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Fuzzy Normed Rings

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Received: 12 September 2018; Accepted: 8 October 2018; Published: 16 October 2018

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Abstract: In this paper, the concept of fuzzy normed ring is introduced and some basic properties related to it are established. Our definition of normed rings on fuzzy sets leads to a new structure, which we call a fuzzy normed ring. We define fuzzy normed ring homomorphism, fuzzy normed subring, fuzzy normed ideal, fuzzy normed prime ideal, and fuzzy normed maximal ideal of a normed ring, respectively. We show some algebraic properties of normed ring theory on fuzzy sets, prove theorems, and give relevant examples.

Keywords: Fuzzy sets; ring; normed space; fuzzy normed ring; fuzzy normed ideal

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1. Introduction

Normed rings attracted attention of researchers after the studies by Naimark [1], a generalization of normed rings [2] and commutative normed rings [3]. Naimark defined normed rings in an algebraic fashion, while Gel'fand addressed them as complex Banach spaces and introduced the notion of commutative normed rings. In Reference [4], Jarden defined the ultrametric absolute value and studied the properties of normed rings in a more topological perspective. During his invaluable studies, Zadeh [5] presented fuzzy logic theory, changing the scientific history forever by making a modern definition of vagueness and using the sets without strict boundaries. As, in almost every aspect of computational science, fuzzy logic also became a convenient tool in classical algebra. Zimmermann [6] made significant contributions to the fuzzy set theory. Mordeson, Bhutani, and Rosenfeld [7] defined fuzzy subgroups, Liu [8], Mukherjee, and Bhattacharya [9] examined normal fuzzy subgroups. Liu [8] also discussed fuzzy subrings and fuzzy ideals. Wang, Ruan and Kerre [10] studied fuzzy subrings and fuzzy rings. Swamy and Swamy [11] defined and proved major theorems on fuzzy prime ideals of rings. Gupta and Qi [12] are concerned with T-norms, T-conorms and T-operators. In this study, we use the definitions of Kolmogorov, Silverman, and Formin [13] on linear spaces and norms. Uluçay, Şahin, and Olgun [14] worked out on normed Z-Modules and also on soft normed rings [15]. Şahin, Olgun, and Uluçay [16] defined normed quotient rings while Şahin and Kargın [17] presented neutrosophic triplet normed space. In Reference [18], Olgun and Şahin investigated fitting ideals of the universal module and while Olgun [19] found a method to solve a problem on universal modules. Şahin and Kargin proposed neutrosophic triplet inner product [20] and Florentin, Şahin, and Kargin introduced neutrosophic triplet G-module [21]. Şahin and et al defined isomorphism theorems for soft G-module in [22]. Fundamental homomorphism theorems for neutrosophic extended triplet groups [23] were introduced by Mehmet, Moges, and Olgun in 2018. In Reference [24], Bal, Moges, and Olgun introduced neutrosophic triplet cosets and quotient groups, and deal with its application areas in neutrosophic logic.

This paper anticipates a normed ring on R and fuzzy rings are defined in the previous studies. Now, we use that norm on fuzzy sets, hence a fuzzy norm is obtained and by defining our fuzzy norm on fuzzy rings, we get fuzzy normed rings in this study. The organization of this paper is as follows. In Section 2, we give preliminaries and fuzzy normed rings. In Section 3, consists of further definitions and relevant theorems on fuzzy normed ideals of a normed ring. Fuzzy normed prime and fuzzy normed maximal ideals of a normed ring are introduced in Section 4. The conclusions are summarized in Section 5.

2. Preliminaries

In this section, definition of normed linear space, normed ring, Archimedean strict T-norm and concepts of fuzzy sets are outlined.

