$ of u with respect to W is the $k-$ tuple $\{d(u, v_1), d(u, v_2), d(u, v_3), \ldots, d(u, v_k)\}$, where W is called a resolving set or locating set if distinct vertices of G have distinct representations with respect to W. See for more results [1,2].

A resolving set of minimum cardinality is called a metric basis for G and the cardinality of a metric basis is said the metric dimension of G, denoted by $dim(G)$, see [3]. The motivation for this topic stems from chemistry [4]. A common but important problem in the study of chemical structures is to determine ways of representing a set of chemical compounds such that distinct compounds have distinct representations. Moreover the application of this invariant to the navigation of robots in networks are discussed in [5]. The application to problems of pattern recognition and image processing, some of which involve the use of hierarchical data structures are given in [6].

For a given ordered set of vertices $W = \{v_1, v_2, \ldots, v_k\}$ of a graph G, the i^{th} component of $r(u|W)$ is 0 if and only if $u = v_i$. Thus, to show that W is a resolving set it suffices to verify that $r(y|W) \neq r(z|W)$ for each pair of distinct vertices $y, z \in V(G)\backslash W$.

Motivated by the problem of determining uniquely the location of an intruder in a network, the concept of metric dimension was introduced by Slater in [7] and studied independently by Harary and Melter in [8].

Let Ω be a family of connected graphs $F_m : \Omega = (F_m)_{m\geq 1}$ depending on m as follows: $\psi(m)$ = cardinality of the set of vertices of any member F of Ω and $\lim_{m\to\infty} \psi(m) = \infty$. If $\forall\, m \geq 1, \exists\, C > 0$ such that $dim(F_m) \leq C$, then we shall say that Ω has bounded metric dimension, otherwise Ω has unbounded metric dimension. If all graphs in Ω have the same metric dimension then F is called a family with constant metric dimension [9].

A connected graph G has $dim(G) = 1$ if and only if G is a path [5], cycle C_n have metric dimension 2 for every $n \geq 3$. Other families of graphs with unbounded metric dimension are regular bipartite graphs [10], wheel graph [11]. The metric dimensions of m-level wheel graphs, convex polytope graphs and antiweb gear graphs are computed in [12]. The metric dimension of honeycomb networks are computed in [13] and t he metric dimension of generators of graphs in [14]. In the following section, some results related to m-level generalized gear graph are given.

2. The Metric Dimension of Double Gear Graph $J_{2n,m}$

Definition 1. *The joining of two graphs G_1 and G_2 is denoted by $G_1 + G_2$ with the following vertex and edge sets:*

$$V(G_1 + G_2) = V(G_1) \cup V(G_2)$$

$$E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{uv; u \in V(G_1), v \in V(G_2)\}.$$

Definition 2. *In graph theory, an isomorphism of graphs G_1 and G_2 is a bijection between the vertex sets of G_1 and G_2, $f : V(G_1) \to V(G_2)$ such that any two vertices u and v of G_1 are adjacent in G_1 if and only if $f(u)$ and $f(v)$ are adjacent in G_2. If an isomorphism exists between two graphs, then the graphs are called isomorphic and denoted as $G_1 \cong G_2$.*

Note that the the graph $C_n + K_1$ is isomorphic to wheel graph W_n. In addition, note that $2C_n + K_1$ mean union of two copies of C_n that are joined with K_1.

Definition 3. *A double-wheel graph $W_{n,2}$ can be obtained as join of $2C_n + k_1$ and inductively an m-level wheel graph denoted by $W_{n,m}$ can be constructed as $W_{n,m} \cong mC_n + k_1$.*

Definition 4. *A double gear graph denoted by $J_{2n,2}$ can be obtained from double-wheel $W_{2n,2} = 2C_{2n} + k_1$ by alternatively deleting n spokes of each copy of C_{2n} and inductively an m-level gear graph $J_{2n,m}$ can be constructed from m-level wheel $W_{2n,m} \cong mC_{2n} + k_1$ by alternatively deleting n spokes of each C_{2n} (see [15]). A double gear graph is depicted in Figure 1.*

Construction and Observations

A double gear graph $J_{2n,2}$ (see in Figure 1) is constructed if we consider two even cycles with $n \geq 2$,

$$C_{2n,1} : v_1^1, v_2^1, v_3^1, \ldots, v_{2n}^1, v_1^1 \ \ \textit{and} \ \ C_{2n,2} : v_1^2, v_2^2, v_3^2, \ldots, v_{2n}^2, v_1^2$$

Now take a new vertex v adjacent to n vertices of $C_{2n,1}$:$v_2^1, v_4^1, \ldots, v_{2n}^1$ as well as v is also adjacent to n vertices of $C_{2n,2}$:$v_2^2, v_4^2, \ldots, v_{2n}^2$. Inductively we can construct an m-level gear graph denoted by $J_{2n,m}$ by taking m even cycles $C_{2n,1}, C_{2n,2}, \ldots, C_{2n,m}$.

