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On the Distinguishing Number of Functigraphs

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Received: 3 July 2018; Accepted: 4 August 2018; Published: 9 August 2018

Abstract: Let G_1 and G_2 be disjoint copies of a graph G and $g : V(G_1) \to V(G_2)$ be a function. A functigraph F_G consists of the vertex set $V(G_1) \cup V(G_2)$ and the edge set $E(G_1) \cup E(G_2) \cup \{uv : g(u) = v\}$. In this paper, we extend the study of distinguishing numbers of a graph to its functigraph. We discuss the behavior of distinguishing number in passing from G to F_G and find its sharp lower and upper bounds. We also discuss the distinguishing number of functigraphs of complete graphs and join graphs.

Keywords: distinguishing number; functigraph; complete graph

1. Introduction

Given a key ring of apparently identical keys to open different doors, how many colors are needed to identify them? This puzzle was given by Rubin [1] for the first time. In this puzzle, there is no need for coloring to be a proper one. Indeed, one cannot find a reason why adjacent keys must be assigned different colors, whereas in other problems, like storing chemicals and scheduling meetings, a proper coloring is needed with a small number of colors required.

Inspired by this puzzle, Albertson and Collins [2] introduced the concept of the distinguishing number of a graph as follows: a labeling $f : V(G) \to \{1, 2, 3, ..., t\}$ is called *t-distinguishing* if no non-trivial automorphism of a graph G preserves the vertex labels. The least integer t such that a graph G has a labeling which is t-distinguishing for the graph G, is called the *distinguishing number* of G and it is denoted by $Dist(G)$. For example, the distinguishing number of a complete graph K_n is n, the distinguishing number of a path graph P_n is 2 and the distinguishing number of a cyclic graph C_n, $n \geq 6$ is 2. For a graph G of order n, $1 \leq Dist(G) \leq n$ [2].

Harary [3] gave different methods (orienting some of the edges, coloring some of the vertices with one or more colors and same for the edges, labeling vertices or edges, adding or deleting vertices or edges) of destroying the symmetries of a graph. Collins and Trenk [4] defined the distinguishing chromatic number where the authors used proper t-distinguishing for vertex labeling. The authors have given a comparison between the distinguishing number, the distinguishing chromatic number and the chromatic number of families like complete graphs, path graphs, cyclic graphs, Petersen graph and trees, etc. Kalinowski and Pilsniak [5] defined similar graph parameters, the distinguishing index and the distinguishing chromatic index where the authors labeled edges instead of vertices. The authors also gave a comparison between the distinguishing number and the distinguishing index of a connected graph G of order $n \geq 3$. Boutin [6] introduced the concept of determining sets. Albertson and Boutin [7] proved that a graph has a $(t-1)$-distinguishable determining set if and only if the graph is t-distinguishable. The authors also proved that every Kneser graph $K_{n:k}$ with $n \geq 6$ and $k \geq 2$ is 2-distinguishable. A considerable amount of literature has been developed in this area—for example, see [8–12].

Unless otherwise specified, all graphs considered in this paper are simple, non-trivial and connected. The set of all vertices that are adjacent to a vertex $u \in V(G)$ is called the *open neighborhood* of u and it is denoted by $N(u)$. The set of vertices $\{u\} \cup N(u)$ is called the *closed neighborhood* of u and it is denoted by $N[u]$. If two distinct vertices u, v of a graph G have the same open neighborhood, then these are called *non-adjacent twins*. If the two vertices have the same closed neighborhood, then these are called *adjacent twins*. In the both cases, u and v are called *twins*. A vertex v of a graph G is called *saturated*, if it is adjacent to all other vertices of G. A graph H whose vertex set $V(H)$ and edge set $E(H)$ are subsets of $V(G)$ and $E(G)$, respectively, then H is called a *subgraph* of graph G. Let $S \subset V(G)$ be any subset of vertices of G. The *induced subgraph*, denoted by $< S >$, is the graph whose vertex set is S and whose edge set is the set of all those edges in $E(G)$ which have both end vertices in S. A *spanning subgraph* H of a graph G is a subgraph such that $V(H) = V(G)$ and $E(H) \subseteq E(G)$. An *automorphism* α of G, $\alpha : V(G) \to V(G)$, is a bijective mapping such that $\alpha(u)\alpha(v) \in E(G)$ if and only if $uv \in E(G)$. Thus, each automorphism α of G is a permutation of the vertex set $V(G)$ which preserves adjacencies and non-adjacencies. The *automorphism group* of a graph G, denoted by $\Gamma(G)$, is the set of all automorphisms of a graph G. A graph with a trivial automorphism group is called a *rigid (or asymmetric)* graph. The minimum number of vertices in a rigid graph is 6 [13]. The distinguishing number of a rigid graph is 1.

