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Hyperbolicity on Graph Operators

J. A. Méndez-Bermúdez [1], Rosalío Reyes [2], José M. Rodríguez [2] and José M. Sigarreta [1,3,*]

1 Instituto de Física, Benemérita Universidad Autónoma de Puebla, Apartado Postal J-48, Puebla 72570, Mexico; jmendezb@ifuap.buap.mx
2 Departamento de Matemáticas, Universidad Carlos III de Madrid, Avenida de la Universidad 30, 28911 Leganés, Madrid, Spain; rreyes@math.uc3m.es (R.R.); jomaro@math.uc3m.es (J.M.R.)
3 Facultad de Matemáticas, Universidad Autónoma de Guerrero, Carlos E. Adame No.54 Col. Garita, Acapulco Gro. 39650, Mexico
* Correspondence: josemariasigarretaalmira@hotmail.com; Tel.: +52-1-7441592272

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Abstract: A graph operator is a mapping $F : \Gamma \to \Gamma'$, where Γ and Γ' are families of graphs. The different kinds of graph operators are an important topic in Discrete Mathematics and its applications. The symmetry of this operations allows us to prove inequalities relating the hyperbolicity constants of a graph G and its graph operators: line graph, $\Lambda(G)$; subdivision graph, $S(G)$; total graph, $T(G)$; and the operators $R(G)$ and $Q(G)$. In particular, we get relationships such as $\delta(G) \leq \delta(R(G)) \leq \delta(G) + 1/2$, $\delta(\Lambda(G)) \leq \delta(Q(G)) \leq \delta(\Lambda(G)) + 1/2$, $\delta(S(G)) \leq 2\delta(R(G)) \leq \delta(S(G)) + 1$ and $\delta(R(G)) - 1/2 \leq \delta(\Lambda(G)) \leq 5\delta(R(G)) + 5/2$ for every graph which is not a tree. Moreover, we also derive some inequalities for the Gromov product and the Gromov product restricted to vertices.

Keywords: graph operators; gromov hyperbolicity; geodesics

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1. Introduction

In [1], J. Krausz introduced the concept of graph operators. These operators have applications in studies of graph dynamics (see [2,3]) and topological indices (see [4–6]). Many large graphs can be obtained by applying graph operators on smaller ones, thus some of their properties are strongly related. Motivated by the above works, we study here the hyperbolicity constant of several graph operators.

Along this paper, we denote by $G = (V(G), E(G))$ a connected simple graph with edges of length 1 (unless edge lengths are explicitly given) and $V \neq \emptyset$. Given an edge $e = uv \in E(G)$ with endpoints u and v, we write $V(e) = \{u, v\}$. Next, we recall the definition of some of the main graph operators.

The *line graph*, $\Lambda(G)$, is the graph constructed from G with vertices the set of edges of G, and and two 19 vertices are adjacent if and only if their corresponding edges are incident in G.

The *subdivision graph*, $S(G)$, is the graph constructed from G substituting each of its edges by a path of length 2.

The graph $Q(G)$ is the graph constructed from $S(G)$ byadding edges between adjacent vertices in $\Lambda(G)$.

The graph $R(G)$ is constructed from $S(G)$ by adding edges between adjacent vertices in G.

The *total graph*, $T(G)$,is constructed from $S(G)$ by adding edges between adjacent vertices in G or $\Lambda(G)$.

We define:

$$E_E(G) := \{\{e_1, e_2\} : e_1, e_2 \in E(G), e_1 \neq e_2, |V(e_1) \cap V(e_2)| = 1\},$$

and

$$E_V(G) := \{\{e, u\} : e \in E(G), u \in V(e)\}.$$

So, we have the following:
$\Lambda(G) := (E(G), E_E(G))$.
$S(G) := (V(G) \cup E(G), E_V(G))$.
$T(G) := (V(G) \cup E(G), E(G) \cup E_E(G) \cup E_V(G))$.
$R(G) := (V(G) \cup E(G), E(G) \cup E_V(G))$.
$Q(G) := (V(G) \cup E(G), E_E(G) \cup E_V(G)$.

The Gromov hyperbolic spaces have multiple applications both theoretical and practical (see [7–10]). A space is geodesic if any two points in it can be joined by a curve whose length is the distance between them. In this paper we will consider a graph G as a geodesic metric space and any geodesic joining x and y will be denote by $[xy]$.

