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Q-Filters of Quantum B-Algebras and Basic Implication Algebras

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Abstract: The concept of quantum B-algebra was introduced by Rump and Yang, that is, unified algebraic semantics for various noncommutative fuzzy logics, quantum logics, and implication logics. In this paper, a new notion of q-filter in quantum B-algebra is proposed, and quotient structures are constructed by q-filters (in contrast, although the notion of filter in quantum B-algebra has been defined before this paper, but corresponding quotient structures cannot be constructed according to the usual methods). Moreover, a new, more general, implication algebra is proposed, which is called basic implication algebra and can be regarded as a unified frame of general fuzzy logics, including nonassociative fuzzy logics (in contrast, quantum B-algebra is not applied to nonassociative fuzzy logics). The filter theory of basic implication algebras is also established.

Keywords: fuzzy implication; quantum B-algebra; q-filter; quotient algebra; basic implication algebra

1. Introduction

For classical logic and nonclassical logics (multivalued logic, quantum logic, t-norm-based fuzzy logic [1–6]), logical implication operators play an important role. In the study of fuzzy logics, fuzzy implications are also the focus of research, and a large number of literatures involve this topic [7–16]. Moreover, some algebraic systems focusing on implication operators are also hot topics. Especially with the in-depth study of noncommutative fuzzy logics in recent years, some related implication algebraic systems have attracted the attention of scholars, such as pseudo-basic-logic (BL) algebras, pseudo- monoidal t-norm-based logic (MTL) algebras, and pseudo- B, C, K axiom (BCK)/ B, C, I axiom (BCI) algebras [17–23] (see also References [5–7]).

For formalizing the implication fragment of the logic of quantales, Rump and Yang proposed the notion of quantum B-algebras [24,25], which provide a unified semantic for a wide class of nonclassical logics. Specifically, quantum B-algebras encompass many implication algebras, like pseudo-BCK/BCI algebras, (commutative and noncommutative) residuated lattices, pseudo- MV/BL/MTL algebras, and generalized pseudo-effect algebras. New research articles on quantum B-algebras can be found in References [26–28]. Note that all hoops and pseudo-hoops are special quantum B-algebras, and they are closely related to L-algebras [29].

Although the definition of a filter in a quantum B-algebra is given in Reference [30], quotient algebraic structures are not established by using filters. In fact, filters in special subclasses of quantum B-algebras are mainly discussed in Reference [30], and these subclasses require a unital element. In this paper, by introducing the concept of a q-filter in quantum B-algebras, we establish the quotient structures using q-filters in a natural way. At the same time, although quantum B-algebra has generality, it cannot include the implication structure of non-associative fuzzy logics [31,32], so we propose a wider concept, that is, basic implication algebra that can include a wider range of implication operations, establish filter theory, and obtain quotient algebra.

2. Preliminaries

Definition 1. *Let $(X, \leq)$ be partially ordered set endowed with two binary operations $\rightarrow$ and $\rightsquigarrow$ [24,25]. Then, $(X, \rightarrow, \rightsquigarrow, \leq)$ is called a quantum B-algebra if it satisfies: $\forall x, y, z \in X$,*

(1) $y \rightarrow z \leq (x \rightarrow y) \rightarrow (x \rightarrow z)$;
(2) $y \rightsquigarrow z \leq (x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow z)$;
(3) $y \leq z \Rightarrow x \rightarrow y \leq x \rightarrow z$;
(4) $x \leq y \rightarrow z \Longleftrightarrow y \leq x \rightsquigarrow z$.

If $u \in X$ exists, such that $u \rightarrow x = u \rightsquigarrow x = x$ for all x in X, then u is called a unit element of X. Obviously, the unit element is unique. When a unit element exists in X, we call X unital.

Proposition 1. *An algebra structure $(X, \rightarrow, \rightsquigarrow, \leq)$ endowed with a partially order $\leq$ and two binary operations $\rightarrow$ and $\rightsquigarrow$ is a quantum B-algebra if and only if it satisfies [4]: $\forall x, y, z \in X$,*

(1) $x \rightarrow (y \rightsquigarrow z) = y \rightsquigarrow (x \rightarrow z)$;
(2) $y \leq z \Rightarrow x \rightarrow y \leq x \rightarrow z$;
(3) $x \leq y \rightarrow z \Longleftrightarrow y \leq x \rightsquigarrow z$.

