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On p-adic Integral Representation of q-Bernoulli Numbers Arising from Two Variable q-Bernstein Polynomials

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Abstract: The q-Bernoulli numbers and polynomials can be given by Witt's type formulas as p-adic invariant integrals on $\mathbb{Z}_p$. We investigate some properties for them. In addition, we consider two variable q-Bernstein polynomials and operators and derive several properties for these polynomials and operators. Next, we study the evaluation problem for the double integrals on $\mathbb{Z}_p$ of two variable q-Bernstein polynomials and show that they can be expressed in terms of the q-Bernoulli numbers and some special values of q-Bernoulli polynomials. This is generalized to the problem of evaluating any finite product of two variable q-Bernstein polynomials. Furthermore, some identities for q-Bernoulli numbers are found.

Keywords: q-Bernoulli numbers; q-Bernoulli polynomials; two variable q-Bernstein polynomials; two variable q-Bernstein operators; p-adic integral on $\mathbb{Z}_p$

1. Introduction

Let p be a fixed prime number. Throughout this paper, $\mathbb{N}$, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will denote the set of natural numbers, the ring of p-adic integers, the field of p-adic rational numbers, and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. The p-adic norm $|\cdot|_p$ is normalized as $|p|_p = \frac{1}{p}$. Assume that q is an indeterminate in $\mathbb{C}_p$ such that $|1-q|_p < p^{-\frac{1}{p-1}}$.

It is known that the q-number is defined by

$$[x]_q = \frac{1-q^x}{1-q},$$

see [1–20].

Please note that $\lim_{q\to 1}[x]_q = x$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the p-adic q-integral on $\mathbb{Z}_p$ is defined by Kim as

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x)q^x, \tag{1}$$

see [9,10].

As $q \to 1$ in (1), we have the p-adic integral on $\mathbb{Z}_p$ which is given by

$$I_1(f) = \lim_{q\to 1} I_q(f) = \int_{\mathbb{Z}_p} f(x)d\mu_1(x) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \tag{2}$$

see [7–11,17].

From (2), we note that

$$I_1(f_1) - I_1(f) = f'(0), \tag{3}$$

see [9]. Where $f_1(x) = f(x+1)$ and $f'(0) = \frac{df(x)}{dx}|_{x=0}$.

Thus, by (3), we get

$$\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_1(y) = \frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \tag{4}$$

see [6,9], where $B_n(x)$ are the ordinary Bernoulli polynomials.

From (4), we note that

$$\int_{\mathbb{Z}_p} (x+y)^n d\mu_1(y) = B_n(x), (n \geq 0), \tag{5}$$

see [7–11,17,18].

When $x = 0$, $B_n = B_n(0)$, $(n \geq 0)$, are called the ordinary Bernoulli numbers.

The Equation (4) implies the following recurrence relation for Bernoulli numbers:

$$B_0 = 1, \quad (B+1)^n - B_n = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1, \end{cases} \tag{6}$$

with the usual convention about replacing B^n by B_n (see [21]).

In [3,4], L. Carlitz introduced the q-Bernoulli numbers given by the recurrence relation

$$\beta_{0,q} = 1, \quad q(q\beta_q + 1)^n - \beta_{n,q} = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1, \end{cases} \tag{7}$$

with the usual convention about replacing β_q^n by $\beta_{n,q}$.

He also defined q-Bernoulli polynomials as

$$\beta_{n,q}(x) = (q^x \beta_q + [x]_q)^n = \sum_{l=0}^{n} \binom{n}{l} [x]_q^{n-l} q^{lx} \beta_{l,q}, \tag{8}$$

see [3,4].

In 1999, Kim proved the following formula.

$$\int_{\mathbb{Z}_p} [x+y]_q^n d\mu_q(y) = \beta_{n,q}(x), \ (n \geq 0), \tag{9}$$

see [10].

In the view of (5) and (9), we define the q-Bernoulli polynomials, different from Carlitz's q-Bernoulli polynomials, as

$$B_{n,q}(x) = \int_{\mathbb{Z}_p} [x+y]_q^n d\mu_1(y), \ (n \geq 0), \tag{10}$$

see [8,9].