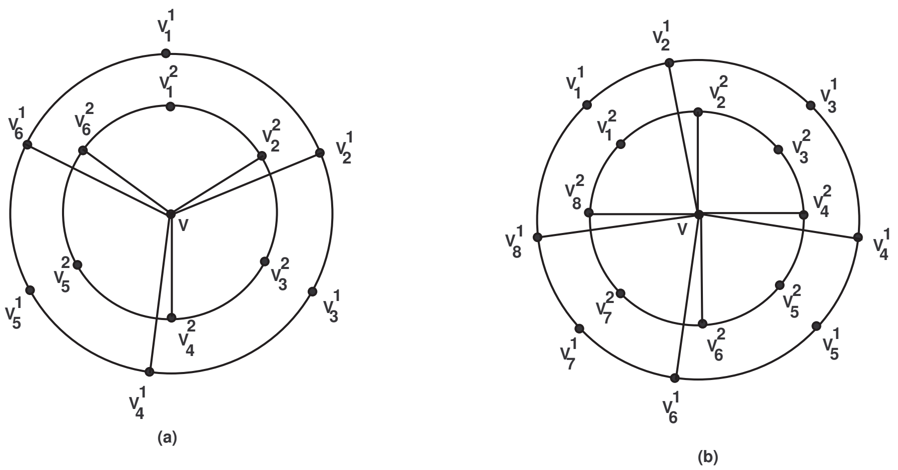

Figure 1. (**a**) The double gear graph $J_{6,2}$; (**b**) The double gear graph $J_{8,2}$.

The vertices of $C_{2n,i}$; $1 \leq i \leq 2$, in the graph $J_{2n,2}$ are of two kinds namely the vertices of degree 2 and the vertices of degree 3. Vertices of degree 2 and 3 will be considered as minor and major vertices respectively. One can easily check that:

- When $n = 2$,
  $dim(J_{4,2}) = 3 + 2$, (central vertex v with one major and minor vertex of each $C_{2n,i}$, $1 \leq i \leq 2$ form basis).
- When $n = 3$,
  $dim(J_{6,2}) = 3 + 2$, (central vertex v with two minor vertices of each $C_{2n,i}, 1 \leq i \leq 2$ form basis).
- When $n = 4$,
  $dim(J_{8,2}) = 2 + 3$, (two minor vertices u^1, w^1 such that $d(u^1, w^1) = 2$ of $C_{2n,1}$ with one minor vertex u^2 and two major vertices w^2, x^2 of $C_{2n,2}$ such that $d(u^2, w^2) = d(u^2, x^2) = 3$ form basis).
- When $n = 5$,
  $dim(J_{10,2}) = 3 + 4$, (three minor vertices u^1, w^1, x^1 satisfying $d(u^1, w^1) = d(w^1, x^1) = 2$, $d(u^1, x^1) = 4$ of $C_{2n,1}$ with three minor vertices u^2, w^2, x^2 and one major vertex z^2 of $C_{2n,2}$ satisfying $d(u^2, w^2) = d(w^2, x^2) = 2$, $d(u^2, x^2) = 4$ and $d(u^2, z^2) = d(w^2, z^2) = d(x^2, z^2) = 3$ form metric basis of $J_{10,2}$).

Consider the gear graph $J_{2n,1}$ in which $C_{2n,1}$ is an outer cycle of length $2n$. If B is a basis of $J_{2n,1}$ then B contains $r \geq 2$ vertices of $C_{2n,1}$ for $n \geq 6$. Suppose $B = \{v_{i_1}, v_{i_2}, \ldots, v_{i_r}\}$ then vertices of B can be ordered as $v_{i_1} < v_{i_2} < \ldots < v_{i_r}$ such that $\{v_{i_t}, v_{i_{t+1}}\}$ for $1 \leq t \leq r-1$ and $\{v_{i_r}, v_{i_1}\}$ are called neighboring vertices. Vertices of $C_{2n,1}$ lying between any two neighboring vertices of B are called gaps which are denoted by G_{i_t} for $1 \leq t \leq r-1$ and G_{i_r}, and their cardinalities are said to be the size of gaps. One can easily observe that every vertex of B has two neighboring vertices; gaps generated by these three vertices are called neighboring gaps following a concept already exist in [2] and [17]. A gap determined by neighboring vertices of basis say v_i and v_j will be called an $\alpha - \beta$ with $\alpha \leq \beta$ when $deg(v_i) = \alpha$ and $deg(v_j) = \beta$ or when $deg(v_i) = \beta$ and $deg(v_j) = \alpha$. Hence we have three kinds of gaps namely, $2-2$ gap, $2-3$ gap and $3-3$ gap.

For the graph $J_{2n,2}, n \geq 4$ central vertex v does not belong to any basis. Since $d(v_i^j, v) \leq 2\forall$, $1 \leq i \leq 2n$, $1 \leq j \leq 2$ and $diam(J_{2n,2}) = 4$, if central vertex v belongs to any metric basis B then there must exists two distinct vertices u_i and u_j for $1 \leq i \neq j \leq 2n$ such that $r(u_i|B) = r(u_J|B)$. Consequently, the basis vertices of $J_{2n,2}$ belong to the cycles induced by $C_{2n,1}$ and $C_{2n,2}$. It is shown in [17] that if B is a basis of $J_{2n,1}$ then B consist only of the vertices of $C_{2n,1}$ that satisfy the following properties.