The idea of a permutation graph was introduced by Chartrand and Harary [14] for the first time. The authors defined a permutation graph as follows: a permutation graph consists of two identical disjoint copies of a graph G, say G_1 and G_2, along with $|V(G)|$ additional edges joining $V(G_1)$ and $V(G_2)$ according to a given permutation on $\{1, 2, ..., |V(G)|\}$. Dorfler [15] defined a mapping graph as follows: a *mapping graph* of a graph G on n vertices consists of two disjoint identical copies of graph G with n additional edges between the vertices of two copies, where the additional edges are defined by a function. The mapping graph was rediscovered and studied by Chen et al. [16], where it was called the functigraph. A functigraph is an extension of a permutation graph. Formally, the functigraph is defined as follows: let G_1 and G_2 be disjoint copies of a connected graph G and let $g : V(G_1) \to V(G_2)$ be a function. A *functigraph* F_G of a graph G consists of the vertex set $V(G_1) \cup V(G_2)$ and the edge set $E(G_1) \cup E(G_2) \cup \{uv : g(u) = v\}$. Linda et al. [17,18] and Kang et al. [19] studied the functigraphs for some graph invariants like metric dimension, domination and zero forcing number. In [20], we have studied the fixing number of some functigraphs. The aim of this paper is to study the distinguishing number of functigraphs.

Network science and graph theory are two interconnected research fields that have synonymous structures, problems and their solutions. The notions 'network' and 'graph' are identical and these can be used interchangeably subject to the nature of application. The roads network, railway network, social networks, scholarly networks, etc are among the examples of networks. In the recent past, the network science has imparted to a functional understanding and the analysis of the complex real world networks. The basic premise in these fields is to relate metabolic networks, proteomic and genomic with disease networks [21] and information cascades in complex networks [22]. Real systems of quite a different nature can have the same network representation. Even though these real systems have different nature, appearance or scope, they can be represented as the same network. Since a functigraph consists of two copies of the same graph (network) with the additional edges described by a function, a mathematical model involving two systems with the same network representation and additional links (edges) between nodes (vertices) of two systems can be represented by a functigraph. The present study is useful in distinguishing the nodes of such pair of the same networks (systems) that can be represented by a functigraph.

Throughout the paper, we denote the functigraph of G by F_G, $V(G_1) = A$, $V(G_2) = B$, g denotes a function $g : A \to B$, f denotes the distinguishing labeling, and $g(V(G_1)) = I$, $|g(V(G_1))| = |I| = s$.

In order to understand the concept of functigraphs, we consider an example of a functigraph of $G = K_9$. Take $V(G_1) = A = \{u_1, ..., u_n\}$ and $V(G_2) = B = \{v_1, ..., v_n\}$ and function g is defined as follows:

$$g(u_i) = \begin{cases} v_1 & if \quad 1 \le i \le 3, \\ v_2 & if \quad 4 \le i \le 5, \\ v_{i-3} & if \quad 6 \le i \le 9. \end{cases} \tag{1}$$

The corresponding functigraph is shown in Figure 1.

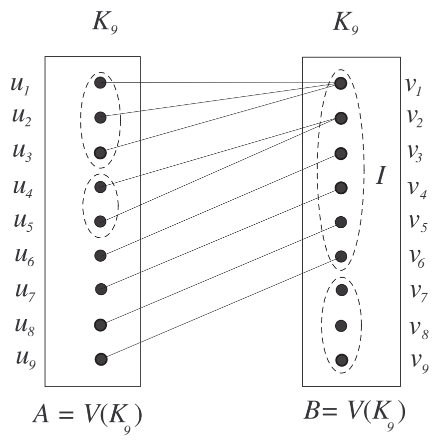

Figure 1. The functigraph F_G when $G = K_9$ and function g is as defined below.

This paper is organized as follows: in Section 2, we give sharp lower and upper bounds for the distinguishing number of functigraphs. This section also establishes connections between the distinguishing number of graphs and their corresponding functigraphs in the form of realizable results. In Section 3, we compute the distinguishing number of functigraphs of complete graphs and joining of path graphs. Some useful results related to these families have also been presented in this section.

2. Bounds and Realizable Results

The sharp lower and upper bounds on the distinguishing number of functigraphs are given in the following result.

Proposition 1. *Let G be a connected graph of order $n \ge 2$; then,*

$$1 \le Dist(F_G) \le Dist(G) + 1.$$

Both bounds are sharp.

Proof. Obviously, $1 \le Dist(F_G)$ by definition. Let $Dist(G) = t$ and f be a t-distinguishing labeling for the graph G. In addition, let $u_i \in A$ and $v_i \in B$, $1 \le i \le n$. We extend labeling f to F_G as: $f(u_i) = f(v_i)$ for all $1 \le i \le n$. We have the following two cases for g:

1. If g is not bijective, then f as defined earlier is a t-distinguishing labeling for F_G. Hence, $Dist(F_G) \le t$.
2. If g is bijective, then f as defined earlier destroys all non-trivial automorphisms of F_G except possible flipping of G_1 and G_2 in F_G. Let F'_G and F_G be the functigraph of G when g is an identity function, i.e., $g(u_i) = v_i$ for all i, $1 \le i \le n$ and when g is not identity function, respectively. The flipping of G_1 and G_2 is possible in the cases when either g is an identity function or when g is not the identity function but the corresponding functigraph F'_G is isomorphic to F_G. In order to break this automorphism (flipping), only one vertex of either G_1 or G_2 must be labeled with an extra color, and hence $Dist(F_G) \le t + 1$.