Proposition 2. *Let $(X, \rightarrow, \rightsquigarrow, \leq)$ be a quantum B-algebra [24–26]. Then, $(\forall\, x, y, z \in X)$*

(1) $y \leq z \Rightarrow x \rightsquigarrow y \leq x \rightsquigarrow z$;
(2) $y \leq z \Rightarrow z \rightsquigarrow x \leq y \rightsquigarrow x$;
(3) $y \leq z \Rightarrow z \rightarrow x \leq y \rightarrow x$;
(4) $x \leq (x \rightsquigarrow y) \rightarrow y$, $x \leq (x \rightarrow y) \rightsquigarrow y$;
(5) $x \rightarrow y = ((x \rightarrow y) \rightsquigarrow y) \rightarrow y$, $x \rightsquigarrow y = ((x \rightsquigarrow y) \rightarrow y) \rightsquigarrow y$;
(6) $x \rightarrow y \leq (y \rightarrow z) \rightsquigarrow (x \rightarrow z)$;
(7) $x \rightsquigarrow y \leq (y \rightsquigarrow z) \rightarrow (x \rightsquigarrow z)$;
(8) *assume that u is the unit of X, then* $u \leq x \rightsquigarrow y \Longleftrightarrow x \leq y \Longleftrightarrow u \leq x \rightarrow y$;
(9) *if $0 \in X$ exists, such that $0 \leq x$ for all x in X, then $0 = 0 \rightsquigarrow 0 = 0 \rightarrow 0$ is the greatest element (denote by 1), and $x \rightarrow 1 = x \rightsquigarrow 1 = 1$ for all $x \in X$;*
(10) *if X is a lattice, then* $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$, $(x \vee y) \rightsquigarrow z = (x \rightsquigarrow z) \vee (y \rightsquigarrow z)$.

Definition 2. *Let $(X, \leq)$ be partially ordered set and $Y \subseteq X$ [24]. If $x \geq y \in Y$ implies $x \in Y$, then Y is called to be an upper set of X. The smallest upper set containing a given $x \in X$ is denoted by $\uparrow x$. For quantum B-algebra X, the set of upper sets is denoted by $U(X)$. For $A, B \in U(X)$, define*

$$A \cdot B = \{x \in X \mid \exists b \in B: b \rightarrow x \in A\}.$$

We can verify that $A \cdot B = \{x \in X \mid \exists a \in A: a \rightsquigarrow x \in B\} = \{x \in X \mid \exists a \in A, b \in B: a \leq b \rightarrow x\} = \{x \in X \mid \exists a \in A, b \in B: b \leq a \rightsquigarrow x\}$.

Definition 3. *Let A be an empty set, $\leq$ be a binary relation on A [17,18], $\rightarrow$ and $\rightsquigarrow$ be binary operations on A, and 1 be an element of A. Then, structure $(A, \rightarrow, \rightsquigarrow, \leq, 1)$ is called a pseudo-BCI algebra if it satisfies the following axioms: $\forall\, x, y, z \in A$,*

(1) $x \rightarrow y \leq (y \rightarrow z) \rightsquigarrow (x \rightarrow z)$, $x \rightsquigarrow y \leq (y \rightsquigarrow z) \rightarrow (x \rightsquigarrow z)$,

(2) $x \le (x \rightsquigarrow y) \to y$, $x \le (x \to y) \rightsquigarrow y$;
(3) $x \le x$;
(4) $x \le y, y \le x \Rightarrow x = y$;
(5) $x \le y \Longleftrightarrow x \to y = 1 \Longleftrightarrow x \rightsquigarrow y = 1$.

If pseudo-BCI algebra A satisfies: $x \to 1 = 1$ *(or* $x \rightsquigarrow 1 = 1$*) for all* $x \in A$*, then A is called a pseudo-BCK algebra.*

Proposition 3. *Let* $(A, \to, \rightsquigarrow, \le, 1)$ *be a pseudo-BCI algebra [18–20]. We have* $(\forall x, y, z \in A)$