Definition 1. *[13] A functional* $\|\|$ *defined on a linear space L is said to be a norm (in L) if it has the following properties:*

N1: $\|x\| \geq 0$ *for all* $x \in L$*, where* $\|x\| = 0$ *if and only if* $x = 0$;
N2: $\|\alpha \cdot x\| = |\alpha|.\|x\|$*; (and hence* $\|x\| = \|-x\|$*), for all* $x \in L$ *and for all* α*;*
N3: Triangle inequality: $\|x + y\| \leq \|x\| + \|y\|$ *for all* $x, y \in L$.
A linear space L, equipped with a norm is called a normed linear space.

Definition 2. *[3] A ring A is said to be a normed ring if A possesses a norm* $\|\|$*, that is, a non-negative real-valued function* $\|\| : A \to R$ *such that for any* $a, b \in A$*,*

1. $\|a\| = 0 \Leftrightarrow a = 0$,
2. $\|a + b\| \leq \|a\| + \|b\|$,
3. $\|-a\| = \|a\|$*, (and hence* $\|1_A\| = 1 = \|-1\|$ *if identity exists), and*
4. $\|ab\| \leq \|a\|\|b\|$.

Definition 3. *[12] Let* $* : [0,1] \times [0,1] \to [0,1]$. $*$ *is an Archimedean strict T-norm iff for all* $x, y, z \in [0,1]$:

(1) $*$ *is commutative and associative, that is,* $*(x,y) = *(y,x)$ *and* $*(x, *(y,z)) = *(*(x,y), z)$,
(2) $*$ *is continuous,*
(3) $*(x,1) = x$,
(4) $*$ *is monotone, which means* $*(x,y) \leq *(x,z)$ *if* $y \leq z$,
(5) $*(x,x) < x$ *for* $x \in (0,1)$*, and*
(6) when $x < z$ *and* $y < t$, $*(x,y) < *(z,t)$ *for all* $x, y, z, t \in (0,1)$.

For convenience, we use the word t-norm shortly and show it as $x * y$ *instead of* $*(x,y)$*. Some examples of t-norms are* $x * y = \min\{x, y\}$, $x * y = \max\{x + y - 1, 0\}$ *and* $x * y = x.y$.

Definition 4. *[12] Let* $\diamond : [0,1] \times [0,1] \to [0,1]$. $*$ *is an Archimedean strict T-conorm iff for all* $x, y, z \in [0,1]$:

(1) $\diamond$ *is commutative and associative, that is,* $\diamond(x,y) = \diamond(y,x)$ *and* $\diamond(x, \diamond(y,z)) = \diamond(\diamond(x,y), z)$,
(2) $\diamond$ *is continuous,*
(3) $\diamond(x,0) = x$,
(4) $\diamond$ *is monotone, which means* $\diamond(x,y) \leq \diamond(x,z)$ *if* $y \leq z$,
(5) $\diamond(x,x) > x$ *for* $x \in (0,1)$*, and*
(6) when $x < z$ *and* $y < t$, $\diamond(z,t) < \diamond(x,y)$ *for all* $x, y, z, t \in (0,1)$.

For convenience, we use the word s-norm shortly and show it as $x \diamond y$ *instead of* $\diamond(x,y)$*. Some examples of s-norms are* $x \diamond y = \max\{x, y\}$, $x * y = \min\{x + y, 1\}$ *and* $x \diamond y = x + y - x.y$.

Definition 5. *[6] The fuzzy set B on a universal set X* is a set of ordered pairs

$$B = \{(x, \mu_B(x) : x \in X)\}$$

Here, $\mu_B(x)$ is the membership function or membership grade of x in B. For all $x \in X$, we have $0 \leq \mu_B(x) \leq 1$. If $x \notin B$, $\mu_B(x) = 0$, and if x is entirely contained in B, $\mu_B(x) = 1$. The membership grade of x in B is shown as $B(x)$ in the rest of this paper.

