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Enumeration of Strongly Regular Graphs on up to 50 Vertices Having S_3 as an Automorphism Group

Marija Maksimović

Department of Mathematics, University of Rijeka, Rijeka 51000, Croatia; mmaksimovic@math.uniri.hr; Tel.: +38-5051-584-665

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Abstract: One of the main problems in the theory of strongly regular graphs (SRGs) is constructing and classifying SRGs with given parameters. Strongly regular graphs with parameters $(37, 18, 8, 9)$, $(41, 20, 9, 10)$, $(45, 22, 10, 11)$, $(49, 24, 11, 12)$, $(49, 18, 7, 6)$ and $(50, 21, 8, 9)$ are the only strongly regular graphs on up to 50 vertices that still have to be classified. In this paper, we give the enumeration of SRGs with these parameters having S_3 as an automorphism group. The construction of SRGs in this paper is a step in the classification of SRGs on up to 50 vertices.

Keywords: strongly regular graph; automorphism group; orbit matrix

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1. Introduction

We assume that the reader is familiar with the basic notions of the theory of finite groups. For basic definitions and properties of strongly regular graphs, we refer the reader to [1–3].

A graph is regular if all its vertices have the same valency; a simple regular graph $\Gamma = (\mathcal{V}, \mathcal{E})$ is strongly regular with parameters (v, k, λ, μ) if it has $|\mathcal{V}| = v$ vertices, valency k, and if any two adjacent vertices are together adjacent to λ vertices, while any two nonadjacent vertices are together adjacent to μ vertices. A strongly regular graph with parameters (v, k, λ, μ) is usually denoted by SRG(v, k, λ, μ). An automorphism of a strongly regular graph Γ is a permutation of vertices of Γ, such that every two vertices are adjacent if and only if their images are adjacent.

By $S(V)$, we denote the symmetric group on the nonempty set V. If $G \leq S(V)$ and $x \in V$, then the set $xG = \{xg | g \in G\}$ is called a G-orbit of x. The set $G_x = \{g \in G | xg = x\}$ is called a stabilizer of x in G. If G is finite, then $|xG| = \frac{|G|}{|G_x|}$. By G_x^g, we denote a conjugate subgroup $g^{-1}G_x g$ of G_x.

One of the main problems in the theory of strongly regular graphs (SRGs) is constructing and classifying SRGs with given parameters. A frequently-used method of constructing combinatorial structures is the construction of combinatorial structures with a prescribed automorphism group. Orbit matrices of block designs have been used for such a construction of combinatorial designs since the 1980s. However, orbit matrices of strongly regular graphs had not been introduced until 2011. Namely, Majid Behbahani and Clement Lam introduced the concept of orbit matrices of strongly regular graphs in [4]. They developed an algorithm for the construction of orbit matrices of strongly regular graphs with an automorphism group of prime order and the construction of corresponding strongly regular graphs.

A method of constructing strongly regular graphs admitting an automorphism group of composite order using orbit matrices is introduced and presented in [5]. Using this method, we classify strongly regular graphs with parameters $(37, 18, 8, 9)$, $(41, 20, 9, 10)$, $(45, 22, 10, 11)$, $(49, 18, 7, 6)$, $(49, 24, 11, 12)$ and $(50, 21, 8, 9)$ having S_3 as an automorphism group. These graphs are the only strongly regular graphs with up to 50 vertices that still have to be classified. Enumeration of SRGs with these parameters having a non-abelian automorphism group of order six, i.e., the construction of SRGs with these parameters in this paper, is a step in that classification. Using this construction, we show that there is no SRG$(37,18,8,9)$ having S_3 as an automorphism group. Furthermore, we show that there are 80 SRGs$(41,20,9,10)$, 288 SRGs$(45,22,10,11)$, 72 SRGs$(49,24,11,12)$, 34 SRGs$(49,18,7,6)$ and 45 SRGs$(50,21,8,9)$ having a non-abelian automorphism group of order six.