When $x = 0$, $B_{n,q} = B_{n,q}(0)$ are called the q-Bernoulli numbers.

From (3) and (10), we have

$$B_{0,q}=1,\ (qB_q+1)^n-B_{n,q}=\begin{cases}\frac{\log q}{q-1} & \text{if } n=1,\\ 0 & \text{if } n>1,\end{cases} \tag{11}$$

with the usual convention about replacing B_q^n by $B_{n,q}$.

By (10), we easily get

$$B_{n,q}(x)=(q^xB_q+[x]_q)^n=\sum_{l=0}^{n}\binom{n}{l}[x]_q^{n-l}q^{lx}B_{l,q}, \tag{12}$$

see [9].

As is known, the p-adic q-Bernstein operator is given by

$$\mathbb{B}_{n,q}(f|x)=\sum_{k=0}^{n}\binom{n}{k}f\left(\frac{k}{n}\right)[x]_q^k[1-x]_{q^{-1}}^{n-k}=\sum_{k=0}^{n}f\left(\frac{k}{n}\right)B_{k,n}(x|q),$$

where $n,k\in\mathbb{N}\cup\{0\}$, $x\in\mathbb{Z}_p$, and f is a continuous function on $\mathbb{Z}_p$ (see [7]). Here

$$B_{k,n}(x|q)=\binom{n}{k}[x]_q^k[1-x]_{q^{-1}}^{n-k},\ (n,k\geq 0),$$

are called the p-adic q-Bernstein polynomials of degree n (see [7]). Please note that $\lim_{q\to 1}B_{k,n}(x|q)=B_{k,n}(x)$, where $B_{k,n}$ are the Bernstein polynomials (see [1,2,18–20,22]).

Here we cannot go without mentioning that Phillips (see [16]) introduced earlier in 1997 a different version of q-Bernstein polynomials from Kim's. Let f be a function defined on $[0,1]$, q any positive real number, and let

$$[n]_q!=[1]_q[2]_q\ldots[n]_q,(n\geq 1),\quad [0]_q!=1,\quad \begin{bmatrix}n\\k\end{bmatrix}_q=\frac{[n]_q!}{[k]_q![n-k]_q!}.$$

Then Phillips' q-Bernstein polynomial of order n for f is given by

$$\mathbb{B}_n(f,q;x)=\sum_{k=0}^{n}f\left(\frac{[k]_q}{[n]_q}\right)\begin{bmatrix}n\\k\end{bmatrix}_q x^k\prod_{s=0}^{n-1-k}(1-q^sx),$$

Many results of Phillips' q-Bernstein polynomials for $q>1$ were obtained for instance in [14,15], while those for $q\in(0,1)$ were derived for example in [12,13]. However, all of these and other related papers deal only with analytic properties of those q-Bernstein polynomials and some applications of them.

The Volkenborn integral and the fermionic p-adic, the p-adic q-invariant and the fermionic p-adic q-invariant integrals introduced by Kim have been studied for more than twenty years. Numberous results of arithmetic or combinatorial nature have been found by Kim and his colleagues around the world.

The present and related paper (see [5,6]) concern about Kim's q-Bernstein polynomials which have some merits over Phillips'. Indeed, by considering p-adic integrals on $\mathbb{Z}_p$ of them we can easily derive integral representations of q-Bernoulli numbers in the present paper, those of a q-analogue of Euler numbers in [5] and those of q-Euler numbers in [6]. These approaches also yield some identities for q-Bernoulli numbers, q-analogue of Euler numbers and q-Euler numbers. In conclusion, the Phillips' q-Bernstein polynomials are more analytic nature, while the Kim's are more arithmetic and combinatorial nature.

In this paper, we will study q-Bernoulli numbers and polynomials, which is introduced as p-adic invariant integrals on $\mathbb{Z}_p$, and investigate some properties for these numbers and polynomials. Also, we will consider two variable q-Bernstein polynomials and operators and derive several properties for these polynomials and operators. Next, we will consider p-adic integrals on $\mathbb{Z}_p$ of any finite product of two variable q-Bernstein polynomials and show that they can be expressed in terms of the q-Bernoulli numbers and some special values of q-Bernoulli polynomials. Furthermore, some identities for q-Bernoulli numbers will be found.