Definition 6. *[6] For the fuzzy sets A and B, the membership functions of the intersection, union and complement are defined pointwise as follows respectively:*

$$(A \cap B)(x) = \min\{A(x), B(x)\},$$

$$(A \cup B)(x) = \max\{A(x), B(x)\},$$

$$\overline{A}(x) = 1 - A(x).$$

Definition 7. *[10] Let $(R, +, .)$ be a ring and $F(R)$ be the set of all fuzzy subsets of R. As $A \in F(R)$, $\wedge$ is the fuzzy intersection and $\vee$ is the fuzzy union functions, for all $x, y \in R$, if A satisfies (1) $A(x - y) \geq A(x) \wedge A(y)$ and (2) $A(x.y) \geq A(x) \wedge A(y)$ then A is called a fuzzy subring of R. If A is a subring of R for all $a \in A$, then A is itself a fuzzy ring.*

Definition 8. *[11] A non-empty fuzzy subset A of R is said to be an ideal (in fact a fuzzy ideal) if and only if, for any $x, y \in R$, $A(x - y) \geq A(x) \wedge A(y)$ and $A(x.y) \geq A(x) \vee A(y)$.*

Note: *The fuzzy operations of the fuzzy subsets $A, B \in F(R)$ on the ring R can be extended to the operations below by t-norms and s-norms:*

For all $z \in R$,

$$(A + B)(z) = \underset{x+y=z}{\diamond}(A(x) * B(y));$$

$$(A - B)(z) = \underset{x-y=z}{\diamond}(A(x) * B(y));$$

$$(A.B)(z) = \underset{x.y=z}{\diamond}(A(x) * B(y)).$$

3. Fuzzy Normed Rings and Fuzzy Normed Ideals

In this section, there has been defined the fuzzy normed ring and some basic properties related to it. Throughout the rest of this paper, R is the set of real numbers, R will denote an associative ring with identity, NR is a normed ring and $F(X)$ is the set of all fuzzy subsets of the set X.

Definition 9. *Let $*$ be a continuous t-norm and $\diamond$ a continuous s-norm, NR a normed ring and let A be a fuzzy set. If the fuzzy set $A = \{(x, \mu_A(x)) : x \in NR\}$ over a fuzzy normed ring $F(NR)$ satisfy the following conditions then A is called a fuzzy normed subring of the normed ring $(NR, +, .)$:*

For all $x, y \in NR$,

(i) $A(x - y) \geq A(x) * A(y)$

(ii) $A(x.y) \geq A(x) * A(y)$.

Let 0 be the zero of the normed ring NR. For any fuzzy normed subring A and for all $x \in NR$, we have $A(x) \leq A(0)$, since $A(x - x) \geq A(x) * A(x) \Rightarrow A(0) \geq A(x)$.

Example 1. *Let A fuzzy set and $R = (Z, +, .)$ be the ring of all integers. Define a mapping $f : A \rightarrow F(NR(Z))$ where, for any $a \in A$ and $x \in Z$,*

$$A_f(x) = \begin{cases} 0 \text{ if } x \text{ is odd} \\ \frac{1}{a} \text{ if } x \text{ is even} \end{cases}$$

Corresponding t-norm $(*)$ *and t-conorm* $(\diamond)$ *are defined as* $a * b = min\{a, b\}$, $a \diamond b = max\{a, b\}$; *then,* A *is a fuzzy set as well as a fuzzy normed ring over* $[(Z, +, .), A]$.

Lemma 1. $A \in F(NR)$ *is a fuzzy normed subring of the normed ring* NR *if and only if* $A - A \subseteq A$ *and* $A.A \subseteq A$.