(1) *if* $1 \le x$*, then* $x = 1$;
(2) *if* $x \le y$*, then* $y \to z \le x \to z$ *and* $y \rightsquigarrow z \le x \rightsquigarrow z$;
(3) *if* $x \le y$ *and* $y \le z$*, then* $x \le z$;
(4) $x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z)$;
(5) $x \le y \to z \Longleftrightarrow y \le x \rightsquigarrow z$;
(6) $x \to y \le (z \to x) \to (z \to y)$; $x \rightsquigarrow y \le (z \rightsquigarrow x) \rightsquigarrow (z \rightsquigarrow y)$;
(7) *if* $x \le y$*, then* $z \to x \le z \to y$ *and* $z \rightsquigarrow x \le z \rightsquigarrow y$;
(8) $1 \to x = 1 \rightsquigarrow x = x$;
(9) $y \to x = ((y \to x) \rightsquigarrow x) \to x$, $y \rightsquigarrow x = ((y \rightsquigarrow x) \to x) \rightsquigarrow x$;
(10) $x \to y \le (y \to x) \rightsquigarrow 1$, $x \rightsquigarrow y \le (y \rightsquigarrow x) \to 1$;
(11) $(x \to y) \to 1 = (x \to 1) \rightsquigarrow (y \to 1)$, $(x \rightsquigarrow y) \rightsquigarrow 1 = (x \rightsquigarrow 1) \to (y \rightsquigarrow 1)$;
(12) $x \to 1 = x \rightsquigarrow 1$.

Proposition 4. *Let* $(A, \to, \rightsquigarrow, \le, 1)$ *be a pseudo-BCK algebra* [17]*, then* $(\forall x, y \in A)$: $x \le y \to x$, $x \le y \rightsquigarrow x$.

Definition 4. *Let X be a unital quantum B-algebra [24]. If there exists* $x \in X$*, such that* $x \to u = x \rightsquigarrow u = u$*, then we call that x integral. The subset of integral element in X is denoted by I(X).*

Proposition 5. *Let X be a quantum B-algebra [24]. Then, the following assertions are equivalent:*

(1) *X is a pseudo-BCK algebra;*
(2) *X is unital, and every element of X is integral;*
(3) *X has the greatest element, which is a unit element.*

Proposition 6. *Every pseudo-BCI algebra is a unital quantum B-algebra [25]. And, a quantum B-algebra is a pseudo-BCI algebra if and only if its unit element u is maximal.*

Definition 5. *Let* $(A, \to, \rightsquigarrow, \le, 1)$ *be a pseudo-BCI algebra [20,21]. When the following identities are satisfied, we call X an antigrouped pseudo-BCI algebra:*

$$\forall x \in A,\ (x \to 1) \to 1 = x \text{ or } (x \rightsquigarrow 1) \rightsquigarrow 1 = x.$$

Proposition 7. *Let* $(A, \to, \rightsquigarrow, \le, 1)$ *be a pseudo-BCI algebra [20]. Then, A is antigrouped if and only if the following conditions are satisfied:*

(G1) *for all* $x, y, z \in A$, $(x \to y) \to (x \to z) = y \to z$*, and*
(G2) *for all* $x, y, z \in A$, $(x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow z) = y \rightsquigarrow z$.

Definition 6. *Let* $(A, \to, \rightsquigarrow, \le, 1)$ *be a pseudo-BCI algebra and* $F \subseteq X$ *[19,20]. When the following conditions are satisfied, we call F a pseudo-BCI filter (briefly, filter) of X:*

(F1) $1 \in F$;

(F2) $x \in F$, $x \to y \in F \Longrightarrow y \in F$;

(F3) $x \in F$, $x \rightsquigarrow y \in F \Longrightarrow y \in F$.

Definition 7. *Let $(A, \to, \rightsquigarrow, \leq, 1)$ be a pseudo-BCI algebra and F be a filter of X [20,21]. When the following condition is satisfied, we call F an antigrouped filter of X:*

$$(GF)\ \forall x \in X,\ (x \to 1) \to 1 \in F \text{ or } (x \rightsquigarrow 1) \rightsquigarrow 1 \in F \Longrightarrow x \in F.$$

Definition 8. *A subset F of pseudo-BCI algebra X is called a p-filter of X if the following conditions are satisfied [20,21]:*

(P1) $1 \in F$,

(P2) $(x \to y) \rightsquigarrow (x \to z) \in F$ *and* $y \in F$ *imply* $z \in F$,

(P3) $(x \rightsquigarrow y) \to (x \rightsquigarrow z) \in F$ *and* $y \in F$ *imply* $z \in F$.

3. Q-Filters in Quantum B-Algebra

In Reference [30], the notion of filter in quantum B-algebra is proposed. If X is a quantum B-algebra and F is a nonempty set of X, then F is called the filter of X if $F \in U(X)$ and $F \cdot F \subseteq F$. That is, F is a filter of X, if and only if: (1) F is a nonempty upper subset of X; (2) $(z \in X, y \in F, y \to z \in F) \Rightarrow z \in F$. We denote the set of all filters of X by $F(X)$.