2. Some Integral Representations of q-Bernoulli Numbers and Polynomials

First, we consider the two variable q-Bernstein operator of order n which is given by

$$\mathbb{B}_{n,q}(f|x_1,x_2) = \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) [x_1]_q^k [1-x_2]_{q^{-1}}^{n-k} = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) B_{k,n}(x_1,x_2|q),$$

where $n \in \mathbb{N}$, and $x_1, x_2 \in \mathbb{Z}_p$.

Here, for $n,k \geq 0$,

$$B_{k,n}(x_1,x_2|q) = \binom{n}{k} [x_1]_q^k [1-x_2]_{q^{-1}}^{n-k} \tag{13}$$

are called two variable q-Bernstein polynomials of degree n (see [6,7]). In particular, this implies that $B_{k,n}(x_1,x_2|q) = 0$, for $0 \leq n < k$. In (13), if $x_1 = x_2 = x$, then $B_{k,n}(x,x|q) = B_{k,n}(x|q)$ are the q-Bernstein polynomials. It is not difficult to show that the generating function of $B_{k,n}(x_1,x_2|q)$ is given by

$$F_q^{(k)}(x_1,x_2|t) = \frac{(t[x_1]_q)^k}{k!} e^{(t[1-x_2]_{q^{-1}})} = \sum_{n=k}^{\infty} B_{k,n}(x_1,x_2|q)\frac{t^n}{n!}, \tag{14}$$

where $k \in \mathbb{N} \cup \{0\}$ (see [6,7]).

From (13), we easily get

$$B_{n-k,n}(1-x_2, 1-x_1|q^{-1}) = B_{k,n}(x_1,x_2|q), \ (0 \leq k \leq n).$$

For $1 \leq k \leq n-1$, we have the following properties (see [6,7]):

$$[1-x_2]_{q^{-1}} B_{k,n-1}(x_1,x_2|q) + [x_1]_q B_{k-1,n-1}(x_1,x_2|q) = B_{k,n}(x_1,x_2|q), \tag{15}$$

$$\frac{\partial}{\partial x_1} B_{k,n}(x_1,x_2|q) = \frac{\log q}{q-1} n \left((q-1)[x_1]_q B_{k-1,n-1}(x_1,x_2|q) + B_{k-1,n-1}(x_1,x_2|q)\right), \tag{16}$$

$$\frac{\partial}{\partial x_2} B_{k,n}(x_1,x_2|q) = \frac{\log q}{1-q} n((q-1)[x_2]_q B_{k,n-1}(x_1,x_2|q) + B_{k,n-1}(x_1,x_2|q)). \tag{17}$$

From (13) and q-Bernstein operator, we note that

$$\mathbb{B}_{n,q}(1|x_1,x_2) = \sum_{k=0}^{n} \binom{n}{k} [x_1]_q^k [1-x_2]_{q^{-1}}^{n-k} = (1+[x_1]_q - [x_2]_q)^n, \tag{18}$$

$$\mathbb{B}_{n,q}(t|x_1,x_2) = [x_1]_q (1+[x_1]_q - [x_2]_q)^{n-1},$$

$$\mathbb{B}_{n,q}(t^2|x_1,x_2) = \frac{n-1}{n} [x_1]_q^2 (1+[x_1]_q - [x_2]_q)^{n-2} + \frac{[x_1]_q}{n} (1+[x_1]_q - [x_2]_q)^{n-1},$$

and

$$\mathbb{B}_{n,q}(f|x_1,x_2) = \sum_{l=0}^{n} \binom{n}{l} [x_2]_q^l \sum_{k=0}^{l} \binom{l}{k} (-1)^{l-k} f\left(\frac{k}{n}\right) \left(\frac{[x_1]_q}{[x_2]_q}\right)^k, \tag{19}$$

where $n \in \mathbb{N}$ and f is a continuous function on $\mathbb{Z}_p$.