Proof. Let A be a fuzzy normed subring of NR. By [10], it is clear that A is a fuzzy group under addition and so $A - A \subseteq A$. Also for all $z \in NR$,

$$(A.A)(z) = \underset{x.y=z}{\diamond}(A(x) * A(y)) \leq \underset{x.y=z}{\diamond} A(xy) = A(z) \Rightarrow A.A \subseteq A$$

Now we suppose $A - A \subseteq A$ and $A.A \subseteq A$. For all $x, y \in NR$,

$$A(x-y) \geq (A-A)(x-y) = \underset{s-t=x-y}{\diamond}(A(s) * A(t)) \geq A(x) * A(y).$$

Similarly,

$$A(xy) \geq (A.A)(xy) = \underset{st=xy}{\diamond}(A(s) * A(t)) \geq A(x) * A(y).$$

Thus, A is a fuzzy normed subring of NR. □

Lemma 2.

i. *Let* A *be a fuzzy normed subring of the normed ring* NR *and let* $f : NR \to NR\prime$ *be a ring homomorphism. Then,* $f(A)$ *is a fuzzy normed subring of* $NR\prime$.

ii. *Let* $f : NR \to NR\prime$ *be a normed ring homomorphism. If* B *is a fuzzy normed subring of* $NR\prime$, *then* $f^{-1}(B)$ *is a fuzzy normed subring of* NR.

Proof. (i) Take $u, v \in NR\prime$. As f is onto, there exists $x, y \in NR$ such that $f(x) = u$ and $f(y) = v$. So,

$$\begin{aligned}
(f(A))(u) * (f(A))(v) &= \left(\underset{f(x)=u}{\diamond} A(x)\right) * \left(\underset{f(y)=v}{\diamond} A(y)\right) \\
&= \underset{f(x)=u, f(y)=v}{\diamond}(A(x) * A(y)) \\
&\leq \underset{f(x)=u, f(y)=v}{\diamond}(A(x-y)) \text{ (as } A \text{ is a fuzzy normed subring of } NR) \\
&\leq \underset{f(x)-f(y)=u-v}{\diamond}(A(x-y)) \\
&= \underset{f(x-y)=u-v}{\diamond}(A(x-y)) \text{ (since } f \text{ is } a \text{ homomorphism)} \\
&= \underset{f(z)=u-v}{\diamond} A(z) \\
&= (f(A))(u-v).
\end{aligned}$$

Similarly, it is easy to see that

$$(f(A))(u.v) \geq (f(A)(u) * f(A)(v)).$$

Therefore, $f(A)$ is a fuzzy normed subring of $NR\prime$.
(ii) Proof is straightforward and similar to the proof of (i). □

Definition 10. *Let* A_1 *and* A_2 *be two fuzzy normed rings over the normed ring* NR. *Then* A_1 *is a fuzzy normed subring of* A_2 if

$$A_1(x) \leq A_2(x)$$

for all $x \in NR$.

Definition 11. *Let NR be a normed ring, $A \in F(NR)$ and let $A \neq \emptyset$. If for all $x, y \in NR$*

(i) $A(x - y) \geq A(x) * A(y)$ *and*
(ii) $A(x.y) \geq A(y)$ $(A(x.y) \geq A(x))$,

then A is called a fuzzy left (right) normed ideal of NR.

Definition 12. *If the fuzzy set A is both a fuzzy normed right and a fuzzy normed left ideal of NR, then A is called a fuzzy normed ideal of NR; i.e., if for all $x, y \in NR$*

(i) $A(x - y) \geq A(x) * A(y)$ *and*
(ii) $A(x.y) \geq A(x) \diamond A(y)$,

then $A \in F(NR)$ is a fuzzy normed ideal of NR.

Remark 1. *Let the multiplicative identity of NR (if exists) be 1_{NR}. As $A(x.y) \geq A(x) \diamond A(y)$ for all $x, y \in NR$, $A(x.1_{NR}) \geq A(x) \diamond A(1_{NR})$ and therefore for all $x \in NR$, $A(x) \geq A(1_{NR})$.*

Example 2. *Let A and B be two (fuzzy normed left, fuzzy normed right) ideals of a normed ring NR. Then, $A \cap B$ is also a (fuzzy normed left, fuzzy normed right) ideal of NR.*

Solution: Let $x, y \in NR$.