The paper is organized as follows: After a brief description of the terminology and some background results, in Section 2, we describe the concept of orbit matrices, based on the work of Behbahani and Lam [4]. In Section 3, we explain the method of construction of strongly regular graphs from their orbit matrices presented in [5]. In Section 4, we apply this method to construct strongly regular graphs with parameters $(37,18,8,9)$, $(41,20,9,10)$, $(45,22,10,11)$, $(49,18,7,6)$, $(49,24,11,12)$ and $(50,21,8,9)$ having a non-abelian automorphism group of order six.

For the construction of orbit matrices and graphs, we have used our own computer programs written for GAP [6]. Isomorphism testing for the obtained graphs and the analysis of their full automorphism groups are conducted using the Grape package for GAP [7].

2. Orbit Matrices of Strongly Regular Graphs

Orbit matrices of block designs have been frequently used for the construction of block designs with a presumed automorphism group, see, e.g., [8–11]. In 2011, Behbahani and Lam introduced the concept of orbit matrices of SRGs (see [4]). While Behbahani and Lam were mostly focused on orbit matrices of strongly regular graphs admitting an automorphism of prime order, a general definition of an orbit matrix of a strongly regular graph is given in [12].

Let Γ be an SRG(v,k,λ,μ) and A be its adjacency matrix. Suppose an automorphism group G of Γ partitions the set of vertices V into b orbits $O_1,\ldots,O_b$, with sizes $n_1,\ldots,n_b$, respectively. The orbits divide A into submatrices $[A_{ij}]$, where A_{ij} is the adjacency matrix of vertices in O_i versus those in O_j. We define matrices $C=[c_{ij}]$ and $R=[r_{ij}]$, $1\le i,j\le b$, such that c_{ij} is the column sum of A_{ij} and r_{ij} is the row sum of A_{ij}. The matrix R is related to C by:

$$r_{ij}n_i = c_{ij}n_j. \tag{1}$$

Since the adjacency matrix is symmetric, it follows that:

$$R = C^T. \tag{2}$$

The matrix R is the row orbit matrix of the graph Γ with respect to G, and the matrix C is the column orbit matrix of the graph Γ with respect to G.

Behbahani and Lam showed that orbit matrices $R=[r_{ij}]$ and $R^T=C=[c_{ij}]$ satisfy the condition:

$$\sum_{s=1}^{b} c_{is}r_{sj}n_s = \delta_{ij}(k-\mu)n_j + \mu n_i n_j + (\lambda-\mu)c_{ij}n_j.$$

Since $R=C^T$, it follows that:

$$\sum_{s=1}^{b} \frac{n_s}{n_j} c_{is}c_{js} = \delta_{ij}(k-\mu) + \mu n_i + (\lambda-\mu)c_{ij} \tag{3}$$

and:

$$\sum_{s=1}^{b} \frac{n_s}{n_j} r_{si}r_{sj} = \delta_{ij}(k-\mu) + \mu n_i + (\lambda-\mu)r_{ji}.$$

Therefore, in [12], we introduced the following definition of orbit matrices of strongly regular graphs.

Definition 1. *A $(b \times b)$-matrix $R = [r_{ij}]$ with entries satisfying conditions:*

$$\sum_{j=1}^{b} r_{ij} = \sum_{i=1}^{b} \frac{n_i}{n_j} r_{ij} = k \tag{4}$$

$$\sum_{s=1}^{b} \frac{n_s}{n_j} r_{si} r_{sj} = \delta_{ij}(k-\mu) + \mu n_i + (\lambda - \mu) r_{ji} \tag{5}$$

where $0 \le r_{ij} \le n_j$, $0 \le r_{ii} \le n_i - 1$ and $\sum_{i=1}^{b} n_i = v$, is called a row orbit matrix for a strongly regular graph with parameters (v, k, λ, μ) and the orbit length distribution $(n_1, \ldots, n_b)$.

Definition 2. *A $(b \times b)$-matrix $C = [c_{ij}]$ with entries satisfying conditions:*

$$\sum_{i=1}^{b} c_{ij} = \sum_{j=1}^{b} \frac{n_j}{n_i} c_{ij} = k \tag{6}$$

$$\sum_{s=1}^{b} \frac{n_s}{n_j} c_{is} c_{js} = \delta_{ij}(k-\mu) + \mu n_i + (\lambda - \mu) c_{ij} \tag{7}$$

where $0 \le c_{ij} \le n_i$, $0 \le c_{ii} \le n_i - 1$ and $\sum_{i=1}^{b} n_i = v$, is called a column orbit matrix for a strongly regular graph with parameters (v, k, λ, μ) and the orbit length distribution $(n_1, \ldots, n_b)$.