In this section, we discuss a new concept of q-filter in quantum B-algebra; by using q-filters, we construct the quotient algebras.

Definition 9. *A nonempty subset F of quantum B-algebra X is called a q-filter of X if it satisfies:*

(1) *F is an upper set of X, that is,* $F \in U(X)$;

(2) *for all* $x \in F$, $x \to x \in F$ *and* $x \rightsquigarrow x \in F$;

(3) $x \in F$, $y \in X$, $x \to y \in F \Longrightarrow y \in F$.

(4) *A q-filter of X is normal if* $x \to y \in F \Longleftrightarrow x \rightsquigarrow y \in F$.

Proposition 8. *Let F be a q-filter of quantum B-algebra X. Then,*

(1) $x \in F$, $y \in X$, $x \rightsquigarrow y \in F \Longrightarrow y \in F$.

(2) $x \in F$ *and* $y \in X \Longrightarrow (x \rightsquigarrow y) \to y \in F$ *and* $(x \to y) \rightsquigarrow y \in F$.

(3) *if X is unital, then Condition (2) in Definition 9 can be replaced by* $u \in F$, *where u is the unit element of X.*

Proof. (1) Assume that $x \in F$, $y \in X$, and $x \rightsquigarrow y \in F$. Then, by Proposition 2 (4), $x \leq (x \rightsquigarrow y) \to y$. Applying Definition 9 (1) and (3), we get that $y \in F$.

(2) Using Proposition 2 (4) and Definition 9 (1), we can get (2).

(3) If X is unital with unit u, then $u \to u = u$. Moreover, applying Proposition 2 (8), $u \leq x \rightsquigarrow x$ and $u \leq x \to x$ from $x \leq x$, for all $x \in X$. Therefore, for unital quantum B-algebra X, Condition (2) in Definition 8 can be replaced by condition "$u \in F$". □

By Definition 6, and Propositions 6 and 8, we get the following result (the proof is omitted).

Proposition 9. *Let $(A, \to, \rightsquigarrow, \leq, 1)$ be a pseudo-BCI algebra. Then, an empty subset of A is a q-filter of A (as a quantum B-algebra) if and only if it is a filter of A (according to Definition 6).*

Example 1. *Let X = {a, b, c, d, e, f}. Define operations $\to$ and $\rightsquigarrow$ on X as per the following Cayley Tables 1 and 2; the order on X is defined as follows: $b \leq a \leq f$; $e \leq d \leq c$. Then, X is a quantum B-algebra (we can verify* *it with the Matlab software (The MathWorks Inc., Natick, MA, USA)), but it is not a pseudo-BCI algebra. Let* $F_1 = \{f\}$, $F_2 = \{a, b, f\}$; *then,* F_1 *is a filter but not a q-filter of X, and* F_2 *is a normal q-filter of X.*

Table 1. Cayley table of operation $\rightarrow$.

$\rightarrow$	**a**	**b**	**c**	**d**	**e**	**f**
a	f	a	c	c	c	f
b	f	a	c	c	c	f
c	c	c	f	a	b	c
d	c	c	f	f	a	c
e	c	c	f	f	a	c
f	a	b	c	d	e	f

Table 2. Cayley table of operation $\rightsquigarrow$.

$\rightsquigarrow$	**a**	**b**	**c**	**d**	**e**	**f**
a	f	a	c	c	d	f
b	f	f	c	c	c	f
c	c	c	f	a	a	c
d	c	c	f	a	a	c
e	c	c	f	f	a	c
f	a	b	c	d	e	f

Theorem 1. *Let X be a quantum B-algebra and F a normal q-filter of X. Define the binary* $\approx_F$ *on X as follows:*

$$x \approx_F y \Longleftrightarrow x \rightarrow y \in F \text{ and } y \rightarrow x \in F, \text{ where } x, y \in X.$$

Then,

(1) $\approx_F$ *is an equivalent relation on X;*
(2) $\approx_F$ *is a congruence relation on X, that is,* $x \approx_F y \Longrightarrow (z \rightarrow x) \approx_F (z \rightarrow y)$, $(x \rightarrow z) \approx_F (y \rightarrow z)$, $(z \rightsquigarrow x) \approx_F (z \rightsquigarrow y)$, $(x \rightsquigarrow z) \approx_F (y \rightsquigarrow z)$, *for all* $z \in X$.

Proof. (1) For any $x \in X$, by Definition 9 (2), $x \rightarrow x \in F$, it follows that $x \approx_F x$.