For the sharpness of bounds, we consider a rigid graph G on $n \geq 6$ vertices. For the sharpness of the lower bound, take a functigraph F_G in which g is a constant function. For the sharpness of the upper bound, take functigraph F_G in which g is an identity function. □

We discuss an example for Proposition 1, where we consider a rigid graph G with the smallest number of vertices i.e., $|V(G)| = 6$ as shown in Figure 2a. Since $Dist(G) = 1$, we label its vertices with a red color. Figure 2b shows F_G, when g is a constant function. In this case, F_G is a rigid graph and hence $Dist(F_G) = 1$. Figure 2c shows F_G, when g is the identity function. In this case, F_G has a non-trivial automorphism i.e., horizontal flipping of F_G. We label vertex v_6 of copy G_2 with blue color to break the non-trivial automorphism. Hence, $Dist(F_G) = 2 = Dist(G) + 1$.

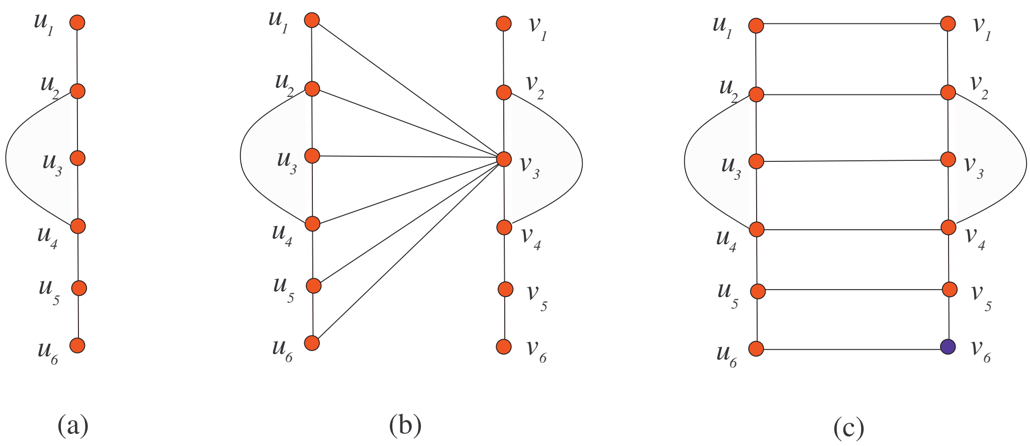

Figure 2. (**a**) a rigid graph G with six vertices; (**b**) a functigraph F_G when g is a constant function; (**c**) a functigraph F_G when g is the identity function i.e., $g(u_i) = v_i$ for all i $(1 \leq i \leq 6)$.

Since at least m colors are required to break all automorphisms of a twin-set of cardinality m, we have the following proposition.

Proposition 2. *Let $U_1, U_2, ..., U_t$ be disjoint twin-sets in a connected graph G of order $n \geq 3$ and $m = max\{|U_i| : 1 \leq i \leq t\}$,*

(i) $Dist(G) \geq m$,

(ii) *If $Dist(G) = m$, then $Dist(F_G) \leq m$.*

Two vertices in a graph G are said to be *similar vertices*, if both can be mapped on each other under some automorphism of graph G.

Lemma 1. *Let G be a connected graph of order $n \geq 2$ and g be a constant function, then $Dist(F_G) = Dist(G)$.*

Proof. Let $I = \{v\} \subset B$. In the functigraph F_G, we label the vertices in copy G_1 of G with $Dist(G)$ colors. Now, v is the only vertex of F_G with the largest degree (as we can see in Figure 2b $I = \{v_3\}$ and v_3 is the vertex of F_G with the largest degree); therefore, it is not similar to any other vertex of F_G and hence it can also be labeled with one of $Dist(G)$ colors. Thus, vertices in $A \cup \{v\}$ are labeled by $Dist(G)$ colors. Since g is a constant function, all vertices in $V(F_G) \setminus \{A \cup \{v\}\}$ are not similar to any vertex in $A \cup \{v\}$ in functigraph F_G. If two disjoint subsets of vertices of a graph are such that every vertex of one set is not similar to any vertex of the other set, then the vertices of both sets can be labeled by the same set of colors; therefore, the vertices in $V(F_G) \setminus \{A \cup \{v\}\}$ and $A \cup \{v\}$ can be labeled by $Dist(G)$ colors. Hence, $Dist(F_G) = Dist(G)$. □

Remark 1. *Let G be a connected graph and $Dist(F_G) = m_1$, if g is constant and $Dist(F_G) = m_2$, if g is not constant; then, $m_1 \geq m_2$.*

Now, we discuss a special type of connected subgraph H of a connected graph G such that $Dist(H) \leq Dist(G)$. We define a set $S(H) = \{u \in V(H) : u \text{ is similar to } v(\neq u) \text{ for some } v \in V(H)\}$. If the graph G has a connected subgraph H in which all vertices in $S(H)$ are either adjacent to all vertices in $V(G) - V(H)$ or non-adjacent to all vertices in $V(G) - V(H)$, then we discuss in Remark 2 that $Dist(G) \geq Dist(H)$.