Let X be a geodesic metric space and $x, y, z \in X$. A *geodesic triangle* with vertices x, y, z, denoted by $T = \{x, y, z\}$, is the union of three geodesics $[xy]$, $[yz]$ and $[zx]$. We write also $T = \{[xy], [yz], [zx]\}$. If the δ-neighborhood of the union of any two sides of T contains the other side, we say that T is *δ-thin*. We define $\delta(T) := \inf\{\delta \geq 0 : \ T \text{ is } \delta\text{-thin}\}$. The space X is *δ-hyperbolic* if all geodesic triangles T in X are δ-thin. Let us denote the sharp hyperbolicity constant of X, by $\delta(X)$, i.e., $\delta(X) := \sup\{\delta(T) : T \text{ is a geodesic triangle in } X\}$. X is *Gromov hyperbolic* if X is δ-hyperbolic for some $\delta \geq 0$; then X is Gromov hyperbolic if and only if $\delta(X) < \infty$.

In this paper we prove inequalities relating the hyperbolicity constants of a graph G and its graph operators $\Lambda(G)$, $S(G)$, $T(G)$, $R(G)$ and $Q(G)$, using their symmetries.

2. Definitions and Background

There are several equivalent definitions for Gromov hyperbolicity (see, e.g., [11–13]), in particular, the definition that we use in this work has an important geometric meaning and serves as a basis for multiple applications (see [14–19]).

Given a graph G, the *Gromov product* of $q_1, q_2 \in G$ with base point $q_0 \in G$ is defined as

$$(q_1, q_2)_{q_0} := \frac{1}{2}\left(d(q_1, q_0) + d(q_2, q_0) - d(q_1, q_2)\right).$$

For every Gromov hyperbolic graph G, we have

$$(q_1, q_3)_{q_0} \geq \min\left\{(q_1, q_2)_{q_0}, (q_2, q_3)_{q_0}\right\} - \delta \tag{1}$$

for every $q_0, q_1, q_2, q_3 \in G$ and some constant $\delta \geq 0$ ([12,13]).

We denote by $\delta^*(G)$ the sharp constant for the inequality (1), i.e.,

$$\delta^*(G) := \sup\left\{\min\left\{(q_1, q_2)_{q_0}, (q_2, q_3)_{q_0}\right\} - (q_1, q_2)_{q_0} : \ q_0, q_1, q_2, q_3 \in G\right\}.$$

Indeed, our definition of Gromov hyperbolicity is equivalent to (1); furthermore, we have $\delta^*(G) \leq 4\delta(G)$ and $\delta(G) \leq 3\delta^*(G)$ ([12,13]). In [20] (Proposition II.20) we found the following improvement of the previous inequality: $\delta^*(G) \leq 2\delta(G)$.

We denote by $\delta_v^*(G)$ the constant of hyperbolicity of the Gromov product restricted to the vertices of G, i.e.,

$$\delta_v^*(G) := \sup\left\{\min\left\{(q_1, q_2)_{q_0}, (q_2, q_3)_{q_0}\right\} - (q_1, q_3)_{q_0} : \ q_0, q_1, q_2, q_3 \in V(G)\right\}.$$

3. Main Results

The following result is immediate from the definition of $S(G)$.

Proposition 1. *Let G be a graph. Then*

$$\delta(S(G)) = 2\delta(G), \qquad \delta^*(S(G)) = 2\delta^*(G).$$

We remark that the equality is not true for $\delta_v^*(G)$ (e.g., $S(C_5) = C_{10}$ but $2\delta_v^*(C_5) = 1 \neq 2 = \delta_v^*(S(G))$), but inequalities may apply. The next result appears in [21].

Theorem 1. *Let $B = (V_0 \cup V_1, E)$ be a bipartite graph. We have $\delta_B(V_i) \le \delta_v^*(B) \le \delta_B(V_i) + 2$, where*

$$\delta_B(V_i) = \sup\{\min\{(x,y)_w, (y,z)_w\} - (x,z)_w : \, x,y,z,w \in V_i\}$$

for every $i \in \{1,2\}$.

Corollary 1. *Let G be a graph. Then*

$$2\delta_v^*(G) \le \delta_v^*(S(G)) \le 2\delta_v^*(G) + 2.$$

Proof. Note that $S(G)$ can be considered as a bipartite graph, where $V(S(G)) = V(G) \cup V(\Lambda(G))$. Theorem 1 gives $\delta_{S(G)}(V(G)) \le \delta_v^*(S(G)) \le \delta_{S(G)}(V(G)) + 2$. Since $\delta_{S(G)}(V(G)) = 2\delta_v^*(G)$, the desired inequalities hold. □

Proposition 2. *Let G be a graph. Then*

$$\delta_v^*(G) \le \delta^*(G) \le \delta_v^*(G) + 3.$$

Proof. The inequality $\delta_v^*(G) \le \delta^*(G)$ is direct. Let us prove the other inequality.