To see this, we first observe that

$$[1-x_2]_{q^{-1}} = 1 - [x_2]_q, \quad \binom{n}{k}\binom{n-k}{l-k} = \binom{n}{l}\binom{l}{k}.$$

Then (19) can be obtained as follows:

$$\begin{aligned}
\mathbb{B}(f|x_1,x_2) &= \sum_{k=0}^{n} f(\frac{k}{n})\binom{n}{k}[x_1]_q^k(1-[x_2]_q)^{n-k} \\
&= \sum_{k=0}^{n} f(\frac{k}{n})\binom{n}{k}[x_1]_q^k \sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^l[x_2]_q^l \\
&= \sum_{k=0}^{n} f(\frac{k}{n})\binom{n}{k}[x_1]_q^k \sum_{l=k}^{n}\binom{n-k}{l-k}(-1)^{l-k}[x_2]_q^{l-k} \\
&= \sum_{l=0}^{n}\binom{n}{l}[x_2]_q^l \sum_{k=0}^{l}\binom{l}{k}(-1)^{l-k} f\left(\frac{k}{n}\right)\left(\frac{[x_1]_q}{[x_2]_q}\right)^k.
\end{aligned}$$

It is easy to show that

$$\frac{1}{(1+[x_1]_q-[x_2]_q)^{n-j}}\sum_{k=j}^{n}\frac{\binom{k}{j}}{\binom{n}{j}}B_{k,n}(x_1,x_2|q) = [x_1]_q^j, \tag{20}$$

where $j \in \mathbb{N} \cup \{0\}$ and $x_1, x_2 \in \mathbb{Z}_p$.

Indeed, by making use of (18), we see that

$$\begin{aligned}
\sum_{k=j}^{n}\frac{\binom{k}{j}}{\binom{n}{j}}B_{k,n}(x_1,x_2|q) &= \sum_{k=j}^{n}\binom{n-j}{k-j}[x_1]_q^k[1-x_2]_{q^{-1}}^{n-k} \\
&= \sum_{k=0}^{n-j}\binom{n-j}{k}[x_1]_q^{k+j}[1-x_2]_{q^{-1}}^{n-j-k} \\
&= [x_1]^j(1+[x_1]_q-[x_2]_q)^{n-j}.
\end{aligned}$$

From (2), we have

$$\int_{\mathbb{Z}_p}[1-x+y]_{q^{-1}}^n d\mu_1(y) = (-1)^n q^n \int_{\mathbb{Z}_p}[x+y]_q^n d\mu_1(y),\ (n \geq 0). \tag{21}$$

By (10) and (21), we get

$$B_{n,q^{-1}}(1-x) = (-1)^n q^n B_{n,q}(x),\ (n \geq 0). \tag{22}$$

Again, from (11) and (12), we can derive the following equation.

$$B_{n,q}(2) = nq\frac{\log q}{q-1} + (qB_q+1)^n = nq\frac{\log q}{q-1} + B_{n,q},\ (n > 1). \tag{23}$$

Thus, by (23), we obtain the following lemma.

Lemma 1. *For $n \in \mathbb{N}$ with $n > 1$, we have*

$$B_{n,q}(2) = nq\frac{\log q}{q-1} + B_{n,q}.$$

By (2), (10) and (22), we get

$$\begin{aligned}\int_{\mathbb{Z}_p}[1-x]_{q^{-1}}^n d\mu_1(x) &= (-1)^n q^n \int_{\mathbb{Z}_p}[x-1]_q^n d\mu_1(x)\\ &= (-1)^n q^n B_{n,q}(-1)\\ &= B_{n,q^{-1}}(2),\ (n\geq 0).\end{aligned} \tag{24}$$