Lemma 2. *Let H be a connected subgraph of a connected graph G such that all vertices in $S(H)$ are either adjacent to all vertices in $V(G) - V(H)$ or non-adjacent to all vertices in $V(G) - V(H)$, then every automorphism of H can be extended to an automorphism of G.*

Proof. Let $\alpha \in \Gamma(H)$ be an arbitrary automorphism. We define an extension α' of α on $V(G)$ as:

$$\alpha'(w) = \begin{cases} \alpha(w) & if \quad w \in V(H), \\ w & if \quad w \in V(G) - V(H). \end{cases}$$

Since $\alpha'(w) = w$ for all $w \in V(H) - S(H)$, α' being an identity function preserves the relation of adjacency among the vertices in $V(G) - S(H)$. In addition, $\alpha' = \alpha$ being an automorphism of the subgraph H preserves the relation of adjacency among the vertices in $V(H)$. Next, we will prove that α' also preserves the relation of adjacency among the vertices in $\{V(G) - V(H)\} \cup S(H)$. Suppose $u \in S(H)$ and $y \in V(G) - V(H)$, where both y and u are arbitrary vertices of their sets. Since $\alpha \in \Gamma(H)$, $\alpha(u) \in V(H)$. We discuss two cases for the subgraph H in graph G:

1. All vertices in $S(H)$ are adjacent to all vertices in $V(G) - V(H)$; then, u is adjacent to y in G. In addition, $\alpha'(u) = \alpha(u)$ being a vertex of H is adjacent to $\alpha'(y) = y$. Hence, α' preserves the relation of adjacency among the vertices in $\{V(G) - V(H)\} \cup S(H)$.
2. All vertices in $S(H)$ are non-adjacent to all vertices in $V(G) - V(H)$; then, u is non-adjacent to y in G. In addition, $\alpha'(u) = \alpha(u)$ being a vertex of H is non-adjacent to $\alpha'(y) = y$. Hence, α' preserves the relation of adjacency among the vertices in $\{V(G) - V(H)\} \cup S(H)$.

   Thus, α' preserves the relation of adjacency among the vertices of $V(G)$. □

Let H be a connected subgraph of a graph G such that H satisfies the hypothesis of Lemma 2, then every distinguishing labeling of G requires at least $Dist(H)$ colors to break the extended automorphism g' of G, therefore $Dist(G) \geq Dist(H)$ for the subgraph H. It can be seen in Figure 3 that subgraph H of graph G satisfies the hypothesis of Lemma 2 and $Dist(G) = 2 = Dist(H)$. We label the vertices of the graph with red and white colors.

Remark 2. *Let H be a connected subgraph of a connected graph G such that all vertices in $S(H)$ are either adjacent to all vertices in $V(G) - V(H)$ or non-adjacent to all vertices in $V(G) - V(H)$, then $Dist(G) \geq Dist(H)$.*

A vertex v of degree at least three in a connected graph G is called a *major vertex*. Two paths rooted from the same major vertex and having the same length are called the *twin stems*.

We define a function $\psi : \mathbb{N} \setminus \{1\} \to \mathbb{N} \setminus \{1\}$ as $\psi(m) = k$, where k is the least number such that $m \leq 2\binom{k}{2} + k$. For example, $\psi(19) = 5$. Note that ψ is well-defined.
In the following lemma, we find a lower bound of the distinguishing number of a graph having twin stems of length 2 rooted at the same major vertex, in terms of the function ψ.

Lemma 3. *If a graph G has $t \geq 2$ twin stems of length 2 rooted at the same major vertex, then $Dist(G) \geq \psi(t)$.*

Proof. Let $x \in V(G)$ be a major vertex and xu_iu_i' where $1 \leq i \leq t$ are the twin stems of length 2 attached with x. Let $H =< \{x, u_i, u_i'\} >$ and $k = \psi(t)$. Since xu_iu_i' where $1 \leq i \leq t$ are twin stems in the graph G, the subgraph H satisfies the hypothesis of Lemma 2. We define a labeling $f : V(H) \rightarrow \{1, 2, ..., k\}$ as:

$$f(x) = k,$$

$$f(u_i) = \begin{cases} 1 & if \quad 1 \leq i \leq k, \\ 2 & if \quad k+1 \leq i \leq 2k, \\ 3 & if \quad 2k+1 \leq i \leq 3k, \\ \vdots & \qquad \vdots \\ k & if \quad (k-1)k+1 \leq i \leq k^2. \end{cases} \tag{2}$$

$$f(u_i') = \begin{cases} i \operatorname{mod}(k) & if \quad 1 \leq i \operatorname{mod}(k) \leq k-1, \\ k & if \quad i \operatorname{mod}(k) = 0. \end{cases} \tag{3}$$

Using this labeling, one can see that f is a t-distinguishing labeling for H. With permutations with a repetition of k colors, when two of them are taken at a time equal to $2\binom{k}{2} + k$, at least k colors are needed to label the vertices in t-stems. Thus, k is the least integer for which subgraph H has k-distinguishing labeling, and hence $Dist(H) = k$. Thus, $Dist(G) \geq \psi(t)$ by Remark 2. □

It can be seen that the graph G as shown in Figure 3a has four twin stems of length 2 rooted at the same major vertex; therefore, by Lemma 3, $Dist(G) \geq \psi(4) = 2$. The following result gives the existence of a graph G and its functigaph F_G, such that both have the same distinguishing number.