For every $q_0, q_1, q_2 \in G$ there are $q_0', q_1', q_2' \in V(G)$ such that $d(q_i, q_i') \le 1/2$ for $i = 0,1,2$. Then

$$\begin{aligned}
\left|(q_1,q_2)_{q_0} - (q_1',q_2')_{q_0'}\right| &= \frac{1}{2}\left|d(q_0,q_1) + d(q_0,q_2) - d(q_1,q_2) - d(q_0',q_1') - d(q_0',q_2') + d(q_1',q_2')\right| \\
&\le \frac{1}{2}\left|d(q_0,q_1) - d(q_0',q_1')\right| + \frac{1}{2}\left|d(q_0,q_2) - d(q_0',q_2')\right| + \frac{1}{2}\left|d(q_1,q_2) - d(q_1',q_2')\right| \\
&\le \frac{3}{2}.
\end{aligned}$$

Given $q_0, q_1, q_2, q_3 \in G$, let $q_0', q_1', q_2', q_3' \in V(G)$, with $d(q_i, q_i') \le 1/2$ for $i = 0,1,2,3$. We have

$$\begin{aligned}
(q_1,q_3)_{q_0} \ge (q_1',q_3')_{q_0'} - \frac{3}{2} &\ge \min\left\{(q_1',q_2')_{q_0'}, (q_2',q_3')_{q_0'}\right\} - \delta_v^*(G) - \frac{3}{2} \\
&\ge \min\left\{(q_1,q_2)_{q_0} - \frac{3}{2}, (q_2,q_3)_{q_0} - \frac{3}{2}\right\} - \delta_v^*(G) - \frac{3}{2} \\
&= \min\{(q_1,q_2)_{q_0}, (q_2,q_3)_{q_0}\} - \delta_v^*(G) - 3,
\end{aligned}$$

and we conclude $\delta^*(G) \le \delta_v^*(G) + 3$. □

Let H be a subgraph of G, H is *isometric* if $d_H(x,y) = d_G(x,y)$ for every $x,y \in H$. We will need the following well-known result.

Lemma 1. *Let H be an isometric subgraph of G. Then*

$$\delta(H) \le \delta(G),$$

$$\delta^*(H) \le \delta^*(G),$$

$$\delta_v^*(H) \le \delta_v^*(G).$$

Since G is an isometric subgraph of $T(G)$ and $R(G)$, and $\Lambda(G)$ is an isometric subgraph of $T(G)$ and $Q(G)$, we have the following consequence of Lemma 1.

Corollary 2. *For any graph G, we have*

$$\begin{aligned}
\delta(G) &\le \delta(T(G)), & \delta^*(G) &\le \delta^*(T(G)), & \delta_v^*(G) &\le \delta_v^*(T(G)),\\
\delta(G) &\le \delta(R(G)), & \delta^*(G) &\le \delta^*(R(G)), & \delta_v^*(G) &\le \delta_v^*(R(G)),\\
\delta(\Lambda(G)) &\le \delta(T(G)), & \delta^*(\Lambda(G)) &\le \delta^*(T(G)), & \delta_v^*(\Lambda(G)) &\le \delta_v^*(T(G)),\\
\delta(\Lambda(G)) &\le \delta(Q(G)), & \delta^*(\Lambda(G)) &\le \delta^*(Q(G)), & \delta_v^*(\Lambda(G)) &\le \delta_v^*(Q(G)).
\end{aligned}$$

The hyperbolicity of the line graph has been studied previously (see [21–23]). We have the following results.

Theorem 2. *[22] (Corollary 3.12) Let G be a graph. Then*

$$\delta(G) \le \delta(\Lambda(G)) \le 5\delta(G) + 5/2.$$

Furthermore, the first inequality is sharp: the equality is attained by every cycle graph.