- If B is a basis of $J_{2n,1}$, $n \geq 6$ then every $2-2$ gap, $2-3$ gap and $3-3$ gap of B contains at most 5, 4 and 3 vertices respectively.
- If B is a basis of $J_{2n,1}$, $n \geq 6$ then it contains at most one major gap.
- If B is a basis of $J_{2n,1}$, $n \geq 6$ then any two neighboring gaps contain together at most six vertices in which one gap is a major gap.
- If B is a basis of $J_{2n,1}$, $n \geq 6$ then any two minor neighboring gaps contain together at most four vertices.

Lemma 1. *Let B be a basis of $J_{2n,2}$, $n \geq 6$, then every $2-2$ gap, $2-3$ gap and $3-3$ gap of B induced by $C_{2n,1}$ and $C_{2n,2}$ contains at most 5, 4 and 3 vertices respectively.*

Proof. Suppose the result is false and there exists a $2-2$ gap of size 7 say $u_1, u_2, u_3, u_4, u_5, u_6, u_7$ consisting of consecutive vertices of $C_{2n,1}$ or $C_{2n,2}$ with $deg(u_1) = deg(u_7) = 3$ then $r(u_3|B) = r(u_5|B)$ which is a contradiction. If there exists a $2-3$ gap of size 6 then we have a path $u_1, u_2, u_3, u_4, u_5, u_6$ consisting of consecutive vertices of $C_{2n,1}$ or $C_{2n,2}$ with $deg(u_1) = 3$ and $deg(u_6) = 2$ then $r(u_3|B) = r(u_5|B)$ which is again a contradiction. The existence of a $3-3$ gap of size 5 say u_1, u_2, u_3, u_4, u_5 induced by $C_{2n,1}$ or $C_{2n,2}$ with $deg(u_1) = deg(u_5) = 2$, would imply $r(u_2|B) = r(u_4|B)$ a contradiction.

The $2-2$ gap $2-3$ gap and $3-3$ gap containing 5, 4 and 3 vertices respectively will be referred to as major gaps and the remaining gaps are called minor gaps. In the proof of Lemmas 2–4, the major vertices will be labeled by a star (*). □

Lemma 2. *Let B be a basis of $J_{2n,2}$, $n \geq 6$ then it contains at most one major gap induced by the vertices of cycles $C_{2n,1}$ and $C_{2n,2}$.*

Proof. Suppose B contains two distinct major gaps induced by the vertices of cycles $C_{2n,1}$ or $C_{2n,2}$.

Case-(i): When both gaps are $3-3$ then we have two distinct paths consisting consecutive vertices u_1, u_2^*, u_3 and w_1, w_2^*, w_3 of $C_{2n,1}$ and $C_{2n,2}$ respectively in this case $r(u_3^*|B) = r(w_3^*|B)$; a contradiction.

Case-(ii): When both gaps are $2-2$ then we have two distinct paths consisting of consecutive vertices $u_1^*, u_2, u_3^*, u_4, u_5^*$ and $w_1^*, w_2, w_3^*, w_4, w_5^*$ of $C_{2n,1}$ and $C_{2n,2}$ respectively but $r(u_3^*|B) = r(w_3^*|B)$; a contradiction.

Case-(iii): When both gaps are $2-3$ then we have two distinct paths consisting of consecutive vertices u_1^*, u_2, u_3^*, u_4 and w_1^*, w_2, w_3^*, w_4 of $C_{2n,1}$ and $C_{2n,2}$ respectively in this case $r(u_3^*|B) = r(w_3^*|B)$; a contradiction.

Case-(iv): When one gap is $3-3$ and other is $2-2$ gap then we have two distinct paths u_1, u_2^*, u_3 and $w_1^*, w_2, w_3^*, w_4, w_5^*$ induced by $C_{2n,1}$ and $C_{2n,2}$ respectively but $r(u_2^*|B) = r(w_3^*|B)$; a contradiction.

Case-(v): When one gap is $3-3$ and other is $2-3$ gap then we have two distinct paths consisting of consecutive vertices u_1, u_2^*, u_3 and w_1^*, w_2, w_3^*, w_4 of $C_{2n,1}$ and $C_{2n,2}$ respectively but $r(u_2^*|B) = r(w_3^*|B)$; a contradiction.

Case-(vi): When one gap is $2-2$ and other is $2-3$ gap then we have two distinct paths consisting of consecutive vertices $u_1^*, u_2, u_3^*, u_4, u_5^*$ and w_1^*, w_2, w_3^*, w_4 of $C_{2n,1}$ and $C_{2n,2}$ respectively in this case $r(u_3^*|B) = r(w_3^*|B)$; a contradiction.