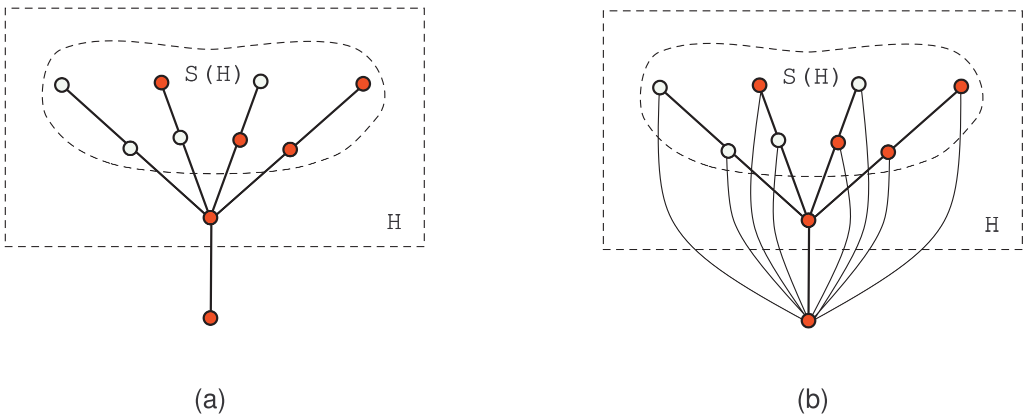

Figure 3. (**a**) a graph G and its subgraph H such that all vertices of $S(H)$ are non-adjacent to all vertices of $V(G) - V(H)$; (**b**) a graph G and its subgraph H such that all vertices of $S(H)$ are adjacent to all vertices of $V(G) - V(H)$.

Lemma 4. *For any integer $t \geq 2$, there exists a connected graph G and a function g such that $Dist(G) = t = Dist(F_G)$.*

Proof. Construct a graph G as follows: let $P_{(t-1)^2+1} : x_1x_2x_3...x_{(t-1)^2+1}$ be a path graph. Join $(t-1)^2 + 1$ twin stems $x_1u_iu_i'$ where $1 \leq i \leq (t-1)^2 + 1$ each of length two with vertex x_1 of $P_{(t-1)^2+1}$. This completes construction of G. We first show that $Dist(G) = t$. For $t = 2$, we have two twin stems attached with x_1, and hence $Dist(G) = 2$. For $t \geq 3$, we define a labeling $f : V(G) \rightarrow \{1, 2, 3, ..., t\}$ as follows: $f(x_i) = t$, for all i, where $1 \leq i \leq (t-1)^2 + 1$:

$$f(u_i) = \begin{cases} 1 & if \ \ 1 \le i \le t-1, \\ 2 & if \ \ t \le i \le 2(t-1), \\ 3 & if \ \ 2t-1 \le i \le 3(t-1), \\ \vdots & \qquad \vdots \\ t-1 & if \ \ (t-1)(t-2)+1 \le i \le (t-1)^2, \\ t & if \ \ i = (t-1)^2+1, \end{cases}$$

$$f(u_i') = \begin{cases} i \bmod (t-1) & if \ \ 1 \le i \bmod (t-1) \le t-2 \ \ and \ \ i \ne (t-1)^2+1, \\ t-1 & if \ \ i \bmod (t-1) = 0, \\ t & if \ \ i = (t-1)^2+1. \end{cases}$$

Using this labeling, one can see the unique automorphism preserving this labeling is the identity automorphism. Hence, f is a t-distinguishing. With permutations with a repetition of $t-1$ colors, when two of them are taken at a time, $2\binom{t-1}{2} + (t-1)$, $(t-1)^2+1$ twin stems can be labeled by at least t colors. Hence, t is the least integer such that G has a t-distinguishing labeling. Now, we denote the corresponding vertices of G_2 as v_i, v_i', y_i for all i, where $1 \le i \le (t-1)^2+1$ and construct a functigraph F_G by defining $g : A \to B$ as follows: $g(u_i) = g(u_i') = y_i$, for all i, where $1 \le i \le (t-1)^2+1$ and $g(x_i) = y_i$, for all i, where $1 \le i \le (t-1)^2+1$ as shown in Figure 4. Thus, F_G has only symmetries of $(t-1)^2+1$ twin stems attached with y_1. Hence, $Dist(F_G) = t$. □

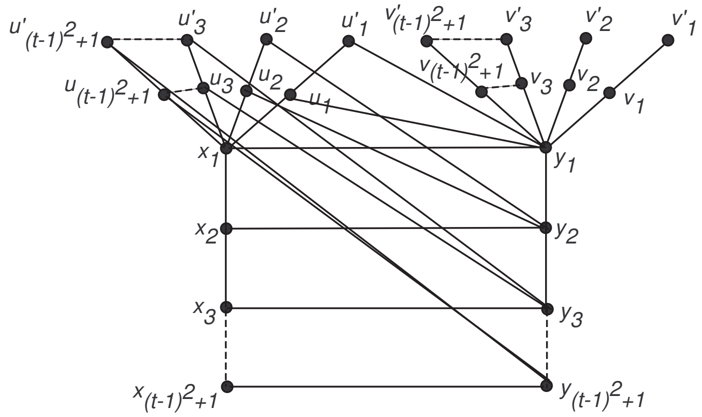

Figure 4. Graph with $Dist(G) = t = Dist(F_G)$.