For all $x, y \in X$, if $x \approx_F y$, we can easily verify that $y \approx_F x$.

Assume that $x \approx_F y$, $y \approx_F z$. Then, $x \rightarrow y \in F$, $y \rightarrow x \in F$, $y \rightarrow z \in F$, and $z \rightarrow y \in F$, since

$$y \rightarrow z \leq (x \rightarrow y) \rightarrow (x \rightarrow z) \text{ by Definition 1 (1).}$$

From this and Definition 9, we have $x \rightarrow z \in F$. Similarly, we can get $z \rightarrow x \in F$. Thus, $x \approx_F z$.
Therefore, $\approx_F$ is an equivalent relation on X.
(2) If $x \approx_F y$, then $x \rightarrow y \in F$, $y \rightarrow x \in F$. Since

$$x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightarrow y), \text{ by Definition 1 (1).}$$

$$y \rightarrow x \leq (z \rightarrow y) \rightarrow (z \rightarrow x), \text{ by Definition 1 (1).}$$

Using Definition 9 (1), $(z \rightarrow x) \rightarrow (z \rightarrow y) \in F$, $(z \rightarrow y) \rightarrow (z \rightarrow x) \in F$. It follows that $(z \rightarrow x) \approx_F (z \rightarrow y)$.
Moreover, since

$$x \rightarrow y \leq (y \rightarrow z) \rightsquigarrow (x \rightarrow z), \text{ by Proposition 2 (6).}$$

$$y \rightarrow x \leq (x \rightarrow z) \rightsquigarrow (y \rightarrow z), \text{ by Proposition 2 (6).}$$

Then, form $x{\rightarrow}y{\in}F$ and $y{\rightarrow}x{\in}F$; using Definition 9 (1), we have $(y{\rightarrow}z){\rightsquigarrow}(x{\rightarrow}z){\in}F$, $(x{\rightarrow}z){\rightsquigarrow}(y{\rightarrow}z){\in}F$. Since F is normal, by Definition 9 we get $(y{\rightarrow}z){\rightarrow}(x{\rightarrow}z){\in}F$, $(x{\rightarrow}z){\rightarrow}(y{\rightarrow}z){\in}F$. Thus, $(x{\rightarrow}z) \approx_F (y{\rightarrow}z)$.

Similarly, we can get that $x \approx_F y \Longrightarrow (z{\rightsquigarrow}x) \approx_F (z{\rightsquigarrow}y)$ and $(x{\rightsquigarrow}z) \approx_F (y{\rightsquigarrow}z)$. □

Definition 10. *A quantum B-algebra X is considered to be perfect, if it satisfies:*

(1) *for any normal q-filter F of X, x, y in X, (there exists an$\in$X, such that $[x{\rightarrow}y]_F = [a{\rightarrow}a]_F$) $\Longleftrightarrow$ (there exists $b{\in}X$, such that $[x{\rightsquigarrow}y]_F = [b{\rightsquigarrow}b]_F$).*

(1) *for any normal q-filter F of X, $(X/{\approx_F} \rightarrow, \rightsquigarrow, \leq)$ is a quantum B-algebra, where quotient operations $\rightarrow$ and $\rightsquigarrow$ are defined in a canonical way, and $\leq$ is defined as follows:*

$$[x]_F \leq [y]_F \Longleftrightarrow (\text{there exists } a{\in}X \text{ such that } [x]_F{\rightarrow}[y]_F = [a{\rightarrow}a]_F)$$
$$\Longleftrightarrow (\text{there exists } b{\in}X \text{ such that } [x]_F{\rightsquigarrow}[y]_F = [b{\rightsquigarrow}b]_F).$$

Theorem 2. *Let $(A, \rightarrow, \rightsquigarrow, \leq, 1)$ be a pseudo-BCI algebra, then A is a perfect quantum B-algebra.*

Proof. By Proposition 6, we know that A is a quantum B-algebra.