Similarly, if both major gaps are induced by $C_{2n,1}$ then we get a contradiction and a similar contradiction arises if $C_{2n,2}$ induced both major gaps. □

Lemma 3. *Let B be a basis of $J_{2n,2}$, $n \geq 6$, then any two neighboring gaps, one of which being a major gap induced by exactly one of two cycles $C_{2n,1}$ or $C_{2n,2}$ contain together at most six vertices.*

Proof. If the major gap is $3-3$ then there is nothing to prove by Lemma 2. Without loss of any generality we can say that only $C_{2n,1}$ induced a major gap by Lemma 2. If the major gap is a $2-2$ gap having five vertices then its neighboring minor gap contains at most one vertex. If this statement is false and $2-2$ gap, $2-3$ minor gaps having three and two vertices respectively are neighboring gaps of $2-2$ major gap, then we have two paths consisting of consecutive vertices of $C_{2n,1}$:$u_1^*, u_2, u_3^*, u_4, u_5^*, u_6, u_7^*, u_8, u_9^*$ and $w_1^*, w_2, w_3^*, w_4, w_5^*, w_6, w_7^*, w_8$, where $u_4, w_6 \in B$ induced by $2-2$ major, $2-2$ minor gaps and $2-2$ major, $2-3$ minor gaps respectively. In this case $r(u_3^*|B) = r(u_5^*|B)$ and $r(w_5^*|B) = r(w_7^*|B)$; a contradiction. The existence of $2-3$ major gap having four vertices is not possible if its neighboring minor gap is a $2-2$ gap with three vertices. If this case holds then we consider the following path: $u_1^*, u_2, u_3^*, u_4, u_5^*, u_6, u_7^*, u_8$, where $u_4 \in B$ then $r(u_4^*|B) = r(u_5^*|B)$; a contradiction. □

Lemma 4. *Let B be a basis of $J_{2n,2}$, $n \geq 6$, then any two minor neighboring gaps induced by $C_{2n,1}$ or $C_{2n,2}$ contain together at most four vertices.*

Proof. To prove the statement, it is sufficient to prove two cases.

Case-(i): $2-2$ minor gap with three vertices cannot be neighboring gap of $2-2$ minor gap having three vertices, otherwise we have a path consisting of consecutive vertices of $C_{2n,1}$ or $C_{2n,2}$:$u_1^*, u_2, u_3^*, u_4, u_5^*, u_6, u_7^*$, where $u_4 \in B$ in this case $r(u_3^*|B) = r(u_5^*|B)$.

Case-(ii): $2-2$ minor gap with three vertices cannot be neighboring gap of $2-3$ minor gap having two vertices, otherwise we have a path consisting of consecutive vertices of $C_{2n,1}$ or $C_{2n,2}$:$w_1^*, w_2, w_3^*, w_4, w_5^*, w_6$ where $w_4 \in B$ in this case $r(w_3^*|B) = r(w_5^*|B)$; a contradiction. □

Theorem 1. *If $J_{2n,2}$ be a double gear graph for $n \geq 4$, then*

$$dim(J_{2n,2}) = dim(J_{2n,1}) + \left\lceil \frac{2n}{3} \right\rceil$$

Proof. We have seen that $dim(J_{8,2}) = 5 = dim(J_{8,1}) + \lceil \frac{8}{3} \rceil$, $dim(J_{10,2}) = 7 = dim(J_{10,1}) + \left\lceil \frac{10}{3} \right\rceil$ and the central vertex v does not belong to any basis B of $J_{2n,2}$. Moreover

$$C_{2n,1} : v_1^1, v_2^1, v_3^1, \ldots, v_{2n}^1, v_1^1$$

and

$$C_{2n,2} : v_1^2, v_2^2, v_3^2, \ldots, v_{2n}^2, v_1^2$$

be the outer cycles of $J_{2n,2}$ at level 1 and 2 respectively. First we prove that $dim(J_{2n,2}) \leq dim(J_{2n,1}) + \lceil \frac{2n}{3} \rceil$ by constructing a resolving set W in $J_{2n,2}$ with $dim(J_{2n,1}) + \lceil \frac{2n}{3} \rceil$ vertices.

We consider three cases according to the residue class modulo 3 to which n belongs.

Case-(i): When $n \equiv 0(mod3)$, then we may write $2n = 3k$, where $k \geq 4$, is even and $dim(J_{2n,1}) + \lceil \frac{2n}{3} \rceil = 2k$, in this case W can be considered as:

$$W = \{v_1^j, v_{2n-1}^j; 1 \leq j \leq 2\} \cup \{v_{6i+1}^1, v_{6i+3}^1, v_{6i-1}^2, v_{6i+1}^2; 1 \leq i \leq \frac{k}{2} - 1\}$$

Case-(ii): When $n \equiv 1(mod3)$, then we may write $2n = 3k+2$, where $k \geq 4$ is even and $dim(J_{2n,1}) + \lceil \frac{2n}{3} \rceil = 2k+1$, in this case W can be considered as:

$$W = \{v_1^1, v_{2n-1}^1, v_1^2\} \cup \{v_{6i+1}^1, v_{6i+3}^1; 1 \leq i \leq \frac{k}{2} - 1\} \cup \{v_{6i-1}^2, v_{6i+1}^2; 1 \leq i \leq \frac{k}{2}\}$$

Case-(iii): When $n \equiv 2(mod3)$, then we may write $2n = 3k+1$, where $k \geq 5$ is even and $dim(J_{2n,1}) + \lceil \frac{2n}{3} \rceil = 2k+1$, in this case W can be considered as:

$$W = \{v_1^1, v_1^2, v_{2n-1}^2\} \cup \{v_{6i+1}^1, v_{6i+3}^1; 1 \leq i \leq \frac{k-1}{2}\} \cup \{v_{6i-1}^2, v_{6i+1}^2; 1 \leq i \leq \frac{k-1}{2}\}$$