3. The Method of Construction

A method of constructing strongly regular graphs admitting an automorphism group of composite order using orbit matrices is introduced and presented in [5]. In this section, we will give a brief overview of this method.

For the construction of strongly regular graphs with parameters (v, k, λ, μ), we first check whether these parameters are feasible (see [2]). Then, we select the group G and assume that it acts as an automorphism group of an SRG(v, k, λ, μ). The construction of strongly regular graphs admitting an action of a presumed automorphism group, using orbit matrices, consists of the following two basic steps:

- Construction of orbit matrices for the presumed automorphism group
- Construction of strongly regular graphs from the obtained orbit matrices (indexing of orbit matrices)

We could use row or column orbit matrices, but since we are constructing matrices row by row, it is more convenient for us to use column orbit matrices. For the construction of orbit matrices for the presumed automorphism group, we need to determine all possible orbit length distributions $(n_1, n_2, \ldots, n_b)$ for an action of the group G. Suppose an automorphism group G of Γ partitions the set of vertices V into b orbits $O_1, \ldots, O_b$, with sizes $n_1, \ldots, n_b$. Obviously, n_i is a divisor of $|G|$, $i = 1, \ldots, b$, and:

$$\sum_{i=1}^{b} n_i = v.$$

When determining the orbit length distribution, we also use the following result that can be found in [13].

Theorem 1. *Let $s < r < k$ be the eigenvalues of an SRG(v, k, λ, μ), then:*

$$\phi \le \frac{\max(\lambda, \mu)}{k - r} v,$$

where ϕ is the number of fixed points for a nontrivial automorphism group G.

For each orbit length distribution we construct column orbit matrices. For the construction of orbit matrices, we first need to find prototypes.

3.1. Prototypes for a Row of a Column Orbit Matrix

A prototype for a row of a column orbit matrix C gives us information about the number of occurrences of each integer as an entry of a particular row of C. Behbahani and Lam [4,13] introduced the concept of a prototype for a row of a column orbit matrix C of a strongly regular graph with a presumed automorphism group of prime order. We will generalize this concept and describe a prototype for a row of a column orbit matrix C of a strongly regular graph under a presumed automorphism group of composite order. Prototypes are useful in the first step of the construction of strongly regular graphs, namely the construction of column orbit matrices.

Suppose an automorphism group G of a strongly regular graph Γ with parameters (v, k, λ, μ) partitions the set of vertices V into b orbits $O_1, \ldots, O_b$, of sizes $n_1, \ldots, n_b$. With $l_i, i = 1, \ldots, \rho$, we denote all divisors of $|G|$ in ascending order ($l_1 = 1, \ldots, l_\rho = |G|$).

3.1.1. Prototypes for a Fixed Row

Consider the row r of a column orbit matrix C. We say that it is a fixed row of a matrix C if $n_r = 1$, i.e., if it corresponds to an orbit of length one. The entries in this row are either zero or one. Let d_{l_i} denote the number of orbits whose length are l_i, $i = 1, \ldots, \rho$.

Let x_e denote the number of occurrences of an element $e \in \{0, 1\}$ at the positions of the row r that correspond to the orbits of length one. It follows that:

$$x_0 + x_1 = d_1, \tag{8}$$

where d_1 is the number of orbits of length one. Since the diagonal elements of the adjacency matrix of a strongly regular graph are equal to zero, it follows that $x_0 \geq 1$.

Let $y_e^{(l_i)}$ denote the number of occurrences of an element $e \in \{0, 1\}$ at the positions of the row r that correspond to the orbits of length l_i ($i = 2, \ldots, \rho$). We have:

$$y_0^{(l_i)} + y_1^{(l_i)} = d_{l_i}, \quad i = 2, \ldots, \rho \tag{9}$$

Because the row sum of an adjacency matrix of Γ is equal to k, it follows that:

$$x_1 + \sum_{i=2}^{\rho} l_i \cdot y_1^{(l_i)} = k. \tag{10}$$

The vector:

$$p_1 = (x_0, x_1; y_0^{(l_2)}, y_1^{(l_2)}; \ldots; y_0^{(l_\rho)}, y_1^{(l_\rho)})$$

whose components are nonnegative integer solutions of the equalities (8), (9) and (10) is called a prototype for a fixed row.