For $n \in \mathbb{N}$ with $n > 1$, by (21), Lemma 1, and (24), we have

$$\begin{aligned}\int_{\mathbb{Z}_p}[1-x]_{q^{-1}}^n d\mu_1(x) &= \int_{\mathbb{Z}_p}[x+2]_{q^{-1}}^n d\mu_1(x)\\ &= (-1)^n q^n \int_{\mathbb{Z}_p}[x-1]_q^n d\mu_1(x)\\ &= n\frac{\log q}{q-1} + \int_{\mathbb{Z}_p}[x]_{q^{-1}}^n d\mu_1(x)\\ &= \frac{n\log q}{q-1} + B_{n,q^{-1}}.\end{aligned} \tag{25}$$

Let us take the double p-adic integral on $\mathbb{Z}_p$ for the two variable q-Bernstein polynomials. Then we have

$$\begin{aligned}&\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} B_{k,n}(x_1,x_2|q)d\mu_1(x_1)d\mu_1(x_2)\\ &= \binom{n}{k}\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}[x_1]_q^k[1-x_2]_{q^{-1}}^{n-k}d\mu_1(x_1)d\mu_1(x_2)\\ &= \binom{n}{k}B_{k,q}\int_{\mathbb{Z}_p}[1-x_2]_{q^{-1}}^{n-k}d\mu_1(x_2)\\ &= \begin{cases}\binom{n}{k}B_{k,q}\left(B_{n-k,q^{-1}}+\frac{\log q}{q-1}(n-k)\right), & \text{if } n>k+1,\\ (k+1)B_{k,q}B_{1,q^{-1}}(2), & \text{if } n=k+1,\\ B_{k,q}, & \text{if } n=k,\\ 0, & \text{if } 0\leq n<k.\end{cases}\end{aligned} \tag{26}$$

Therefore, we obtain the following theorem.

Theorem 1. *For $n, k \in \mathbb{N}\cup\{0\}$, we have*

$$\begin{aligned}&\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} B_{k,n}(x_1,x_2|q)d\mu_1(x_1)d\mu_1(x_2)\\ &= \begin{cases}\binom{n}{k}B_{k,q}\left(B_{n-k,q^{-1}}+\frac{\log q}{q-1}(n-k)\right), & \text{if } n>k+1,\\ (k+1)B_{k,q}B_{1,q^{-1}}(2), & \text{if } n=k+1,\\ B_{k,q} & \text{if } n=k,\\ 0, & \text{if } 0\leq n<k.\end{cases}\end{aligned}$$

For $n, k \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} B_{k,n}(x_1,x_2|q)d\mu_1(x_1)d\mu_1(x_2) \\
&= \int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} \binom{n}{k}[x_1]_q^k[1-x_2]_{q^{-1}}^{n-k}d\mu_1(x_1)d\mu_1(x_2) \\
&= \binom{n}{k}\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} (1-[1-x_1]_{q^{-1}})^k[1-x_2]_{q^{-1}}^{n-k}d\mu_1(x_1)d\mu_1(x_2) \\
&= \binom{n}{k}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} [1-x_1]_{q^{-1}}^{k-l}[1-x_2]_{q^{-1}}^{n-k}d\mu_1(x_1)d\mu_1(x_2) \\
&= \binom{n}{k}\int_{\mathbb{Z}_p}[1-x_2]_{q^{-1}}^{n-k}d\mu_1(x_2) \\
&\times\left(1-k\int_{\mathbb{Z}_p}[1-x_1]_{q^{-1}}d\mu_1(x_1)+\sum_{l=0}^{k-2}\binom{k}{l}(-1)^{k-l}\left((k-l)\frac{\log q}{q-1}+B_{k-l,q^{-1}}\right)\right).
\end{aligned} \tag{27}$$

Thus, by (27), we get

$$\begin{aligned}
&\binom{n}{k}^{-1}\frac{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} B_{k,n}(x_1,x_2|q)d\mu_1(x_1)d\mu_1(x_2)}{\int_{\mathbb{Z}_p}[1-x_2]_{q^{-1}}^{n-k}d\mu_1(x_2)} \\
&= 1-k\int_{\mathbb{Z}_p}[1-x_1]_{q^{-1}}d\mu_1(x_1)+\sum_{l=0}^{k-2}\binom{k}{l}(-1)^{k-l}\left((k-l)\frac{\log q}{q-1}+B_{k-l,q^{-1}}\right) \\
&= 1-k\Big(1-\frac{\log q-q+1}{(q-1)^2}\Big) \\
&\quad +k\sum_{l=0}^{k-2}\binom{k-1}{l}(-1)^{k-l}\frac{\log q}{q-1}+\sum_{l=0}^{k-2}(-1)^{k-l}\binom{k}{l}B_{k-l,q^{-1}}
\end{aligned} \tag{28}$$