The set W contains a unique $2-2$ major gap having at most five vertices and all other gaps are $2-2$ minor gaps which contain at most three vertices. The set W is a resolving set of $J_{2n,2}$ since any two major or any two minor vertices respectively lying in different gaps or in the same gap are separated by at least one vertex in the set of three vertices of W generating these neighboring gaps. When gaps are not neighboring gaps, then the set of four vertices of W which generate two gaps make the representation unique of each vertex of these two gaps. Representation of central vertex v is $(2,2,2,\ldots,2)$, which is different from the representation of all other vertices of $J_{2n,2}$. Hence,

$$dim(J_{2n,2}) \leq dim(J_{2n,1}) + \left\lceil \frac{2n}{3} \right\rceil \tag{1}$$

Now we show that $dim(J_{2n,2}) \geq dim(J_{2n,1}) + \lceil \frac{2n}{3} \rceil$. As the central vertex v does not belong to any basis of J_{3n}. Let B be a basis of $J_{2n,2}$ such that $|B|= r$ then we have r gaps. By lemma 2 B contains at most one major gap, without loss of generality we can say major gap lies on $C_{2n,1}$. Hence B induces $\lfloor \frac{r}{2} \rfloor$ gaps on $C_{2n,1}$ and $\lceil \frac{r}{2} \rceil$ gaps on $C_{2n,2}$.

We denote the gaps on $C_{2n,1}$ by $G^1_1, G^1_2, G^1_3, \ldots, G^1_{\lfloor \frac{r}{2} \rfloor}$ where G^1_i and G^1_{i+1} are called neighboring gaps for $1 \leq i \leq \lfloor \frac{r}{2} \rfloor - 1$ as well as $G^1_{\lfloor \frac{r}{2} \rfloor}$ is also neighboring gap of G^1_1 and the gaps on $C_{2n,2}$ will be denoted by $G^2_1, G^2_2, G^2_3, \ldots, G^2_{\lceil \frac{r}{2} \rceil}$ where G^2_i and G^2_{i+1} are called neighboring gaps for $1 \leq i \leq \lceil \frac{r}{2} \rceil - 1$ as well as $G^2_{\lceil \frac{r}{2} \rceil}$ is also neighboring gap of G^2_1. By Lemma 2, suppose G^1_1 is a major gap. By Lemmas 3 and 4, we can write

$$|G^1_1 + G^1_2| \leq 6, \;\; |G^1_1 + G^1_{\lfloor \frac{r}{2} \rfloor}| \leq 6, \;\; \| G^1_i + G^1_{i+1}| \leq 4, \;\; for \;\; 2 \leq i \leq \left\lfloor \frac{r}{2} \right\rfloor - 1$$

and

$$|G^2_1 + G^2_2| \leq 4, \;\; \| G^2_1 + G^2_{\lceil \frac{r}{2} \rceil}| \leq 4, \;\; \| G^2_i + G^2_{i+1}| \leq 4, \;\; for \;\; 2 \leq i \leq \left\lceil \frac{r}{2} \right\rceil - 1$$

We consider two cases according to the residue class modulo 2 to which r belongs.

Case-(i): When $r \equiv 0(mod 2)$: In this case $\lfloor \frac{r}{2} \rfloor = \lceil \frac{r}{2} \rceil = \frac{r}{2}$
By summing the above inequality we have

$$2(2n - \frac{r}{2}) = 2\sum_{i=1}^{\frac{r}{2}} \left| G^1_i \right| \leq 2r + 4 \Rightarrow \frac{r}{2} \geq \frac{2n-2}{3} \Rightarrow \frac{r}{2} \geq \left\lfloor \frac{2n}{3} \right\rfloor \tag{2}$$

Again

$$2(2n - \frac{r}{2}) = 2\sum_{i=1}^{\frac{r}{2}} \left| G^1_i \right| \leq 2r \Rightarrow \frac{r}{2} \geq \frac{2n}{3} \Rightarrow \frac{r}{2} \geq \left\lceil \frac{2n}{3} \right\rceil \tag{3}$$

From Equations (2) and (3) we have,

$$r \geq \left\lfloor \frac{2n}{3} \right\rfloor + \left\lceil \frac{2n}{3} \right\rceil \Rightarrow dim(J_{2n,2}) \geq dim(J_{2n,1}) + \left\lceil \frac{2n}{3} \right\rceil$$

Case-(ii): When $r \equiv 1(mod 2)$: In this case $\lfloor \frac{r}{2} \rfloor = \frac{r-1}{2}$ and $\lceil \frac{r}{2} \rceil = \frac{r+1}{2}$
By summing the above inequality we have

$$2(2n - \frac{r-1}{2}) = 2\sum_{i=1}^{\frac{r-1}{2}} \left| G^1_i \right| \leq 4 + 4\left(\frac{r-1}{2}\right) \Rightarrow \frac{r-1}{2} \geq \frac{2n-2}{3} \Rightarrow \frac{r-1}{2} \geq \left\lfloor \frac{2n}{3} \right\rfloor \tag{4}$$

and

$$2(2n - \frac{r+1}{2}) = 2\sum_{i=1}^{\frac{r+}{2}} \left| G^2_i \right| \leq 4\left(\frac{r+1}{2}\right) \Rightarrow \frac{r+1}{2} \geq \frac{2n}{3} \Rightarrow \frac{r+1}{2} \geq \left\lceil \frac{2n}{3} \right\rceil \tag{5}$$