3.1.2. Prototypes for a Non-Fixed Row

Let us consider the row r of a column orbit matrix C, where $n_r \neq 1$. Let d_{l_i} denote the number of orbits whose length is l_i, $i = 1, \ldots, \rho$.

If a fixed vertex is adjacent to a vertex from an orbit $O_i, 1 \leq i \leq b$, then it is adjacent to all vertices from the orbit O_i. Therefore, the entries at the positions corresponding to fixed columns are either zero or n_r. Let x_e denote the number of occurrences of an element $e \in \{0, n_r\}$ at those positions of the row r, which correspond to the orbits of length one. We have:

$$x_0 + x_{n_r} = d_1. \tag{11}$$

The entries at the positions corresponding to the orbits whose lengths are greater than one are $0, 1, \dots, n_r - 1$ or n_r. The entry at the position (r, r) is $0 \le c_{rr} \le n_r - 1$, since the diagonal elements of the adjacency matrix of strongly regular graphs are zero.

Let $y_e^{(l_i)}$ denote the number of occurrences of an element $e \in \{0, \dots, n_r\}$ of row r at the positions that correspond to the orbits of length l_i $(i = 2, \dots, \rho)$. From (1) and (2), we conclude that:

$$c_{ri} n_i = c_{ir} n_r, \tag{12}$$

where $c_{ir} \in \{0, \dots, n_i\}$. If $c_{ri} \cdot \frac{n_i}{n_r} \notin \{0, \dots, n_i\}$, then $y_{c_{ri}}^{(n_i)} = 0$. It follows that:

$$\sum_{e=0}^{n_r} y_e^{(l_i)} = d_{l_i}, \quad i = 2, \dots, \rho. \tag{13}$$

Since the row sum of an adjacency matrix is equal to k, we have that:

$$x_{n_r} + \sum_{i=2}^{\rho} \sum_{h=1}^{n_r} y_h^{(l_i)} \cdot h \cdot \frac{n_{l_i}}{n_r} = k, \tag{14}$$

From (3), we conclude that:

$$\sum_{s=1}^{b} c_{rs} c_{rs} n_s = (k - \mu) n_r + \mu n_r^2 + (\lambda - \mu) c_{rr} n_r,$$

where $c_{rr} \in \{0, \dots, n_r - 1\}$. It follows that:

$$n_r^2 x_{n_r} + \sum_{i=2}^{\rho} \sum_{h=1}^{n_r} y_h^{(l_i)} \cdot h^2 \cdot n_{l_i} = (k - \mu) n_r + \mu n_r^2 + (\lambda - \mu) c_{rr} n_r, \tag{15}$$

The vector:

$$p_{n_r} = (x_0, x_{n_r}; y_0^{l_2}, \dots, y_{n_r}^{l_2}; \dots; y_0^{l_\rho}, \dots, y_{n_r}^{l_\rho}),$$

whose components are nonnegative integer solutions of Equalities (11), (13), (14) and (15) is called a prototype for a row corresponding to the orbit of length n_r.

Using prototypes, we construct an orbit matrix row by row.

Not every orbit matrix gives rise to a strongly regular graph, while, on the other hand, a single orbit matrix may produce several nonisomorphic strongly regular graphs. Further, nonisomorphic orbit matrices may produce isomorphic graphs. Therefore, the constructed graphs need to be checked for isomorphism.