Theorem 3. *[21] (Theorem 6) Let G be a graph. Then*

$$\delta_v^*(G) - 1 \le \delta_v^*(\Lambda(G)) \le \delta_v^*(G) + 1.$$

Theorem 4. *Let G be a graph. Then*

$$\delta^*(G) - 4 \le \delta^*(\Lambda(G)) \le \delta^*(G) + 4.$$

Proof. Proposition 2 and Theorem 3 give $\delta^*(G) \le \delta_v^*(G) + 3 \le \delta_v^*(\Lambda(G)) + 4 \le \delta^*(\Lambda(G)) + 4$, and $\delta^*(\Lambda(G)) \le \delta_v^*(\Lambda(G)) + 3 \le \delta_v^*(G) + 4 \le \delta^*(G) + 4$. □

From Proposition 1, and Theorems 2 and 4 we have:

Corollary 3. *Let G be a graph. Then*

$$\begin{aligned}
\delta(S(G)) &\le 2\delta(\Lambda(G)) \le 5\delta(S(G)) + 5,\\
\delta^*(S(G)) - 8 &\le 2\delta^*(\Lambda(G)) \le \delta^*(S(G)) + 8.
\end{aligned}$$

Corollary 2 and Theorems 2, 3 and 4 have the following consequence.

Corollary 4. *Let G be a graph. Then*

$$\begin{aligned}
\delta(G) &\le \delta(Q(G)),\\
\delta_v^*(G) &\le \delta_v^*(Q(G)) + 1,\\
\delta^*(G) &\le \delta^*(Q(G)) + 4.
\end{aligned}$$

Theorem 4 improves the inequality $\delta^*(\Lambda(G)) \le \delta^*(G) + 6$ in [23].

Given a graph G with multiple edges, we define the graph $B(G)$, obtained from G, substituting each multiple edge for one of its simple edges of shorter length (see [23]).

Remark 1. *By argument in the proof of [24](Theorem 8) we have: If in each multiple edge there is at most one edge with length greater than $j := \inf\{d(u,v) : u, v$ are joined by a multiple edge of $G\}$, then $\delta(G) \le \max\left\{\delta(B(G)) + \frac{J-j}{2}, \frac{J+j}{4}\right\}$, where, $J := \sup\{L(e) : e$ is an edge contained in a multiple edge of $G\}$.*

Corollary 5. *Let G be a graph. Then*

$$\max\left\{\delta(G), \frac{3}{4}\right\} \le \delta(R(G)) \le \max\left\{\delta(G) + \frac{1}{2}, \frac{3}{4}\right\}.$$

Proof. Note that $R(G)$ can be obtained by adding an edge of length 2 to each pair of adjacent vertices in G, so the graph becomes a graph with multiple edges, with $j = 1$ and $J = 2$. Then [24] (Theorem 8) and Remark 1 give the result. □

From [25] (Theorem 11), we have the following result.

Lemma 2. *Given the following graphs with edges of length 1, we have*

- *If P_n is a* path graph, *then $\delta(P_n) = 0$ for all $n \ge 1$.*
- *If C_n is a* cycle graph, *then $\delta(C_n) = n/4$ for all $n \ge 3$.*
- *If K_n is a* complete graph, *then $\delta(K_1) = \delta(K_2) = 0$, $\delta(K_3) = 3/4$ and $\delta(K_n) = 1$ for all $n \ge 4$.*

If G is not a tree, we define its *girth* $g(G)$ by

$$g(G) := \inf\{L(C) : C \text{ is a cycle in } G\}.$$

From [26] (Theorem 17), we have:

Theorem 5. *If G is not a tree, then*

$$\delta(G) \ge \frac{g(G)}{4}.$$

Corollary 6. *If G is not a tree, then*

$$\delta(G) \ge \frac{3}{4}.$$

Corollary 7. *If G is not a tree, then*

$$\delta(G) \le \delta(R(G)) \le \delta(G) + \frac{1}{2}.$$

Proof. Since G is not a tree, Corollary 6 gives $\delta(G) \ge 3/4$, and so

$$\max\left\{\delta(G), \frac{3}{4}\right\} = \delta(G), \qquad \max\left\{\delta(G) + \frac{1}{2}, \frac{3}{4}\right\} = \delta(G) + \frac{1}{2},$$

and Corollary 5 gives the inequalities. □

Theorem 2 and Corollary 7 have the following consequence.

Corollary 8. *If G is not a tree, then*

$$\delta(R(G)) - \frac{1}{2} \le \delta(\Lambda(G)) \le 5\delta(R(G)) + \frac{5}{2}.$$

From Proposition 1 and Corollary 7 we have the following result.