Consider an integer $t \ge 4$. We construct a graph G similarly as in the proof of Lemma 4 by taking a path graph $P_{(t-3)^2+1} : x_1x_2...x_{(t-3)^2+1}$ and attach $(t-3)^2+1$ twin stems $x_1u_iu_i'$ where $1 \le i \le (t-3)^2+1$ with any one of its end vertex, say, x_1. Using the similar labeling and arguments as in the proof of Lemma 4, one can see that f is $t-2$ distinguishing and $t-2$ is the least integer such that G has $t-2$ distinguishing labeling. Define functigraph F_G, where $g : A \to B$ is defined by: $g(u_i) = g(u_i') = y_i$, for all i, where $1 \le i \le (t-3)^2+1$, $g(x_i) = v_i$, for all i, where $1 \le i \le (t-3)^2-1$, $g(x_i) = y_i$, for all i, where $(t-3)^2 \le i \le (t-3)^2+1$. From this construction, F_G has only symmetries in which two twin stems attached with y_1 can be mapped on each other under some automorphism of F_G, and hence $Dist(F_G) = 2$. Thus, we have the following result, which shows that $Dist(G) + Dist(F_G)$ can be arbitrarily large:

Lemma 5. *For any integer $t \ge 4$, there exists a connected graph G and a function g such that $Dist(G) + Dist(F_G) = t$.*

Consider $t \ge 3$. We construct a graph G similarly as in the proof of Lemma 4 by taking a path graph $P_{4(t-1)^2+1}$: $x_1x_2...x_{4(t-1)^2+1}$ and attach $4(t-1)^2+1$ twin stems $x_1u_iu_i'$, where $1 \le i \le 4(t-1)^2+1$ with

x_1. Using the similar labeling and arguments as in the proof of Lemma 4, one can see that f is $2t-1$ distinguishing labeling and $2t-1$ is the least integer such that G has $2t-1$ distinguishing labeling. Let us now define g as $g(u_i) = g(u_i') = y_i$, for all i, where $1 \le i \le 4(t-1)^2+1$, $g(x_i) = v_i$, for all i, where $1 \le i \le 3t^2-4t$ and $g(x_i) = y_i$, for all i, where $3t^2-4t+1 \le i \le 4(t-1)^2+1$. Thus, F_G has only symmetries of $(t-2)^2+1$ twin stems attached with y_1, and hence $Dist(F_G) = t-1$. After making this type of construction, we have the following result which shows that $Dist(G) - Dist(F_G)$ can be arbitrarily large:

Lemma 6. *For any integer $t \ge 3$, there exists a connected graph G and a function g such that $Dist(G) - Dist(F_G) = t$.*

3. The Distinguishing Number of Functigraphs of Some Families of Graphs

In this section, we discuss a distinguishing number of functigraphs on complete graphs, edge deletion graphs of complete graph and joining of path graphs.

Let G be the complete graph of order $n \ge 3$. We use the following terminology for F_G in the proof of Theorem 1: Let $I = \{v_1, v_2, ..., v_s\}$ and $n_i = |\{u \in A : g(u) = v_i\}|$ for all i, where $1 \le i \le s$. In addition, let $l = \max\{n_i : 1 \le i \le s\}$ and $m = |\{n_i : n_i = 1, 1 \le i \le s\}|$. From the definitions of l and m, we note that $2 \le l \le n-s+1$ and $0 \le m \le s-1$.

In the next result, we find the distinguishing number of functigraphs of complete graphs, when g is bijective, in terms of function $\psi(m)$ as defined in the previous section.

Lemma 7. *Let G be the complete graph of order $n \ge 3$ and g be a bijective function; then, $Dist(F_G) = \psi(n)$.*

Proof. Let $A = \{u_1, u_2, ..., u_n\}$ and $I = \{g(u_1), g(u_2), ..., g(u_n)\} = B$. In addition, let $k = \psi(n)$. Let $f : V(F_G) \to \{1, 2, ..., k\}$ be a labeling in which $f(u_i)$ is defined as in Equation (2) and $f(g(u_i))$ as in Equation (3) in the proof of Lemma 3. Using this labeling, one can see that f is a k-distinguishing labeling for F_G. With permutations with repetitions of k colors, when two of them are taken at a time equal to $2\binom{k}{2}+k$, at least k colors are needed to label the vertices in F_G. Hence, k is the least integer for which F_G has k-distinguishing labeling. □

Let G be a complete graph and let $g : A \to B$ be a function such that $2 \le m \le s$. Without loss of generality, assume $u_1, u_2, ..., u_m \in A$ are those vertices of A such that $g(u_i) \ne g(u_j)$, where $1 \le i \ne j \le m$ in B. In addition, $(u_iu_j)(g(u_i)g(u_j)) \in \Gamma(F_G)$ for all $i \ne j$, where $1 \le i, j \le m$. By using the similar labeling f as defined in Lemma 7, at least $\psi(m)$ colors are needed to break these automorphisms in F_G. Thus, we have the following proposition:

Proposition 3. *Let G be a complete graph of order $n \ge 3$ and g be a function such that $2 \le m \le s$; then, $Dist(F_G) \ge \psi(m)$.*

The following result gives the distinguishing number of functigraphs of complete graphs.