From Equations (4) and (5) we have

$$r \geq \left\lfloor \frac{2n}{3} \right\rfloor + \left\lceil \frac{2n}{3} \right\rceil \Rightarrow dim(J_{2n,2}) \geq dim(J_{2n,1}) + \left\lceil \frac{2n}{3} \right\rceil \quad (6)$$

Now from Equations (1) and (6) we conclude that,

$$dim(J_{2n,2}) = dim(J_{2n,1}) + \left\lceil \frac{2n}{3} \right\rceil$$

which complete the proof. □

Theorem 2. *If $J_{2n,m}$ be a double gear graph for $n \geq 4$, $m \geq 3$, then*

$$dim(J_{2n,m}) = dim(J_{2n,1}) + (m-1)\left\lceil \frac{2n}{3} \right\rceil$$

Proof. We will prove this result by induction on levels of gear graph denoted by $J_{2n,m}$.

When $m = 1$, then $dim(J_{2n,1}) = \lfloor \frac{2n}{3} \rfloor$ is obtained in [17].

When $m = 2$, then $dim(J_{2n,2}) = dim(J_{2n,1}) + \lceil \frac{2n}{3} \rceil$ by Theorem 1.

Now we assume that the statement is true for $m = k$, $dim(J_{2n,k}) = dim(J_{2n,1}) + (k-1)\lceil \frac{2n}{3} \rceil$. we will show the result for $m = k+1$, by using concept of Theorem 1 we have $dim(J_{2n,k+1}) = dim(J_{2n,k}) + \lceil \frac{2n}{3} \rceil$.

Now $dim(J_{2n,k+1}) = dim(J_{2n,k}) + \lceil \frac{2n}{3} \rceil = dim(J_{2n,1}) + (k-1)\lceil \frac{2n}{3} \rceil + \lceil \frac{2n}{3} \rceil$. $\Rightarrow dim(J_{2n,k+1}) = dim(J_{2n,1}) + k\lceil \frac{2n}{3} \rceil$. Hence the result is true for all positive integers $m \geq 3$. □

3. The Metric Dimension of Generalized Gear Graph J_{3n}

Definition 5. *To define the generalized gear graph J_{3n}: consider a cycle C_{3n} having vertices $v_1, v_2, v_3, \ldots, v_{3n}, v_1$ with $n \geq 2$, take a new vertex v adjacent to n vertices $v_3, v_6, v_9, \ldots, v_{3n}$ of C_{3n}. The generalized gear graph J_{3n} has order $3n + 1$ and size $4n$. It can be obtained from wheel graph W_{3n} by alternately deleting $2n$ spokes.*

Construction and Observations

The vertices of C_{3n} in the graph J_{3n} are of two kinds: vertices of degree 2 and 3. Vertices of degree 2 and 3 will be considered as minor and major vertices respectively. The graph J_{3n} is a bipartite graph in which one bipartition class contains minor vertices together with central vertex v and the second bipartition class contain major vertices. In the proof of Lemmas 5–9, major vertices will be represented by a star. One can easily check that:

- When $n = 2$
  $dim(J_6) = 2$, (one minor vertex of C_6 together with central vertex v form basis).
- When $n = 3$
  $dim(J_9) = 2 = dim(J_{12})$, (two minor vertices w_1 and w_2 such that $d(w_1, w_2) = 3$ form basis).
- When $n = 5$
  $dim(J_{15}) = 3$, (three minor vertices w_1,w_2 and w_3 such that $d(w_1, w_2) = d(w_2, w_3) = d(w_3, w_4) = 4$ form basis).

For the graph $J_{3n}, n \geq 4$ central vertex v does not belong to any basis. Since $d(v_i, v) \leq 2\forall$, $1 \leq i \leq 3n$, and $diam(J_{3n}) = 4$ if central vertex v belongs to any metric basis B then there must exist two distinct vertices u_i and u_j for $1 \leq i \neq j \leq 3n$ such that $r(u_i|B) = r(u_j|B)$. If B is a basis of J_{3n} and central vertex v does not belong to B then by using the concept of gap given in Section 2, we have again three kinds of gaps i.e $2-2$ gpa, $2-3$ gap, and $3-3$ gap.

Lemma 5. *If B is a basis of J_{3n}, $n \geq 6$ then every $2-2$ gap, $2-3$ gap and $3-3$ gap of B contains at most 8, 7 and 5 points respectively.*

Proof. Suppose the basis set B contains a $2-2$ gap of nine consecutive vertices $u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9$ of C_{3n} such that $deg(u_1) = deg(u_9)$ we have $r(u_4|B) = r(u_6|B)$ in this case. If $2-3$ gap contains more than 7 vertices then it contains 9 consecutive vertices $u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9$ of C_{3n} such that $deg(u_1) = 3$ and $deg(u_9) = 2$ we have $r(u_4|B) = r(u_7|B)$, a contradiction in this case. If a $3-3$ gap contains more then 5 vertices, then it contains 8 consecutive vertices $u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8$ such that $deg(u_1) = deg(u_8) = 2$ then $r(u_3|B) = r(u_6|B)$ which is again a contradiction.