Corollary 9. *If G is not a tree, then*

$$\delta(S(G)) \le 2\delta(R(G)) \le \delta(S(G)) + 1.$$

Theorem 6. *Let G be a graph. Then*

$$\begin{aligned}
\delta^*(\Lambda(G)) &\le \delta^*(Q(G)) \le \delta_v^*(\Lambda(G)) + 6 \le \delta^*(\Lambda(G)) + 6,\\
\delta_v^*(\Lambda(G)) &\le \delta_v^*(Q(G)) \le \delta_v^*(\Lambda(G)) + 6,\\
\delta^*(\Lambda(G)) &\le \delta^*(T(G)) \le \delta_v^*(\Lambda(G)) + 9 \le \delta^*(\Lambda(G)) + 9,\\
\delta_v^*(\Lambda(G)) &\le \delta_v^*(T(G)) \le \delta_v^*(\Lambda(G)) + 6,\\
\delta^*(G) &\le \delta^*(R(G)) \le \delta_v^*(G) + 6 \le \delta^*(G) + 6,\\
\delta_v^*(G) &\le \delta_v^*(R(G)) \le \delta_v^*(G) + 6,\\
\delta^*(G) &\le \delta^*(T(G)) \le \delta_v^*(G) + 9 \le \delta^*(G) + 9,\\
\delta_v^*(G) &\le \delta_v^*(T(G)) \le \delta_v^*(G) + 6.
\end{aligned}$$

Proof. The lower bounds follow from Corollary 2. We consider the map $P : Q(G) \to \Lambda(G)$ such that $P(q) = q$ if $q \in \Lambda(G)$, $P(q) = v_q$ if $q \notin \Lambda(G)$, where $v_q \in V(\Lambda(G))$ and $d_{Q(G)}(q, v_q) \le 1$. If $q_0, q_1, q_2, q_3 \in Q(G)$, then

$$\big|d_{Q(G)}(q_i, q_j) - d_{\Lambda(G)}(P(q_i), P(q_j))\big| = \big|d_{Q(G)}(q_i, q_j) - d_{Q(G)}(P(q_i), P(q_j))\big| \le 2,$$

since $\Lambda(G)$ is an isometric subgraph of $Q(G)$ and

$$\begin{aligned}
&\big|(q_i, q_j)_{q_0} - (P(q_i), P(q_j))_{P(q_0)}\big|\\
&\quad = \frac{1}{2}\big|d_{Q(G)}(q_0, q_i) + d_{Q(G)}(q_0, q_j) - d_{Q(G)}(q_i, q_j)\\
&\quad -d_{\Lambda(G)}(P(q_0), P(q_i)) - d_{\Lambda(G)}(P(q_0), P(q_j)) + d_{\Lambda(G)}(P(q_i), P(q_j))\big| \le 3,
\end{aligned}$$

for $i, j \in \{1, 2, 3\}$. Thus,

$$\begin{aligned}
(q_1, q_3)_{q_0} &\ge (P(q_1), P(q_3))_{P(q_0)} - 3\\
&\ge \min\{(P(q_1), P(q_2))_{P(q_0)}, (P(q_2), P(q_3))_{P(q_0)}\} - \delta_v^*(\Lambda(G)) - 3\\
&\ge \min\{(q_1, q_2)_{q_0} - 3, (q_2, q_3)_{q_0} - 3\} - \delta_v^*(\Lambda(G)) - 3\\
&= \min\{(q_1, q_2)_{q_0}, (q_2, q_3)_{q_0}\} - \delta_v^*(\Lambda(G)) - 6.
\end{aligned}$$

Therefore,

$$\delta^*(\Lambda(G)) + 6 \ge \delta_v^*(\Lambda(G)) + 6 \ge \delta^*(Q(G)) \ge \delta_v^*(Q(G)).$$

These inequalities allow us to obtain the result for upper bounds of $\delta^*(Q(G))$ and $\delta_v^*(Q(G))$. The other upper bounds can be obtained similarly. □

From Theorems 3 and 6 and Corollary 4 we have:

Corollary 10. *For all graph G, we have*

$$\begin{aligned}
\delta_v^*(G) - 1 &\le \delta_v^*(Q(G)) \le \delta_v^*(G) + 7,\\
\delta^*(G) - 4 &\le \delta^*(Q(G)) \le \delta_v^*(G) + 7 \le \delta^*(G) + 7.
\end{aligned}$$

From Corollaries 2, 4 and 10, Theorem 6 and the inequalities $\delta(G) \le 3\delta^*(G)$ and $\delta^*(G) \le 2\delta(G)$, we have:

Corollary 11. *Let G be a graph. Then*

$$\delta(\Lambda(G)) \le \delta(Q(G)) \le 6\delta(\Lambda(G)) + 18,$$
$$\delta(\Lambda(G)) \le \delta(T(G)) \le 6\delta(\Lambda(G)) + 27,$$
$$\delta(G) \le \delta(T(G)) \le 6\delta(G) + 27,$$
$$\delta(G) \le \delta(Q(G)) \le 6\delta(G) + 21.$$