Theorem 2. *Let $\Gamma = (V, E)$ be a strongly regular graph, $G \le Aut(\Gamma)$, and let $(b \times b)$-matrix C be a column orbit matrix of the graph Γ with respect to the group G. Further, let α be an element of $S(V)$ with the following property: if $\alpha(i) = j$, then the stabilizer G_{x_i} is conjugate to G_{x_j}, where $x_i, x_j \in V$ and $O_i = x_i G, O_j = x_j G$. Then, there exists permutation $g^* \in C_{S(V)}(G)$ such that $\alpha(i) = j \iff g^*(O_i) = O_j$.*

Definition 3. *Let $A = (a_{ij})$ be an $(b \times b)$-matrix and $\alpha \in S_b$. The matrix $B = A\alpha$ is the $(b \times b)$-matrix $B = (b_{ij})$, where $b_{\alpha(i)\alpha(j)} = a_{ij}$. If $A\alpha = A$, then α is called an automorphism of the matrix A.*

Definition 4. *Let an $(b \times b)$-matrix $A = (a_{ij})$ be the orbit matrix of a strongly regular graph Γ with respect to the group $G \le Aut(\Gamma)$. A mapping $\alpha \in S_b$ is called an isomorphism from A to $B = A\alpha$ if the following condition holds: if $\alpha(i) = j$, then the stabilizer G_{x_i} is conjugate to G_{x_j}. We say that the orbit matrices A and B are isomorphic. If $A\alpha = A$, then α is called an automorphism of the orbit matrix A. All automorphisms of an orbit matrix A form the full automorphism group of A, denoted by $Aut(A)$.*

During the construction of orbit matrices, for the elimination of isomorphic structures, we use permutations that satisfy the conditions from Theorem 2, i.e., isomorphisms from Definition 4.

The next big step of the construction of graphs, called indexing, often cannot be performed in a reasonable amount of time. To make such a construction possible, for a refinement of constructed orbit matrices, we use the composition series:

$$\{1\} = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_n = G,$$

of a solvable automorphism group G of a strongly regular graph. Let Γ be a strongly regular graph and $H \trianglelefteq G \leq \mathrm{Aut}(\Gamma)$. Each G-orbit of Γ decomposes to H-orbits of the same size (see [9]). Therefore, each orbit matrix for the group G decomposes to orbit matrices for the group H, and the following theorem holds [5].

Theorem 3. *Let Ω be a finite nonempty set, $H \triangleleft G \leq S(\Omega)$, $x \in \Omega$ and $xG = \bigsqcup_{i=1}^{h} x_i H$. Then, a group G/H acts transitively on the set $\{x_i H \mid i = 1, 2, \ldots, h\}$.*

Therefore, after we have constructed corresponding orbit matrices for the group G, we continue until we find all refinements for the normal subgroup $H_{n-1} \trianglelefteq G$. In the next step, we obtain orbit matrices for the group H_{n-2}, H_{n-3}, and so on. Our last step is the construction of the corresponding orbit matrices for the subgroup $H_0 = \{1\}$, i.e., construction of adjacency matrices of the strongly regular graphs. The concept of the G-isomorphism of two-block designs was introduced in [14]. For the elimination of mutually-isomorphic structures, we use the concept of G-isomorphism.

Definition 5. *Let $\Gamma_1 = (V, E_1)$ and $\Gamma_2 = (V, E_2)$ be strongly regular graphs, and let $G \leq Aut(\Gamma_1) \cap Aut(\Gamma_2) \leq S(V)$. An isomorphism $\alpha : \Gamma_1 \to \Gamma_2$ is called a G-isomorphism from Γ_1 onto Γ_2 if there is an automorphism $\tau : G \to G$ such that for each $x, y \in V$ and each $g \in G$, the following holds:*

$$(\alpha x)(\tau g) = \alpha y \Leftrightarrow xg = y.$$

If α is a $G-$isomorphism from Γ_1 to Γ_2, then the vertices x_i and x_j are in the same G-orbit if and only if the vertices $\alpha(x_i)$ and $\alpha(x_j)$ are in the same G-orbit.

Lemma 1. *Let $\Gamma_1 = (V, E_1)$ and $\Gamma_2 = (V, E_2)$ be strongly regular graphs, and let $G \leq Aut(\Gamma_1) \cap Aut(\Gamma_2) \leq S = S(V)$. A permutation $\alpha \in S$ is a G-isomorphism from Γ_1 onto Γ_2 if and only if α is an isomorphism from Γ_1 to Γ_2 and $\alpha \in N_S(G)$, where $N_S(G)$ is the normalizer of G in S.*

In each step of refinement of an orbit matrix A, we eliminate isomorphic orbit matrices using the automorphisms from $Aut(A)$, because each automorphism of an orbit matrix determines an G-isomorphism.