(1) For any normal q-filter F of A, x, $y{\in}A$, if there exists $a{\in}A$, such that $[x{\rightarrow}y]_F = [a{\rightarrow}a]_F$, then

$$[x{\rightarrow}y]_F = [a{\rightarrow}a]_F = [1]_F.$$

It follows that $(x{\rightarrow}y){\rightarrow}1{\in}F$, $1{\rightarrow}(x{\rightarrow}y) = x{\rightarrow}y{\in}F$. Applying Proposition 3 (11) and (12), we have

$$(x{\rightarrow}1){\rightsquigarrow}(y{\rightarrow}1) = (x{\rightarrow}y){\rightarrow}1{\in}F.$$

Since F is normal, from $(x{\rightarrow}1){\rightsquigarrow}(y{\rightarrow}1){\in}F$ and $x{\rightarrow}y{\in}F$ we get that

$$(x{\rightarrow}1){\rightarrow}(y{\rightarrow}1){\in}F \text{ and } x{\rightsquigarrow}y{\in}F.$$

Applying Proposition 3 (11) and (12) again, $(x{\rightsquigarrow}y){\rightarrow}1 = (x{\rightarrow}1){\rightarrow}(y{\rightarrow}1)$. Thus,

$$(x{\rightsquigarrow}y){\rightarrow}1 = (x{\rightarrow}1){\rightarrow}(y{\rightarrow}1){\in}F \text{ and } 1{\rightarrow}(x{\rightsquigarrow}y) = x{\rightsquigarrow}y{\in}F.$$

This means that $[x{\rightsquigarrow}y]_F = [1]_F = [1{\rightsquigarrow}1]_F$. Similarly, we can prove that the inverse is true. That is, Definition 10 (1) holds for A.

(2) For any normal q-filter F of pseudo-BCI algebra A, binary $\leq$ on $A/{\approx_F}$ is defined as the following:

$$[x]_F \leq [y]_F \Longleftrightarrow [x]_F{\rightarrow}[y]_F = [1]_F.$$

We verify that $\leq$ is a partial binary on $A/{\approx_F}$.

Obviously, $[x]_F \leq [x]_F$ for any $x{\in}A$.

If $[x]_F \leq [y]_F$ and $[y]_F \leq [x]_F$, then $[x]_F{\rightarrow}[y]_F = [x{\rightarrow}y]_F = [1]_F$, $[y]_F{\rightarrow}[x]_F = [y{\rightarrow}x]_F = [1]_F$. By the definition of equivalent class, $x{\rightarrow}y = 1{\rightarrow}(x{\rightarrow}y) \in F$, $y{\rightarrow}x = 1{\rightarrow}(y{\rightarrow}x){\in}F$. It follows that $x \approx_F y$; thus, $[x]_F = [y]_F$.

If $[x]_F \leq [y]_F$ and $[y]_F \leq [z]_F$, then $[x]_F{\rightarrow}[y]_F = [x{\rightarrow}y]_F = [1]_F$, $[y]_F{\rightarrow}[z]_F = [y{\rightarrow}z]_F = [1]_F$. Thus,

$$x{\rightarrow}y = 1{\rightarrow}(x{\rightarrow}y){\in}F,\ (x{\rightarrow}y){\rightarrow}1{\in}F;$$

$$y{\rightarrow}z = 1{\rightarrow}(y{\rightarrow}z){\in}F,\ (y{\rightarrow}z){\rightarrow}1{\in}F.$$

Applying Definition 3 and Proposition 3,

$$y{\rightarrow}z \leq (x{\rightarrow}y){\rightarrow}(x{\rightarrow}z),$$

$$(x{\to}y){\to}1 = (x{\to}1){\rightsquigarrow}(y{\to}1) \leq ([(y{\to}1){\rightsquigarrow}(z{\to}1)]{\to}[(x{\to}1){\rightsquigarrow}(z{\to}1)]) = [(y{\to}z){\to}1]{\to}[(x{\to}z){\to}1].$$

By Definition 9,

$$1{\to}(x{\to}z) = x{\to}z{\in}F,\ (x{\to}z){\to}1{\in}F.$$

This means that $(x{\to}z) \approx_F 1$, $[x{\to}z]_F = [1]_F$. That is, $[x]_F{\to}[z]_F = [x{\to}z]_F = [1]_F$, $[x]_F \leq [z]_F$.

Therefore, applying Theorem 1, we know that $(A/{\approx_F} \to, \rightsquigarrow, [1]_F)$ is a quantum B-algebra and pseudo-BCI algebra. That is, Definition 10 (2) holds for A.

Hence, we know that A is a perfect quantum B-algebra. □

The following examples show that there are some perfect quantum B-algebras that may not be a pseudo-BCI algebra.