$$\begin{aligned} (A \cap B)(x - y) &= \min\{A(x - y), B(x - y)\} \\ &\geq \min\{A(x) * A(y), B(x) * B(y)\} \\ &\geq \min\{(A \cap B)(x), (A \cap B)(y)\}. \end{aligned}$$

On the other hand, as A and B are fuzzy normed left ideals, using $A(x.y) \geq A(y)$ and $B(x.y) \geq B(y)$ we have

$$(A \cap B)(x.y) = \min\{A(x.y), B(x.y)\} \geq \min\{A(y), B(y)\} = (A \cap B)(y).$$

So $A \cap B$ is a fuzzy normed left ideal. Similarly, it is easy to show that $A \cap B$ is a fuzzy normed right ideal. As a result $A \cap B$ is an fuzzy normed ideal of NR.

Example 3. *Let A be a fuzzy ideal of NR. The subring $A^0 = \{x : \mu_A(x) = \mu_A(0_{NR})\}$ is a fuzzy normed ideal of NR, since for all $x \in NR$, $A^0(x) \leq A^0(0)$.*

Theorem 1. *Let A be a fuzzy normed ideal of NR, $X = \{a_1, a_2, \ldots, a_m\} \subseteq NR$, $x, y \in NR$ and let $FN(X)$ be the fuzzy normed ideal generated by the set X in NR.* Then,

(i) $w \in FN(X) \Rightarrow A(w) \geq \underset{1 \leq i \leq m}{*} (A(a_i))$,
(ii) $x \in (y) \Rightarrow A(x) \geq A(y)$,
(iii) $A(0) \geq A(x)$ *and*
(iv) *if 1 is the multiplicative identity of NR, then $A(x) \geq A(1)$.*

Proof. (ii), (iii), and (iv) can be proved using (i). The set $FN(X)$ consists of the finite sums in the form $ra + as + uav + na$ where $a \in X$, $r, s, u, v \in NR$ and n is an integer. Let $w \in FN(X)$. So there exists an integer n and $r, s, u, v \in NR$ such that $w = ra_i + a_is + ua_iv + na_i$ where $1 \leq i \leq m$. As A is a fuzzy normed ideal,

$$A(ra_i + a_is + ua_iv + na_i) \geq A(ra_i) * A(a_is) * A(ua_iv) * A(na_i) \geq A(a_i).$$

Therefore

$$A(w) \geq \underset{1 \leq i \leq m}{*} (A(a_i)).$$

□

4. Fuzzy Normed Prime Ideal and Fuzzy Normed Maximal Ideal

In this section, fuzzy normed prime ideal and fuzzy normed maximal ideal are outlined.

Definition 13. *Let A and B be two fuzzy subsets of the normed ring NR. We define the operation $A \circ B$* as follows:

$$A \circ B(x) = \begin{cases} \underset{x=yz}{\diamond}(A(y) * B(z)) & \text{, if } x \text{ can be defined as } x = yz \\ 0 & \text{, otherwise .} \end{cases}$$

If the normed ring NR has a multiplicative inverse, namely if $NR.NR = NR$, then the second case does not occur.

Lemma 3. *If A and B are a fuzzy normed right and a fuzzy normed left ideal of a normed ring NR, respectively, $A \circ B \subseteq A \cap B$ and hence $(A \circ B)(x) \leq (A \cap B)(x)$ for all $x \in NR$.*

Proof. It is shown in Example 2 that if A and B are fuzzy normed left ideals of NR, then $A \cap B$ is also a fuzzy normed left ideal. Now, let A and B be a fuzzy normed right and a fuzzy normed left ideal of NR, respectively. If $A \circ B(x) = 0$, the proof is trivial.