Proof. Corollaries 2 and 4 give the lower bounds. On the other hand, Theorem 6 gives $\delta(Q(G)) \le 3\delta^*(Q(G)) \le 3\delta^*(\Lambda(G)) + 18 \le 6\delta(\Lambda(G)) + 18$, $\delta(T(G)) \le 3\delta^*(T(G)) \le 3(\delta^*(\Lambda(G)) + 9) \le 6\delta(\Lambda(G)) + 27$; we obtain the third upper bound in a similar way. Corollary 10 gives $3\delta^*(Q(G)) \le 3(\delta^*(G) + 7) \le 6\delta(G) + 21$, obtaining the last upper bound. □

Let G be a graph, a family of subgraphs $\{G_s\}_s$ of G is a *T-decomposition* if $\cup_s G_s = G$ and $G_s \cap G_r$ is either a *cut-vertex* or the empty set for each $s \neq r$ (see [25]).

The following result was proved in [24] (Theorem 3).

Lemma 3. *Given a graph G and $\{G_s\}_s$ any T-decomposition of G, then*

$$\delta(G) = \sup_s \delta(G_s).$$

The following results improve the inequality $\delta(Q(G)) \le 6\delta(\Lambda(G)) + 18$ in Corollary 11.

Theorem 7. *Let G be a path graph, then*

$$0 = \delta(\Lambda(G)) \le \delta(Q(G)) \le 3/4.$$

Proof. Since G is a path graph, $\Lambda(G)$ is also a path graph, and so $0 = \delta(\Lambda(G)) \le \delta(Q(G))$.

Consider the T-decomposition $\{G_n\}$ of $Q(G)$. Since each connected component G_n is either a cycle C_3 or a path of length 1, we have $\delta(Q(G)) = \sup_n\{\delta(G_n)\} \le 3/4$, by Lemmas 2 and 3. □

The union of the set of the midpoints of the edges of a graph G and the set of vertices, $V(G)$, will be denote by $N(G)$. Let $\mathbb{T}_1$ be the set of geodesic triangles T in G such that every vertex of T belong to $N(G)$ and $\delta_1(G) := \inf\{\lambda : \text{every triangle in } \mathbb{T}_1 \text{ is } \lambda\text{-thin}\}$.

Lemma 4. *[27] (Theorems 2.5 and 2.7) For every graph G, we have $\delta_1(G) = \delta(G)$. Furthermore, if G is hyperbolic, then there exists $T \in \mathbb{T}_1$ with $\delta(T) = \delta(G)$.*

The previous lemma allows to reduce the study of the hyperbolicity constant of a graph G to study only the geodetic triangles of G, whose vertices are vertices of G (i.e., belong to $V(G)$) or midpoints of the edges of G.

Theorem 8. *If G is not a path graph, then*

$$\delta(\Lambda(G)) \le \delta(Q(G)) \le \delta(\Lambda(G)) + 1/2.$$

Proof. By Corollary 2 we have the first inequality. We will prove the second one. If $\delta(Q(G)) = \infty$, then Theorem 6 gives $\delta(\Lambda(G)) = \infty$, and the second inequality holds. Assume now that $\delta(Q(G)) < \infty$ (and so, $\delta(\Lambda(G)) < \infty$ by Theorem 6). If G is not a path, then $\Lambda(G)$ is not a tree and Corollary 6 gives $\delta(\Lambda(G)) \ge 3/4$.

For each $v \in V(G)$, let us define $V_v := \{u \in V(Q(G)) : uv \in E(Q(G))\} = \{u \in V(\Lambda(G)) : uv \in E(Q(G))\}$. Denote by G_v and G_v^* the subgraphs of $Q(G)$ induced by the sets $V_v \cup \{v\}$ and V_v, respectively. Note that both G_v and G_v^* are complete graphs for every $v \in V(G)$, and if

G^* is a complete graph with r vertices, then G_v is a complete graph with $r+1$ vertices. Also, $Q(G) = \Lambda(G) \cup (\cup_{v \in V(G)} G_v)$.

By Lemma 4 there exists a geodesic triangle $T \in \mathbb{T}_1$ in $Q(G)$ with $\delta(T) = \delta(Q(G))$. Denote by $\gamma_1, \gamma_2, \gamma_3$ the sides of T. Without loss of generality we can assume that there exists $p \in \gamma_1$ with $d_{Q(G)}(p, \gamma_2 \cup \gamma_3) = \delta(T) = \delta(Q(G))$. Thus, T is a cycle and each vertex of T is either the midpoint of some edge of $Q(G)$ or a vertex of $Q(G)$.