4. SRGs with up to 50 Vertices Having S_3 as an Automorphism Group

SRGs with parameters $(37, 18, 8, 9)$, $(41, 20, 9, 10)$, $(45, 22, 10, 11)$, $(49, 24, 11, 12)$, $(49, 18, 7, 6)$ and $(50, 21, 8, 9)$ are the only strongly regular graphs on up to 50 vertices that still have to be classified [2,15]. According to [2], it is known that strongly regular graphs with these parameters exist, but their final enumeration result is not known. In this section, we present the results of the constructed strongly regular graphs with parameters $(37, 18, 8, 9)$, $(41, 20, 9, 10)$, $(45, 22, 10, 11)$, $(49, 18, 7, 6)$, $(49, 24, 11, 12)$ and $(50, 21, 8, 9)$ having $S_3 \cong Z_3 : Z_2 \cong \langle \rho\phi | \rho^3 = 1, \phi^2 = 1, \phi\rho\phi = \rho^{-1} \rangle$ as an automorphism group. In each case, we construct strongly regular graphs by using the algorithm described in Section 3. The orbit lengths for an action of the group G at the set of points of a graph can get values from the set $\{1, 2, 3, 6\}$. Using the program Mathematica [16], we get all possible orbit length distributions

(d_1, d_2, d_3, d_6) for the action of S_3 on a particular SRG that satisfy Theorem 1. For each orbit length distribution, we find the corresponding prototypes using Mathematica. Using our own programs, which are written for GAP [6], we construct all orbit matrices for a given orbit length distribution. Having in mind the action of the whole group, we refine the constructed orbit matrices. For the refinement, we use the composition series

$$\{1\} \trianglelefteq \langle\rho\rangle \trianglelefteq S_3$$

and obtain orbit matrices for the action of the subgroup $Z_3 \triangleleft S_3$. In this step, each orbit of length two and six decomposes to two orbits of length one and three, respectively. In the final step of the construction, we obtain adjacency matrices of the strongly regular graphs with particular parameters admitting a non-abelian automorphism group of order six. Finally, we check isomorphisms of strongly regular graphs and determine orders of the full automorphism groups using the Grape package for GAP [7].

4.1. SRGs(37,18,8,9)

In this section, we present the results of SRGs(37,18,8,9) having S_3 as an automorphism group. According to [17], there are at least 6760 SRGs(37,18,8,9), and none of them have S_3 as an automorphism group. We show that there are no strongly regular graphs with parameters (37,18,8,9) having a non-abelian automorphism group of order six.

We get 176 possibilities for orbit length distributions, but only three give rise to orbit matrices. In Table 1, we present the number of mutually-nonisomorphic orbit matrices for each orbit length distribution, the number of orbit matrices for Z_3 (obtained by the refinement) and the number of constructed SRGs with parameters (37,18,8,9). These calculations prove Theorem 4.

Table 1. Number of orbit matrices and SRGs(37,18,8,9) for the automorphism group S_3.

Distribution	#OM-S_3	#OM-Z_3	#SRGs
(1,0,0,6)	3	6	0
(1,0,4,4)	3	3	0
(1,0,8,2)	3	3	0

Theorem 4. *There are no strongly regular graphs with parameters* $(37, 18, 8, 9)$ *having an automorphism group isomorphic to the symmetric group* S_3.

4.2. SRGs(41,20,9,10)

In this section, we present the results of SRGs$(41, 20, 9, 10)$ having S_3 as an automorphism group. We show that there are exactly 80 strongly regular graphs with parameters $(41, 20, 9, 10)$ having a non-abelian automorphism group of order six.

We get 216 possibilities for orbit length distributions, but only one gives rise to any orbit matrices. In Table 2, we present the number of mutually-nonisomorphic orbit matrices for each orbit length distribution, the number of orbit matrices for Z_3 (obtained by the refinement) and the number of constructed SRGs with parameters $(41, 20, 9, 10)$. These calculations prove Theorem 5. Information about the orders of the full automorphism groups is presented in Table 3.

Table 2. Number of orbit matrices and SRGs(41,20,9,10) for the automorphism group S_3.