Let

$$(A \circ B)(x) = \underset{x=yz}{\diamond}(A(y) * B(z)).$$

As A is a fuzzy normed right ideal and B is a fuzzy normed left ideal, we have

$$A(y) \leq A(yz) = A(x)$$

and

$$B(y) \leq B(yz) = B(x)$$

Thus,

$$\begin{aligned} (A \circ B)(x) &= \underset{x=yz}{\diamond}(A(y) * B(z)) \\ &\leq \min(A(x), B(x)) \\ &= (A \cap B)(x) \end{aligned}$$

□

Definition 14. *Let A and B be fuzzy normed ideals of a normed ring NR and let FNP be a non-constant function, which is not an ideal of NR. If*

$$A \circ B \subseteq FNP \;\Rightarrow\; A \subseteq FNP \text{ or } B \subseteq FNP,$$

then FNP is called a fuzzy normed prime ideal of NR.

Example 4. *Show that if the fuzzy normed ideal I ($I \neq NR$) is a fuzzy normed prime ideal of NR, then the characteristic function λ_I* is also a fuzzy normed prime ideal.

Solution: As $I \neq NR$, λ_I is a non-constant function on NR. Let A and B be two fuzzy normed ideals on NR such that $A \circ B \subseteq \lambda_I$, but $A\lambda_I$ and $B\lambda_I$. There exists $x, y \in NR$ such that $A(x) \leq \lambda_I(x)$ and $B(y) \leq \lambda_I(y)$. In this case, $A(x) \neq 0$ and $B(y) \neq 0$, but $\lambda_I(x) = 0$ and $\lambda_I(y) = 0$. Therefore

$x \notin I, y \notin I$. As I is a fuzzy normed prime ideal, there exists an $r \in NR$, such that $xry \notin I$. This is obvious, because if I is fuzzy normed prime, $A \circ B(xry) \subseteq I \Rightarrow A(x) \subseteq I$ or $B(ry) \subseteq I$ and therefore as $(NRxNR)(\underline{NR}ryNR) = (NRxNR)(NRyNR) \subseteq I$, we have either $NRxNR \subseteq I$ or $NRyNR \subseteq I$. Assume $NRxNR \subseteq I$. Then $xxx = (x)^3 \in I \Rightarrow x \subseteq I$, but this contradicts with the fact that $\lambda_I(x) = 0$. Now let $a = xry$. $\lambda_I(a) = 0$. Thus, $A \circ B(a) = 0$. On the other hand,

$$\begin{aligned} A \circ B(a) &= \underset{a=cd}{\diamond}(A(c) * B(d)) \\ &\geq A(x) * B(ry) \\ &\geq A(x) * B(y) \\ &\geq 0 \text{ (as } A(x) \neq 0 \text{ and } B(y) \neq 0). \end{aligned}$$

This is a contradiction, since $A \circ B(a) = 0$. Therefore if A and B are fuzzy normed ideals of a normed ring NR, then $A \circ B \subseteq \lambda_I \Rightarrow A \subseteq \lambda_I$ or $B \subseteq \lambda_I$. As a result, the characteristic function λ_I is a fuzzy normed prime ideal.

Theorem 2. *Let FNP be a fuzzy normed prime ideal of a normed ring NR. The ideal defined by $FNP^0 = \{x : x \in NR, FNP(x) = FNP(0)\}$ and is also a fuzzy normed prime ideal of NR.*