Theorem 1. *Let $G = K_n$ be the complete graph of order $n \ge 3$, and let $1 < s \le n-1$; then,*

$$Dist(F_G) \in \{n-s, n-s+1, \psi(m)\}.$$

Proof. We discuss the following cases for l:

1. If $l = n-s+1 > 2$, then A contains $n-s+1$ twin vertices and B contains $n-s$ twin vertices (except for $n = 3, 4$ where B contains no twin vertices). In addition, there are $m(= s-1)$ vertices in A which have distinct images in B. By Proposition 3, these m vertices and their distinct images are labeled by at least $\psi(m)$ colors (only 1 color if $m = 1$). Since $n-s+1$ is the largest

among $n-s+1$, $n-s$ and $\psi(m)$, $n-s+1$ is the least number such that F_G has $(n-s+1)$—distinguishing labeling. Thus, $Dist(F_G) = n-s+1$.

2. If $l = n-s+1 = 2$, then $\psi(m) \geq \max\{n-s+1, n-s\}$, and hence $Dist(F_G) = \psi(m)$.
3. If $l < n-s$, then B contains the largest set of $n-s$ twin vertices in F_G. In addition, there are $m(\leq s-2)$ vertices in A, each of which have distinct images in B. Since $n-s \geq \psi(m)$, $Dist(F_G) = n-s$.
4. If $l = n-s > 2$, then both A and B contain the largest set of $n-s$ twin vertices in F_G. In addition, there are $m(= s-2)$ vertices in A that have distinct images in B. Since $n-s \geq \psi(m)$, $Dist(F_G) = n-s$.
5. If $l = n-s = 2$, then we have the following two subcases:

   (a) If $1 < s \leq \lfloor \frac{n}{2} \rfloor + 1$, then both A and B contain the largest set of $n-s$ twin vertices in F_G. In addition, there are $m(= s-2)$ vertices in A that have distinct images in B. Since $n-s \geq \psi(m)$ (if $\psi(m)$ exists), $Dist(F_G) = n-s$.

   (b) If $\lfloor \frac{n}{2} \rfloor + 1 < s \leq n-1$, then $\psi(m) \geq \max\{n-s+1, n-s\}$, and hence $Dist(F_G) = \psi(m)$.

□

We define a function $\phi : \mathbb{N} \to \mathbb{N} \setminus \{1\}$ as $\phi(i) = k$, where k is the least number such that $i \leq \binom{k}{2}$. For instance, $\phi(32) = 9$. Note that ϕ is well defined.

The following result gives the distinguishing number of functigraphs of a family of spanning subgraphs of complete graphs.

Theorem 2. *For a complete graph G of order $n \geq 5$ and G_i, where $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$ is the graph deduced from G by deleting i edges with no common end vertices that join two saturated vertices of G for all i. If g is a constant function, then*

$$Dist(F_{G_i}) = max\{n-2i, \phi(i)\}.$$

Proof. On deleting i edges from G, we have $n-2i$ saturated vertices and i twin-sets each of cardinality two (as shown in Figure 5 where G is the complete graph on 7 vertices, $i = 2$ and g is a constant function). We will now show that exactly $\phi(i)$ colors are required to label vertices of all i twin-sets. We observe that all vertices in twin sets of cardinality 2 are similar to each other in G. Since two vertices in a twin-set are labeled by a unique pair of colors out of $\binom{k}{2}$ pairs of k colors, at least k colors are required to label vertices of i twin-sets. Now, we discuss the following two cases for $\phi(i)$:

1. If $\phi(i) \leq n-2i$, then the number of colors required to label $n-2i$ saturated vertices is greater than or equal to the number of colors required to label the vertices of i twin-sets. Thus, we label $n-2i$ saturated vertices with exactly $n-2i$ colors and out of these $n-2i$ colors, $\phi(i)$ colors will be used to label vertices of i twin-sets.
2. If $\phi(i) > n-2i$, then the number of colors required to label $n-2i$ saturated vertices is less than the number of colors required to label vertices of i twin-sets. Thus, we label vertices of i twin-sets with $\phi(i)$ colors and, out of these $\phi(i)$ colors, $n-2i$ colors will be used to label saturated vertices in G_i.

Since g is constant, by using the same arguments as in the proof of Lemma 1, $Dist(F_{G_i}) = Dist(G_i)$. □

Suppose that $G = (V_1, E_1)$ and $G^* = (V_2, E_2)$ are two graphs with disjoint vertex sets V_1 and V_2 and disjoint edge sets E_1 and E_2. The *join* of G and G^* is the graph $G + G^*$, in which $V(G+G^*) = V_1 \cup V_2$ and $E(G+G^*) = E_1 \cup E_2 \cup \{ uv\text{: } u \in V_1, v \in V_2\}$.