Distribution	#OM-S_3	#OM-Z_3	#SRGs
(1,2,4,4)	10	10	80

Theorem 5. *Up to isomorphism, there are exactly 80 strongly regular graphs with parameters* $(41, 20, 9, 10)$ *having an automorphism group isomorphic to the symmetric group* S_3.

Table 3. SRGs with parameters (41,20,9,10) having S_3 as an automorphism group.

|Aut(Γ)|	**#SRGs**
6	80

The adjacency matrices of the constructed SRGs can be found at [18].

4.3. SRGs(45,22,10,11)

In this section, we present the results of SRGs$(45, 22, 10, 11)$ having S_3 as an automorphism group. We show that there are exactly 288 strongly regular graphs with parameters $(45, 22, 10, 11)$ having a non-abelian automorphism group of order six.

We get 309 possibilities for orbit length distributions, but only one gives rise to any orbit matrices. In Table 4, we present the number of mutually-nonisomorphic orbit matrices for each orbit length distribution, the number of orbit matrices for Z_3 (obtained by the refinement) and the number of constructed SRGs with parameters $(45, 22, 10, 11)$. These calculations prove Theorem 6. Information about orders of the full automorphism groups is presented in Table 5.

Table 4. Number of orbit matrices and SRGs(45,22,10,11) for the automorphism group S_3.

Distribution	**#OM-S_3**	**#OM-Z_3**	**#SRGs**
(1,4,4,4)	7	7	288

Table 5. SRGs with parameters (45,22,10,11) having S_3 as an automorphism group

|Aut(Γ)|	**#SRGs**
6	288

Theorem 6. *Up to isomorphism, there are exactly 288 strongly regular graphs with parameters* $(45, 22, 10, 11)$ *having an automorphism group isomorphic to the symmetric group* S_3.

The adjacency matrices of the constructed SRGs can be found at [19].

4.4. SRGs(49,18,7,6)

In the paper [5], we proved the following theorem.

Theorem 7. *Up to isomorphism, there are exactly* 36 *strongly regular graphs with parameters* $(49, 18, 7, 6)$ *having an automorphism group isomorphic to the symmetric group* S_3.

Two of these graphs have not been constructed in [4,20]. The adjacency matrices of the constructed SRGs can be found at [21].

4.5. SRGs(49,24,11,12)

In this section, we present the results of SRGs$(49, 24, 11, 12)$ having S_3 as an automorphism group. We show that there are exactly 72 strongly regular graphs with parameters $(49, 24, 11, 12)$ having a non-abelian automorphism group of order six.

We get 435 possibilities for orbit length distributions, but only a few give rise to orbit matrices. In Table 6, we present the number of mutually-nonisomorphic orbit matrices for each orbit length distribution, the number of orbit matrices for Z_3 (obtained by the refinement) and the number of constructed SRGs with parameters $(49, 24, 11, 12)$. Thus, we prove Theorem 8. Information about orders of the full automorphism groups is presented in Table 7.

Table 6. Number of orbit matrices and SRGs$(49, 24, 11, 12)$ for the automorphism group S_3.

Distribution	#OM-S_3	#OM-Z_3	#SRGs	Distribution	#OM-S_3	#OM-Z_3	#SRGs
(0,2,3,6)	8	16	6	(1,3,0,7)	6	6	0
(0,2,5,5)	4	0	0	(1,3,2,6)	10	2	0
(0,2,7,4)	8	0	0	(1,3,6,4)	2	2	0
(1,0,0,8)	2	15	2	(1,6,0,6)	1	0	0
(1,0,2,7)	20	32	0	(3,2,0,7)	4	10	0
(1,0,8,4)	26	24	12	(3,2,6,4)	6	16	0
(1,0,10,3)	2	0	0	(4,0,9,3)	6	0	0
(1,0,12,2)	16	0	0	(5,1,0,7)	2	4	0
(1,0,14,1)	12	0	0	(5,1,6,4)	2	2	12
				(7,0,0,7)	2	2	40

Table 7. SRGs with parameters $(49, 24, 11, 12)$ having S_3 as an automorphism group.

$\|\mathrm{Aut}(\Gamma)\|$	#SRGs
6	42
18	22
24	4
126	4

Theorem 8. *Up to isomorphism, there are exactly 72 strongly regular graphs with parameters* $(49, 24, 11, 12)$ *having an automorphism group isomorphic to the symmetric group* S_3.