$$\begin{aligned}
&= 1-k\Big(1-\frac{\log q-q+1}{(q-1)^2}\Big)+k\frac{\log q}{q-1}+\sum_{l=0}^{k-2}(-1)^{k-l}\binom{k}{l}B_{k-l,q^{-1}} \\
&= 1-kq\left(\frac{q-\log q-1}{(q-1)^2}\right)+\sum_{l=0}^{k-2}(-1)^{k-l}\binom{k}{l}B_{k-l,q^{-1}}.
\end{aligned} \tag{29}$$

Therefore, by (28), we obtain the following theorem.

Theorem 2. *For $n, k \in \mathbb{N} \cup \{0\}$ with $k > 1$, we have*

$$\begin{aligned}
&\frac{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} B_{k,n}(x_1,x_2|q)d\mu_1(x_1)d\mu_1(x_2)}{\binom{n}{k}\int_{\mathbb{Z}_p}[1-x_2]_{q^{-1}}^{n-k}d\mu_1(x_2)} \\
&= \binom{n}{k}\left(1-kq\left(\frac{q-\log q-1}{(q-1)^2}\right)+\sum_{l=0}^{k-2}(-1)^{k-l}\binom{k}{l}B_{k-l,q^{-1}}\right).
\end{aligned}$$

Therefore, by Theorems 1 and 2, we obtain the following corollary.

Corollary 1. *For $k \in \mathbb{N}$ with $k > 1$, we have*

$$B_{k,q} = 1-kq\left(\frac{q-\log q-1}{(q-1)^2}\right)+\sum_{l=0}^{k-2}(-1)^{k-l}\binom{k}{l}B_{k-l,q^{-1}}.$$

For $m, n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned}&\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} B_{k,n}(x_1,x_2|q)B_{k,m}(x_1,x_2|q)d\mu_1(x_1)d\mu_1(x_2)\\&=\binom{n}{k}\binom{m}{k}\int_{\mathbb{Z}_p}[x_1]_q^{2k}d\mu_1(x_1)\int_{\mathbb{Z}_p}[1-x_2]_{q^{-1}}^{n+m-2k}d\mu_1(x_2).\end{aligned}\tag{30}$$

Thus, by (29), we get

$$\begin{aligned}&\frac{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} B_{k,n}(x_1,x_2|q)B_{k,m}(x_1,x_2|q)d\mu_1(x_1)d\mu_1(x_2)}{\int_{\mathbb{Z}_p}[1-x_2]_{q^{-1}}^{n+m-2k}d\mu_1(x_2)}\\&=\binom{n}{k}\binom{m}{k}B_{2k,q}.\end{aligned}$$

Hence, we have the following proposition.

Proposition 1. *For $m, n, k \in \mathbb{N} \cup \{0\}$, we have*

$$\begin{aligned}&\frac{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} B_{k,n}(x_1,x_2|q)B_{k,m}(x_1,x_2|q)d\mu_1(x_1)d\mu_1(x_2)}{\int_{\mathbb{Z}_p}[1-x_2]_{q^{-1}}^{n+m-2k}d\mu_1(x_2)}\\&=\binom{n}{k}\binom{m}{k}B_{2k,q}.\end{aligned}$$

Let $m, n, k \in \mathbb{N} \cup \{0\}$. Then we get

$$\begin{aligned}&\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} B_{k,n}(x_1,x_2|q)B_{k,m}(x_1,x_2|q)d\mu_1(x_1)d\mu_1(x_2)\\&=\sum_{l=0}^{2k}\binom{n}{k}\binom{m}{k}\binom{2k}{l}(-1)^{2k-l}\\&\quad\times\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}[1-x_1]_{q^{-1}}^{2k-l}[1-x_2]_{q^{-1}}^{n+m-2k}d\mu_1(x_1)d\mu_1(x_2)\end{aligned}\tag{31}$$