If G_v contains to T for some $v \in V(G)$, then $\delta(Q(G)) = \delta(T) \leq \delta(G_v) \leq 1 < 3/4 + 1/2 \leq \delta(\Lambda(G)) + 1/2$ by Lemma 2, since G_v is an isometric subgraph of $Q(G)$.

If $\Lambda(G)$ contains to T, then $\delta(Q(G)) = \delta(T) \leq \delta(\Lambda(G))$ by Lemma 1, since $\Lambda(G)$ is isometric.

Suppose that T is not contained either in $\Lambda(G)$ nor G_v with $v \in V(G)$.

Note that if $T \cap (G_v \setminus G_v^*) \neq \emptyset$ for some $v \in V(G)$, then there exists at least one vertex of T in $G_v \setminus \Lambda(G)$. In order to form a triangle $T^* \subset \Lambda(G)$ from T, we define $\gamma_i^* := \gamma_i \cap \Lambda(G)$. Note that, for $i \in \{1,2,3\}$, γ_i^* is a geodesic, since $\Lambda(G)$ is a isometric subgraph of $Q(G)$.

We denote by $x_{i,j}$ the common vertex of γ_i and γ_j and by u_i and u_j the other vertices of γ_i and γ_j respectively.

We consider the following cases:

Case A. We assume that exactly one vertex of T belongs to $Q(G) \setminus \Lambda(G)$. Thus, there exists $v \in V(G)$ such that $T \cap (G_v \setminus G_v^*) \neq \emptyset$. By Lemma 4, we have two possibilities: the vertex of T is a vertex of G or a midpoint of an edge in $G_v \setminus G_v^*$.

We can suppose that $x_{i,j} \in T \setminus \Lambda(G)$. Let v be a vertex of $V(G)$ such that $x_{i,j} \in G_v \setminus \Lambda(G)$. Let x_i (respectively, x_j) be the closest point of γ_i^* (respectively, γ_j^*) to $x_{i,j}$. Thus, $x_i x_j \in E(\Lambda(G))$. Let v^* be the midpoint of the edge $x_i x_j$. Let T_1 be the connected component of $T \setminus \Lambda(G)$ joining x_i and x_j. Note that $L(T_1) = 2$. We analyze the two possibilities:

Case A1. Assume that $x_{i,j} \in V(Q(G))$. Let us define $\sigma_i := \gamma_i^* \cup [x_i v^*]$ and $\sigma_j := \gamma_j^* \cup [x_j v^*]$. We are going to prove that σ_i and σ_j are geodesics in $\Lambda(G)$. In fact, we prove now that if $\gamma_j^* = [z_j x_j]$, then $d_{Q(G)}(z_j, x_j) \leq d_{Q(G)}(z_j, x_i)$. Seeking for a contradiction assume that $d_{Q(G)}(z_j, x_j) > d_{Q(G)}(z_j, x_i)$. Thus,

$$d_{Q(G)}(z_j, x_i) + d_{Q(G)}(x_i, x_{i,j}) = d_{Q(G)}(z_j, x_i) + 1 \leq d_{Q(G)}(z_j, x_j) + d_{Q(G)}(x_j, x_{i,j})$$

therefore γ_j is not a geodesic obtaining the desired contradiction and we conclude $d_{Q(G)}(z_j, x_j) \leq d_{Q(G)}(z_j, x_i)$. Hence, σ_i is a geodesic in $\Lambda(G)$.

Case A2. There is an edge $e \in E(Q(G)) \setminus E(\Lambda(G))$ such that $x_{i,j}$ is the midpoint of e, thus without loss of generality we can assume that $e = x_i v$, and we define $\sigma_i := \gamma_i^*$ and $\sigma_j := \gamma_j^* \cup x_j x_i$. Thus, σ_i is a geodesic in $\Lambda(G)$.

Note that $\gamma_j^* \cup x_j v \cup [v x_{i,j}]$ and $\sigma_j \cup [x_i x_{i,j}] = \gamma_j^* \cup x_j x_i \cup [x_i x_{i,j}]$ have the same endpoints and length; therefore, σ_j is also a geodesic in $\Lambda(G)$.

Case B. Assume that there are two vertices of T in some connected component of $T \setminus \Lambda(G)$. Thus, there exists $v \in V(G)$ such that $T \cap (G_v \setminus G_v^*) \neq \emptyset$. By Lemma 4, we have two possibilities again: both vertices of T are midpoints of edges or one vertex of T is a vertex of G and the other is a midpoint of an edge.