The adjacency matrices of the constructed SRGs can be found at [22].

4.6. SRGs(50,21,8,9)

In this section, we present the results of SRGs$(50, 21, 8, 9)$ having S_3 as an automorphism group. According to [17], there are 18 graphs obtained from the 18 Steiner (2,4,25) systems, and three of them have S_3 as an automorphism group. We show that there are exactly 45 strongly regular graphs with parameters $(50, 21, 8, 9)$ having a non-abelian automorphism group of order six. Hence, to our best knowledge, 42 of the constructed strongly regular graphs are new.

We get 340 possibilities for orbit length distributions, but only a few give rise to orbit matrices. In Table 8, we present the number of mutually-nonisomorphic orbit matrices for each orbit length distribution, the number of orbit matrices for Z_3 (obtained by the refinement) and the number of constructed SRGs with parameters $(50, 21, 8, 9)$. Thus, we prove Theorem 9. Information about the orders of the full automorphism groups is presented in Table 9.

Theorem 9. *Up to isomorphism, there are exactly 45 strongly regular graphs with parameters* $(50, 21, 8, 9)$ *having an automorphism group isomorphic to the symmetric group* S_3.

The adjacency matrices of the constructed SRGs can be found at [23].

In Table 10, we summarize the obtained results, i.e., give a list of all the obtained strongly regular graphs and orders of their full automorphism groups.

Table 8. Number of orbit matrices and SRGs$(50, 21, 8, 9)$ for the automorphism group S_3.

Distribution	#OM-S_3	#OM-Z_3	#SRGs	Distribution	#OM-S_3	#OM-Z_3	#SRGs
(0,1,2,7)	10	3	2	(2,0,2,7)	10	16	0
(0,1,4,6)	10	4	6	(2,0,8,4)	22	24	12
(0,1,6,5)	12	21	6	(2,0,10,3)	2	0	0
(0,1,8,4)	8	8	1	(2,0,12,2)	27	0	0
(0,4,2,6)	2	2	0	(2,0,14,1)	14	0	0
(0,4,4,5)	4	3	16	(2,3,0,7)	2	3	0
(0,4,6,4)	3	4	0	(2,3,2,6)	6	1	0
(0,4,8,2)	4	6	0	(2,3,6,4)	2	4	0
(1,2,3,6)	10	20	5	(4,2,6,4)	6	12	0
(1,2,5,5)	2	0	0	(5,0,9,3)	2	0	0
(1,2,7,4)	4	0	0	(6,1,6,4)	1	1	4

Table 9. SRGs with parameters $(50, 21, 8, 9)$ having S_3 as an automorphism group.

$\|\mathrm{Aut}(\Gamma)\|$	#SRGs
6	35
18	6
72	1
150	1
336	1
504	1

Table 10. SRGs on up to 50 vertices having S_3 as an automorphism group.

(v, k, λ, μ)	$\|\mathrm{Aut}(\Gamma)\|$	#SRGs
$(41, 20, 9, 10)$	6	80
$(45, 22, 10, 11)$	6	288
$(49, 18, 7, 6)$	6	18
$(49, 18, 7, 6)$	12	2
$(49, 18, 7, 6)$	18	2
$(49, 18, 7, 6)$	24	4
$(49, 18, 7, 6)$	48	1
$(49, 18, 7, 6)$	72	4
$(49, 18, 7, 6)$	126	1
$(49, 18, 7, 6)$	144	2
$(49, 18, 7, 6)$	1008	1
$(49, 18, 7, 6)$	1764	1
$(49, 24, 11, 12)$	6	42
$(49, 24, 11, 12)$	18	22
$(49, 24, 11, 12)$	24	4
$(49, 24, 11, 12)$	126	4
$(50, 21, 8, 9)$	6	35
$(50, 21, 8, 9)$	18	6
$(50, 21, 8, 9)$	72	1
$(50, 21, 8, 9)$	150	1
$(50, 21, 8, 9)$	336	1
$(50, 21, 8, 9)$	504	1

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Abbreviations

The following abbreviations are used in this manuscript:

SRG Strongly regular graph

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