Example 2. *Let X = {a, b, c, d, e, 1}. Define operations $\to$ and $\rightsquigarrow$ on X as per the following Cayley Tables 3 and 4, the order on X is defined as the following: $b \leq a \leq 1$; $e \leq d \leq c$. Then, X is a pseudo-BCI algebra (we can verify it with Matlab). Denote F_1 = {1}, F_2 = {a, b, 1}, F_3 = X, then F_i (i = 1, 2, 3) are all normal q-filters of X, and quotient algebras $(X/{\approx_{Fi}} \to, \rightsquigarrow, [1]_{Fi})$ are pseudo-BCI algebras. Thus, X is a perfect quantum B-algebra.*

Table 3. Cayley table of operation $\to$.

$\to$	***a***	***b***	***c***	***d***	***e***	***1***
a	1	a	c	c	c	1
b	1	1	c	c	c	1
c	c	c	1	a	b	c
d	c	c	1	1	a	c
e	c	c	1	1	1	c
1	a	b	c	d	e	1

Table 4. Cayley table of operation $\rightsquigarrow$.

$\rightsquigarrow$	***a***	***b***	***c***	***d***	***e***	***1***
a	1	a	c	c	d	1
b	1	1	c	c	c	1
c	c	c	1	a	a	c
d	c	c	1	1	a	c
e	c	c	1	1	1	c
1	a	b	c	d	e	1

Example 3. *Let X = {a, b, c, d, e, f}. Define operations $\to$ and $\rightsquigarrow$ on X as per the following Cayley Tables 5 and 6, the order on X is defined as follows: $b \leq a \leq f$; $e \leq d \leq c$. Then, X is a quantum B-algebra (we can verify it with Matlab), but it is not a pseudo-BCI algebra, since $e{\rightsquigarrow}e \neq e{\to}e$. Denote F = {a, b, f}, then F, X are all normal q-filters of X, quotient algebras $(X/{\approx_F} \to, \rightsquigarrow, \leq)$, $(X/{\approx_X} \to, \rightsquigarrow, \leq)$ are quantum B-algebras, and X is a perfect quantum B-algebra.*

Table 5. Cayley table of operation $\to$.

$\to$	***a***	***b***	***c***	***d***	***e***	***f***
a	f	a	c	c	c	f
b	f	f	c	c	c	f
c	c	c	f	a	b	c
d	c	c	f	f	a	c
e	c	c	f	f	f	c
f	a	b	c	d	e	f

Table 6. Cayley table of operation $\rightsquigarrow$.

$\rightsquigarrow$	**a**	**b**	**c**	**d**	**e**	**f**
a	f	a	c	c	d	f
b	f	f	c	c	c	f
c	c	c	f	a	a	c
d	c	c	f	f	a	c
e	c	c	f	f	a	c
f	a	b	c	d	e	f

4. Basic Implication Algebras and Filters

Definition 11. *Let* $(A, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ *be a type-(2, 2, 2, 2, 0, 0) algebra [32]. A is called a nonassociative residuated lattice, if it satisfies:*

(A1) $(A, \vee, \wedge, 0, 1)$ *is a bounded lattice;*

(A2) $(A, \otimes, 1)$ *is a commutative groupoid with unit element 1;*

(A3) $\forall x, y, z \in A$, $x \otimes y \leq z \Longleftrightarrow x \leq y \rightarrow z$.

Proposition 10. *Let* $(A, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ *be a nonassociative residuated lattice [32]. Then,* $(\forall\, x, y, z \in A)$

(1) $x \leq y \Longleftrightarrow x \rightarrow y = 1$;

(2) $x \leq y \Rightarrow x \otimes z \leq y \otimes z$;

(3) $x \leq y \Rightarrow y \rightarrow z \leq x \rightarrow z$;

(4) $x \leq y \Rightarrow z \rightarrow x \leq z \rightarrow y$;

(5) $x \otimes (y \vee z) = (x \otimes y) \vee (x \otimes z)$;

(6) $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$;

(7) $(y \vee z) \rightarrow x = (y \rightarrow x) \wedge (z \rightarrow x)$;

(8) $(x \rightarrow y) \otimes x \leq x, y$;

(9) $(x \rightarrow y) \rightarrow y \geq x, y$.