Thus, from (30), we have

$$\begin{aligned}&\binom{n}{k}^{-1}\binom{m}{k}^{-1}\frac{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} B_{k,n}(x_1,x_2|q)B_{k,m}(x_1,x_2|q)d\mu_1(x_1)d\mu_1(x_2)}{\int_{\mathbb{Z}_p}[1-x_2]_{q^{-1}}^{n+m-2k}d\mu_1(x_2)}\\&=1-2k\int_{\mathbb{Z}_p}[1-x_1]_{q^{-1}}d\mu_1(x_1)+\sum_{l=0}^{2k-2}\binom{2k}{l}(-1)^{2k-l}\int_{\mathbb{Z}_p}[1-x_1]_{q^{-1}}^{2k-l}d\mu_1(x_1)\\&=1-2k\Big(1-\frac{\log q-q+1}{(q-1)^2}\Big)+2k\frac{\log q}{q-1}+\sum_{l=0}^{2k-2}\binom{2k}{l}(-1)^{2k-l}B_{2k-l,q^{-1}}\\&=1-2kq\left(\frac{q-\log q-1}{(q-1)^2}\right)+\sum_{l=0}^{2k-2}\binom{2k}{l}(-1)^{2k-l}B_{2k-l,q^{-1}}.\end{aligned}\tag{32}$$

By (31), we have the following proposition.

Proposition 2. *For $m, n, k \in \mathbb{N} \cup \{0\}$, with $k \geq 1$, we have*

$$\frac{\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} B_{k,n}(x_1, x_2|q) B_{k,m}(x_1, x_2|q) d\mu_1(x_1) d\mu_1(x_2)}{\int_{\mathbb{Z}_p} [1 - x_2]_{q^{-1}}^{n+m-2k} d\mu_1(x_2)}$$
$$= \binom{n}{k}\binom{m}{k}\left(1 - 2kq\left(\frac{q - \log q - 1}{(q-1)^2}\right) + \sum_{l=0}^{2k-2}\binom{2k}{l}(-1)^{2k-l} B_{2k-l,q^{-1}}\right).$$

Therefore, by Propositions 1 and 2, we obtain the following corollary.

Corollary 2. *For $k \in \mathbb{N}$, we have*

$$B_{2k,q} = 1 - 2k\left(\frac{q^2 - q - \log q}{(q-1)^2}\right)$$
$$+ \sum_{l=0}^{2k-2}\binom{2k}{l}(-1)^{2k-l}\left((2k-l)\frac{\log q}{q-1} + B_{2k-l,q^{-1}}\right).$$

For $m \in \mathbb{N}$, let $n_1, n_2, \cdots, n_m, k \in \mathbb{N} \cup \{0\}$. Then we note that

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\left(\prod_{i=1}^{m} B_{k,n_i}(x_1, x_2|q)\right) d\mu_1(x_1) d\mu_1(x_2) \\
&= \sum_{l=0}^{mk}\left(\prod_{i=1}^{m}\binom{n_i}{k}\right)\binom{mk}{l}(-1)^{mk-l} \\
&\times \int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} [1 - x_1]_{q^{-1}}^{mk-l}[1 - x_2]_{q^{-1}}^{n_1+n_2+\cdots+n_m-mk} d\mu_1(x_1) d\mu_1(x_2)
\end{aligned} \tag{33}$$