We can assume that $u_i, u_j \in G_v \setminus G_v^*$ for some v. We denote by x_i' (respectively, x_j') the closest point in γ_i^* (respectively, γ_j^*) to u_i (respectively, u_j); then $x_i' x_j' \in E(\Lambda(G))$. Let v' be the midpoint of the edge $x_i' x_j'$. Let T_2 be the connected component of $T \setminus \Lambda(G)$ joining x_i' and x_j'. Note that $L(T_2) = 2$.

We analyze the two possibilities again:

Case B1. The vertices u_i, u_j of T are the midpoints of $x_i' v$ and $x_j' v$. Thus, $\sigma_i := \gamma_i^*$, $\sigma_j := \gamma_j^*$ and $\sigma_k := x_i' x_j'$ are geodesics in $\Lambda(G)$.

Case B2. Otherwise, we can assume without loss of generality that $u_j = v$ and u_i is the midpoint of $x_i v$. We have $d_{Q(G)}(u_i, x_j) = d_{Q(G)}(u_i, x_i) + 1$ and so, $\sigma_i := \gamma_i^*$ and $\sigma_j := \gamma_j^* \cup x_j' x_i'$ are geodesics in $\Lambda(G)$. In this case we define $\sigma_k := \{x_i'\}$.

Note that the most general possible case is the following: there are at most three vertices $v_1, v_2, v_3 \in V(G)$ such that $T \cap (G_{v_i} \setminus G_{v_i^*}) \neq \emptyset$, for $i = 1,2,3$. Repeating the previous process at most three times we obtain a geodesic triangle T^* in $\Lambda(G)$ with sides γ_1', γ_2' and γ_3' containing γ_1^*, γ_2^* and γ_3^*, respectively.

If $p \in \Lambda(G)$, then one can check that $\delta(Q(G)) = d_{Q(G)}(p, \gamma_2 \cup \gamma_3) \leq d_{Q(G)}(p, \gamma_2' \cup \gamma_3') + 1/2 \leq \delta(\Lambda(G)) + 1/2$. If $p \notin \Lambda(G)$, then $\delta(Q(G)) = d_{Q(G)}(p, \gamma_2 \cup \gamma_3) \leq 5/4$; since $\delta(\Lambda(G)) \geq 3/4$, we have $\delta(\Lambda(G)) + 1/2 \geq 5/4 \geq \delta(Q(G))$. This finishes the proof. □

Proposition 1, Theorems 2 and 8, and Corollary 3 have the following consequence.

Corollary 12. *Let G be a graph. If G is not a path graph, then*

$$\delta(S(G)) \leq 2\delta(Q(G)) \leq 5\delta(S(G)) + 6.$$

4. Conclusions

In this paper, we obtained several inequalities and closed formulas relating the hyperbolicity constants of a graph G and its graph operators $\Lambda(G)$, $S(G)$, $T(G)$, $R(G)$ and $Q(G)$, by the use of their symmetries. As a first step, as the basis of our research, we found relations among the Gromov hyperbolicity constant (satisfying the Rips condition), the Gromov product and the Gromov product restricted to vertices. In the same direction, we derived inequalities between Gromov products and graph operators; as examples we mention: $\delta_v^*(G) \leq \delta^*(G) \leq \delta_v^*(G) + 3$, $\delta_v^*(G) \leq \delta_v^*(Q(G)) + 1$ and $\delta^*(G) \leq \delta^*(R(G)) \leq \delta_v^*(G) + 6 \leq \delta^*(G) + 6$.

Then, we studied relations between the Gromov hyperbolicity constant of a graph and the application of given operators to that graph. In this context, we obtained inequalities such as: $\delta(G) \leq \delta(R(G)) \leq \delta(G) + 1/2$, $\delta(\Lambda(G)) \leq \delta(Q(G)) \leq \delta(\Lambda(G)) + 1/2$, $\delta(S(G)) \leq 2\delta(R(G)) \leq \delta(S(G)) + 1$ and $\delta(R(G)) - 1/2 \leq \delta(\Lambda(G)) \leq 5\delta(R(G)) + 5/2$, where G not a tree.

We believe that our work may motivate the investigation of related open problems such as: (i) the computation of the hyperbolicity constant on geometric graphs; (ii) the analysis of hyperbolicity on the graph operators reported here (i.e., $\Lambda(G)$, $S(G)$, $T(G)$, $R(G)$ and $Q(G)$) when applied to geometric graphs; (iii) the study of the hyperbolicity constants of additional graph operators; and (iv) the identification of the properties of graph operations that break or preserve hyperbolicity.

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