Proposition 4. *Let P_n be a path graph of order $n \geq 2$; then, for all $m, n \geq 2$ and $1 < s < m+n$, $1 \leq Dist(F_{P_m+P_n}) \leq 3$.*

Proof. Let $P_m : v_1, ..., v_m$ and $P_n : u_1, ..., u_n$. We discuss the following cases for m, n.

1. If $m = 2$ and $n = 2$, then $P_2 + P_2 = K_4$, and hence $1 \leq Dist(F_{K_4}) \leq 3$ by Theorem 1.
2. If $m = 2$ and $n = 3$, then $P_2 + P_3$ has three saturated vertices. Thus, $1 \leq Dist(F_{P_2+P_3}) \leq 4$ by Proposition 1. However, for all s where $2 \leq s \leq 4$ and all possible definitions of g in $F_{P_2+P_3}$, one can see $1 \leq Dist(F_{P_2+P_3}) \leq 3$.
3. If $m = 3$ and $n = 3$, then a labeling $f : V(P_3 + P_3) \to \{1, 2, 3\}$ defined as:

$$f(x) = \begin{cases} 1 & if \quad x = v_1, v_2, \\ 2 & if \quad x = v_3, u_3, \\ 3 & if \quad x = u_1, u_2, \end{cases}$$

   is a distinguishing labeling for $P_3 + P_3$, and hence $Dist(P_3 + P_3) = 3$. Thus, $1 \leq Dist(F_{P_3+P_3}) \leq 4$ by Proposition 1. However, for all s where $2 \leq s \leq 5$ and all possible definitions of g in $F_{P_3+P_3}$, one can see $1 \leq Dist(F_{P_3+P_3}) \leq 3$.
4. If $m \geq 2$ and $n \geq 4$, then a labeling $f : V(P_m + P_n) \to \{1, 2\}$ defined as:

$$f(x) = \begin{cases} 1 & if \quad x = v_1, u_2, ..., u_n, \\ 2 & if \quad x = u_1, v_2, ..., v_m, \end{cases}$$

   is a distinguishing labeling for $P_m + P_n$, and hence $Dist(P_m + P_n) = 2$. Thus, the result follows by Proposition 1.

□

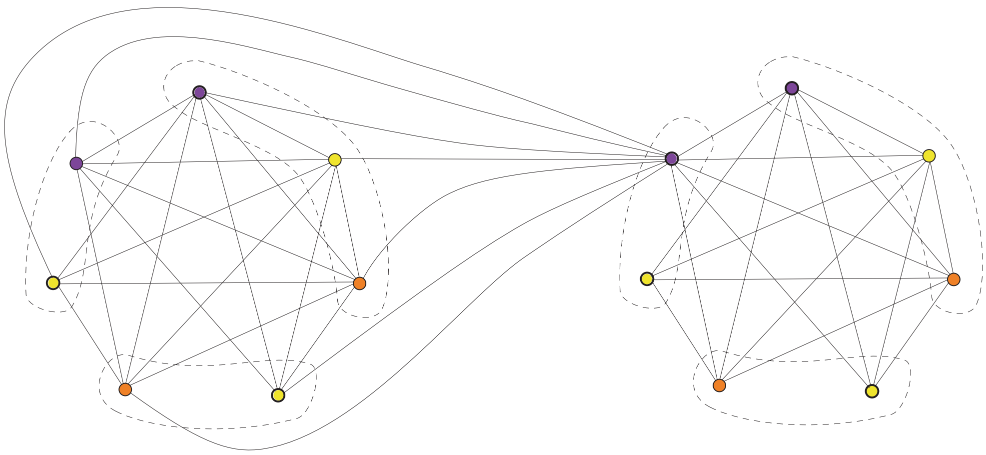

Figure 5. Graph $G = K_7$ and F_{G_2}, when g is a constant function. $Dist(F_{G_2}) = \phi(2) = n - 2i = 3$.

4. Conclusions

In this paper, we have studied the distinguishing number of functigraphs, which is an extension of the permutation graphs. We have given sharp lower and upper bounds of a distinguishing number of functigraphs. We have established the connection between the distinguishing number of graphs and its corresponding functigraph in the form of realizable results. We have computed the distinguishing number of functigraphs of a complete graph and the joining of path graphs. Furthermore, we have defined a function $\phi : \mathbb{N} \to \mathbb{N} \setminus \{1\}$ as $\phi(i) = k$, where k is the least number such that $i \leq \binom{k}{2}$. By using this function ϕ, we have found the distinguishing number of functigraphs of spanning subgraphs of complete graphs. In the future, it would be interesting to work on the distinguishing number of functigraphs of some well known families of graphs.

Author Contributions: M.F., M.M., Z.U., U.A. and I.J. have contributed equally in the writing of this article.

Acknowledgments: The authors are very grateful to the anonymous reviewers for the evaluation and the constructive critiques for the improvement of this manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

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