Proof: Let $x, y \in FNP^0$. As FNP is an fuzzy normed ideal, $FNP(x - y) \geq FNP(x) * FNP(y) = FNP(0)$. On the other hand, by Theorem 1, we have $FNP(0) \geq FNP(x - y)$. So, $FNP(x - y) = FNP(0)$ and $x - y \in FNP^0$. Now, let $x \in FNP^0$ and $r \in NR$. In this case, $FNP(rx) \geq FNP(x) = FNP(0)$ and thus $FNP(rx) = FNP(0)$. Similarly, $FNP(xr) = FNP(0)$. Now, for all $x \in FNP^0$ and $r \in NR$, $rx, xr \in FNP^0$. Therefore, FNP^0 is a fuzzy normed ideal of NR. Let I and J be two ideals of NR, such that $IJ \subseteq FNP^0$. Now, we define fuzzy normed ideals $A = FNP^0\lambda_I$ and $B = FNP^0\lambda_J$. We will show that $(A \circ B)(x) \leq FNP(x)$ for all $x \in NR$. Assume $(A \circ B)(x) \neq 0$. Recall $A \circ B = \underset{x=yz}{\diamond}(A(y) * B(z))$, so we only need to take the cases of $A(y) * B(z) \neq 0$ under consideration. However, in all these cases, $A(y) = FNP(0)$ or $A(y) = 0$ and similarly $B(z) = FNP(0)$ or $B(z) = 0$ and hence $A(y) = B(z) = FNP(0)$. Now, $\lambda_I(y) = 1$ and $\lambda_J(z) = 1$ implies $y \in I$, $z \in J$ and $x \in IJ \subseteq FNP^0$. Thus, $FNP(x) = FNP(0)$ and for all $x \in NR$, we get $(A \circ B)(x) \leq FNP(x)$. As FNP is a fuzzy normed prime ideal and A and B are fuzzy normed ideals, either $A \subseteq FNP$ or $B \subseteq FNP$. Assume $A = FNP^0\lambda_I \subseteq FNP$. We need to show that $I \subseteq FNP^0$. Let $IFNP$. Then, there exists an $a \in I$, such that $a \notin FNP^0$; i.e., $FNP(a) \neq FNP(0)$. It is evident that $FNP(0) \geq FNP(a)$. Thus, $FNP(a) < FNP(0)$. However, $A(a) = FNP^0\lambda_I(a) = FNP(0) > FNP(a)$ and this is a contradiction to the assumption $A \subseteq FNP$. So, $I \subseteq FNP^0$. Similarly, one can show that $B \subseteq FNP$ and $J \subseteq FNP^0$. Thus, FNP^0 is a fuzzy normed prime ideal. □

Definition 15. *Let A be a fuzzy normed ideal of a normed ring NR. If A is non-constant and for all fuzzy normed ideals B of NR, $A \subseteq B$ implies $A^0 = B^0$ or $B = \lambda_{NR}$, A is called a fuzzy normed maximal ideal of the normed ring NR.* Fuzzy normed maximal left(right) ideals are defined similarly.

Example 5. *Let A be a fuzzy normed maximal left (right) ideal of a normed ring NR. Then, $A^0 = \{x \in NR : A(x) = A(0)\}$ is a fuzzy normed maximal left (right) ideal of NR.*

Theorem 3. *If A is a fuzzy normed left(right) maximal ideal of a normed ring NR, then $A(0) = 1$.*

Proof. Assume $A(0) \neq 1$. Let $A(0) < t < 1$ and let B be a fuzzy subset of NR such that $B(x) = t$ for all $x \in NR$. B is trivially an ideal of NR. Also it is easy to verify that $A \subset B$, $B \neq \lambda_{NR}$ and $B^0 = \{x \in NR : B(x) = B(0)\} = NR$. But, despite the fact that $A \subset B$, $A^0 \neq B^0$ and $B \neq \lambda_{NR}$ is a contradiction to the fuzzy normed maximality of A. Thus, $A(0) = 1$. □

5. Conclusions

In this paper, we defined a fuzzy normed ring. Here we examine the algebraic properties of fuzzy sets in ring structures. Some related notions, e.g., the fuzzy normed ring homomorphism, fuzzy normed subring, fuzzy normed ideal, fuzzy normed prime ideal and fuzzy normed maximal ideal are proposed. We hope that this new concept will bring a new opportunity in research and development of fuzzy set theory. To extend our work, further research can be done to study the properties of fuzzy normed rings in other algebraic structures such as fuzzy rings and fuzzy fields.

Author Contributions: All authors contributed equally.

Acknowledgments: We thank Vakkas Uluçay for the arrangement.

Conflicts of Interest: The authors declare no conflict of interest.

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