The $2-2$ gap, $2-3$ gap and $3-3$ gap containing 8, 7 and 5 vertices respectively will be referred to as major gaps and the remaining gaps are called minor gaps. □

Lemma 6. *If B is a basis of J_{3n}, $n \geq 6$, then it contains at most one major gap.*

Proof. Suppose B is basis of J_{3n} and it contains two distinct major gaps.

Case-(i): When both gaps are $3-3$ then we have two distinct paths $u_1, u_2, u_3^*, u_4, u_5$ and $w_1, w_2, w_3^*, w_4, w_5$ but $r(u_3^*|B) = r(w_3^*|B)$.

Case-(ii): When both gaps are $2-2$ then we have two distinct paths $u_1, u_2^*, u_3, u_4, u_5^*, u_6, u_7, u_8^*$ and $w_1, w_2^*, w_3, w_4, w_5^*, w_6, w_7, w_8^*$ but $r(u_5^*|B) = r(w_5^*|B)$.

Case-(iii): When both gaps are $2-3$ then we have two distinct paths $u_1, u_2^*, u_3, u_4, u_5^*, u_6, u_7$ and $w_1, w_2^*, w_3, w_4, w_5^*, w_6, w_7$ but $r(u_5^*|B) = r(w_5^*|B)$.

Case-(iv): When one gap is $3-3$ and other is $2-2$ gap then we have two distinct paths $u_1, u_2, u_3^*, u_4, u_5$ and $w_1, w_2^*, w_3, w_4, w_5^*, w_6, w_7, w_8^*$ but $r(u_3^*|B) = r(w_5^*|B)$.

Case-(v): When one gap is $3-3$ and other is $2-3$ gap then we have two distinct paths $u_1, u_2, u_3^*, u_4, u_5$ and $w_1, w_2^*, w_3, w_4, w_5^*, w_6, w_7$ but $r(u_3^*|B) = r(w_5^*|B)$.

Case-(vi): When one gap is $2-2$ and other is $2-3$ gap then we have two distinct paths $u_1^*, u_2, u_3, u_4^*, u_5, u_6, u_7^*, u_8$ and $w_1, w_2^*, w_3, w_4, w_5^*, w_6, w_7$ but $r(u_4^*|B) = r(w_5^*|B)$. □

Lemma 7. *If B is a basis of J_{3n}, $n \geq 6$, containing one major gap either $2-2$ gap or $2-3$ gap then it does not contain $2-2$ gap and $2-3$ minor gap having 7 and 6 vertices respectively.*

Proof. ***Case-(i)***: When one gap is $2-2$ major gap and the other is $2-2$ minor gap having 7 vertices, then we have two distinct paths $u_1, u_2^*, u_3, u_4, u_5^*, u_6, u_7, u_8^*$ and $w_1^*, w_2, w_3, w_4^*, w_5, w_6, w_7^*$ but $r(u_5^*|B) = r(w_4^*|B)$.

Case-(ii): When one gap is $2-2$ major gap and the other is $2-3$ minor gap having 6 vertices, then we have two distinct paths $u_1, u_2^*, u_3, u_4, u_5^*, u_6, u_7, u_8^*$ and $w_1^*, w_2, w_3, w_4^*, w_5, w_6$ but $r(u_5^*|B) = r(w_4^*|B)$.

Case-(iii): When one gap is $2-3$ major gap and the other is $2-2$ minor gap having 7 vertices, then we have two distinct paths $u_1, u_2^*, u_3, u_4, u_5^*, u_6, u_7$ and $w_1^*, w_2, w_3, w_4^*, w_5, w_6, w_7^*$ but $r(u_5^*|B) = r(w_4^*|B)$.

Case-(iv): When one gap is $2-3$ major gap and the other is $2-3$ minor gap having 6 vertices, then we have two distinct paths $u_1, u_2^*, u_3, u_4, u_5^*, u_6, u_7$ and $w_1^*, w_2, w_3, w_4^*, w_5, w_6$ but $r(u_5^*|B) = r(w_4^*|B)$. □

Lemma 8. *If B is a basis of J_{3n}, $n \geq 6$ then any two neighboring gaps contain together at most 13 vertices in which one gap is a major gap.*

Proof. To show the statement, it is sufficient to show that a $2-2$ major gap with 8 vertices has a neighboring $2-2$ minor gap in which 6 vertices cannot occur. If it holds then we have the path $u_1, u_2, u_3^*, u_4, u_5, u_6^*, u_7, u_8, u_9^*, u_{10}, u_{11}, u_{12}^*, u_{13}, u_{14}, u_{15}^*, u_{16}, u_{17}$ with $u_1, u_{10}, u_{17} \in B$ in this case $r(u_7|B) = r(w_{13}|B)$, a contradiction. □