Thus, by (32), we have

$$\begin{aligned}
&\prod_{i=1}^{m}\binom{n_i}{k}^{-1}\frac{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\left(\prod_{i=1}^{m} B_{k,n_i}(x_1, x_2|q)\right) d\mu_1(x_1) d\mu_1(x_2)}{\int_{\mathbb{Z}_p} [1 - x_2]_{q^{-1}}^{n_1+n_2+\cdots+n_m-mk} d\mu_1(x_2)} \\
&= \sum_{l=0}^{mk}\binom{mk}{l}(-1)^{mk-l}\int_{\mathbb{Z}_p} [1 - x_1]_{q^{-1}}^{mk-l} d\mu_1(x_1) \\
&= 1 - mk\int_{\mathbb{Z}_p} [1 - x_1]_{q^{-1}} d\mu_1(x_1) \\
&\quad + \sum_{l=0}^{mk-2}\binom{mk}{l}(-1)^{mk-l}\int_{\mathbb{Z}_p} [1 - x_1]_{q^{-1}}^{mk-l} d\mu_1(x_1)
\end{aligned}$$

Therefore we obtain the following theorem.

Theorem 3. *For $n_1, n_2, \cdots, n_m \in \mathbb{N} \cup \{0\}$, and $k, m \in \mathbb{N}$ with $mk > 1$, we have*

$$\begin{aligned}
&\prod_{i=1}^{m}\binom{n_i}{k}^{-1}\frac{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\left(\prod_{i=1}^{m} B_{k,n_i}(x_1, x_2|q)\right) d\mu_1(x_1) d\mu_1(x_2)}{\int_{\mathbb{Z}_p} [1 - x_2]_{q^{-1}}^{n_1+n_2+\cdots+n_m-mk} d\mu_1(x_2)} \\
&= 1 - mk\left(\frac{q^2 - q - \log q}{(q-1)^2}\right) \\
&\quad + \sum_{l=0}^{mk-2}\binom{mk}{l}(-1)^{mk-l}\left((mk-l)\frac{\log q}{q-1} + B_{mk-l,q^{-1}}\right).
\end{aligned}$$

On the other hand, we easily get

$$\frac{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\left(\prod_{i=1}^{m} B_{k,n_i}(x_1,x_2|q)\right)d\mu_1(x_1)d\mu_1(x_2)}{\int_{\mathbb{Z}_p}[1-x_2]_{q^{-1}}^{n_1+n_2+\cdots+n_m-mk}d\mu_1(x_2)}=\prod_{i=1}^{m}\binom{n_i}{k}B_{mk,q}. \tag{34}$$

Therefore, by Theorem 3 and (33), we obtain the following corollary.

Corollary 3. *For $m,k\in\mathbb{N}$ with $mk>1$, we have*

$$\begin{aligned}B_{mk,q}=&\,1-mk\left(\frac{q^2-q-\log q}{(q-1)^2}\right)\\&+\sum_{l=0}^{mk-2}\binom{mk}{l}(-1)^{mk-l}\left((mk-l)\frac{\log q}{q-1}+B_{mk-l,q^{-1}}\right).\end{aligned}$$

3. Conclusions

Here we studied q-Bernoulli numbers and polynomials which are different from the classical Carlitz q-Bernoulli numbers $\beta_{n,q}$ and polynomials $\beta_{n,q}(x)$, and arise naturally from some p-adic invariant integrals on $\mathbb{Z}_p$, as was shown in (10). After investigating some of their properties, we turned our attention to two variable q-Bernstein polynomials and operators, which was introduced by Kim and generalizes the single variable q-Bernstein polynomials and operators in [6]. As a preparation, we derived several properties of these polynomials and operators.

Next, we considered the evaluation problem for the double integrals on $\mathbb{Z}_p$ of two variable q-Bernstein polynomials and showed that they can be expressed in terms of the q-Bernoulli numbers and some special values of q-Bernoulli polynomials. This was further generalized to the problem of evaluating the product of two and that of an arbitrary number of two variable q-Bernstein polynomials. It was shown again that they can be expressed in terms of the q-Bernoulli numbers. Also, some identities for q-Bernoulli numbers were found along the way.

Finally, we would like to mention that, along the same line, in [5] we studied some properties of a q-analogue of Euler numbers and polynomials arising from the p-adic fermionic integrals on $\mathbb{Z}_p$. Then we considered p-adic fermionic integrals on $\mathbb{Z}_p$ of the two variable q-Bernstein polynomials and of products of the two variable q-Bernstein polynomials, and showed that they can be expressed in terms of the q-analogues of Euler numbers.

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