Lemma 9. *If B is a basis of J_{3n}, $n \geq 6$, then any two minor neighboring gaps contain together at most 11 vertices.*

Proof. To show the statement, it is sufficient to show that a $2-2$ gap with 6 vertices has a neighboring $2-2$ gap with 6 vertices cannot occur. Since gap is $2-2$, both base elements must have degree 2. For two consecutive $2-2$ gaps having 6 vertices, we have two possible paths. (i) First possible path is $u_1, u_2, u_3^*, u_4, u_5, u_6^*, u_7, u_8, u_9^*, u_{10}, u_{11}, u_{12}^*, u_{13}, u_{14}, u_{15}^*$ with $u_1, u_8, u_{15}^* \in B$ which is not possible as $d(u_1) = 2 = d(u_8)$ but $d(u_{15}^*) = 3 \neq 2$.

(ii) Second possible path is $u_2, u_3^*, u_4, u_5, u_6^*, u_7, u_8, u_9^*, u_{10}, u_{11}, u_{12}^*, u_{13}, u_{14}, u_{15}^*, u_{16}$ with $u_2, u_9^*, u_{16} \in B$ which is not possible as $d(u_2) = 2 = d(u_{16})$ but $d(u_9^*) = 3 \neq 2$. Hence two minor gap contain at most 11 vertices. □

Theorem 3. *If J_{3n} be the generalized gear graph for $n \geq 6$, then $dim(J_{3n}) = \lfloor \frac{n}{2} \rfloor$.*

Proof. First we prove that $dim(J_{3n}) \leq \lfloor \frac{n}{2} \rfloor$ by constructing a resolving set W in J_{3n} with $\lfloor \frac{n}{2} \rfloor$ vertices. We consider two cases according to the residue class modulo 2 to which n belongs.

Case-(i): When $n \equiv 0(mod2)$ then W can be considered as:

$$W = \{v_1, v_{10}, v_{16}\} \cup \{v_{6i+5}; 3 \leq i \leq \frac{n}{2} - 1\}$$

Case-(ii): When $n \equiv 1(mod2)$ then W can be considered as:

$$W = \{v_1, v_{10}, v_{16}\} \cup \{v_{6i+5}; 3 \leq i \leq \frac{n-1}{2} - 1\}$$

□

The set W contains a unique $2-2$ major gap and all other gaps are $2-2$ minor gap which contain at most five vertices, only one $2-2$ minor gap contains six vertices. The set W is a resolving set of J_{3n} since any two major or any two minor vertices lying in different gaps or in the same gap are separated by at least one vertex in the set of three vertices of W generating these neighboring gaps; when gaps are not neighboring gaps then the set of four vertices of W which generate two gaps make the representation of each vertex of these two gaps unique. Representation of central vertex is $(2, 2, 2, \ldots, 2)$, which is different from the representation of all other vertices of J_{3n}. Hence

$$dim(J_{3n}) \leq \left\lfloor \frac{n}{2} \right\rfloor \tag{7}$$

Now we show that $dim(J_{3n}) \geq \lfloor \frac{n}{2} \rfloor$. By Lemma 5 the central vertex v does not belong to any basis of J_{3n}. Let B be a basis of J_{3n} such that $|B|= r$. We have r gaps on C_{3n} generated by elements of B. We denote these gaps by $G_1, G_2, G_3, \ldots, G_r$ where G_i and G_{i+1} are called neighboring gaps for $1 \leq i \leq r-1$ as well as G_r is also a neighboring gap of G_1. By Lemma 6 at most one of them say G_1 is a major gap. By Lemmas 6 and 7, we have

$$|G_1 + G_2| \leq 13, \;\; |G_1 + G_r| \leq 13$$

and by Lemmas 8 and 9, we have,

$$|G_2 + G_3| \leq 11, \;\; |G_3 + G_4| \leq 11, \;\; |G_i + G_{i+1}| \leq 10, \;\; for \;\; all \;\; 4 \leq i \leq r-1$$

By summing these inequalities, we get,

$$2(3n-r) = 2\sum_{i=1}^{r}|G_i| \le 8+10r \Rightarrow 6n-2r \le 8+10r$$

$$\Rightarrow 6n-8 \le 12r \Rightarrow r \ge \frac{n}{2} - \frac{2}{3}$$

Hence $r = \lfloor \frac{n}{2} \rfloor$.

$$\Rightarrow dim(J_{3n}) \ge \left\lfloor \frac{n}{2} \right\rfloor \tag{8}$$

So from Equations (7) and (8), we get

$$dim(J_{3n}) = \left\lfloor \frac{n}{2} \right\rfloor$$

which complete the proof.

4. Conclusions

In the foregoing section, m-level gear graph $J_{2n,m}$ and generalized gear graph J_{3n} are constructed. It is proved that metric dimension of $J_{2n,m}$ is $dim(J_{2n,1}) + (m-1)\lceil \frac{2n}{3} \rceil$ for every $n \ge 4$ and metric dimension of J_{3n} is $\lfloor \frac{n}{2} \rfloor$ for every $n \ge 6$. This section is closed by raising the following open problem.

Open Problem. Determine the metric dimension of m-level generalized gear graph $J_{2n,k,m}$.

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