Example 4. *Let* $A = [0, 1]$*, operation* $\otimes$ *on A is defined as follows:*

$$x \otimes y = 0.5xy + 0.5max\{0, x + y - 1\}, x, y \in A.$$

Then, $\otimes$ *is a nonassociative t-norm on A (see Example 1 in Reference [32]). Operation* $\rightarrow$ *is defined as follows:*

$$x \rightarrow y = max\{z \in [0, 1] \mid z \otimes x \leq y\}, x, y \in A.$$

Then, $(A, max, min, \otimes, \rightarrow, 0, 1)$ *is a nonassoiative residuated lattice (see Theorem 5 in Reference [32]). Assume that* $x = 0.55$, $y = 0.2$, $z = 0.1$*, then*

$$y \rightarrow z = 0.2 \rightarrow 0.1 = \max\{a \in [0,1|a \otimes 0.2 \ \leq \ 0.1\} = \frac{5}{6}.$$

$$x \rightarrow y = 0.55 \rightarrow 0.2 = \max\{a \in [0,1|a \otimes 0.55 \ \leq \ 0.2\} = \frac{17}{31}.$$

$$x \rightarrow z = 0.55 \rightarrow 0.1 = \max\{a \in [0,1|a \otimes 0.55 \ \leq \ 0.1\} = \frac{4}{11}.$$

$$(x \rightarrow y) \ \rightarrow \ (x \rightarrow z) = \frac{17}{31} \rightarrow \frac{4}{11} = \max\{a \in [0,1|a \otimes \frac{17}{31} \leq \frac{4}{11}\} = \frac{67}{88}.$$

Therefore,

$$y \rightarrow z \nleq (x \rightarrow y) \rightarrow (x \rightarrow z).$$

Example 4 shows that Condition (1) in Definition 1 is not true for general non-associative residuated lattices, that is, quantum B-algebras are not common basic of non-associative fuzzy logics. So, we discuss more general implication algebras in this section.

Definition 12. *A basic implication algebra is a partially ordered set* $(X, \leq)$ *with binary operation* $\rightarrow$*, such that the following are satisfied for x, y, and z in X:*

(1) $x \leq y \Rightarrow z \rightarrow x \leq z \rightarrow y$;
(2) $x \leq y \Rightarrow y \rightarrow z \leq x \rightarrow z$.

A basic implication algebra is considered to be normal, if it satisfies:

(3) *for any* $x, y \in X$, $x \rightarrow x = y \rightarrow y$;
(4) *for any* $x, y \in X$, $x \leq y \Longleftrightarrow x \rightarrow y = e$, *where* $e = x \rightarrow x = y \rightarrow y$.

We can verify that the following results are true (the proofs are omitted).

Proposition 11. *Let* $(X, \rightarrow, \leq)$ *be a basic implication algebra. Then, for all* $x, y, z \in X$,

(1) $x \leq y \Rightarrow y \rightarrow x \leq x \rightarrow x \leq x \rightarrow y$;
(2) $x \leq y \Rightarrow y \rightarrow x \leq y \rightarrow y \leq x \rightarrow y$;
(3) $x \leq y$ *and* $u \leq v \Rightarrow y \rightarrow u \leq x \rightarrow v$;
(4) $x \leq y$ *and* $u \leq v \Rightarrow v \rightarrow x \leq u \rightarrow y$.

Proposition 12. *Let* $(X, \rightarrow, \leq, e)$ *be a normal basic implication algebra. Then for all* $x, y, z \in X$,

(1) $x \rightarrow x = e$;
(2) $x \rightarrow y = y \rightarrow x = e \Rightarrow x = y$;
(3) $x \leq y \Rightarrow y \rightarrow x \leq e$;
(4) *if e is unit (that is, for all x in X,* $e \rightarrow x = x$*), then e is a maximal element (that is,* $e \leq x \Rightarrow e = x$*).*

Proposition 13. *(1) If* $(X, \rightarrow, \rightsquigarrow, \leq)$ *is a a quantum B-algebra, then* $(X, \rightarrow, \leq)$ *and* $(X, \rightsquigarrow, \leq)$ *are basic implication algebras; (2) If* $(A, \rightarrow, \rightsquigarrow, \leq, 1)$ *is a pseudo-BCI algebra, then* $(A, \rightarrow, \leq, 1)$ *and* $(A, \rightsquigarrow, \leq, 1)$ *are normal basic implication algebras with unit 1; (3) If* $(A, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ *is a non-associative residuated lattice, then* $(A, \rightarrow, \leq, 1)$ *is a normal basic implication algebra.*

The following example shows that element *e* may not be a unit.

Example 5. *Let* $X = \{a, b, c, d, 1\}$*. Define* $a \leq b \leq c \leq d \leq 1$ *and operation* $\rightarrow$ *on X as per the following Cayley Table 7. Then, X is a normal basic implication algebra in which element 1 is not a unit.* $(X, \rightarrow, \leq)$ *is not a commutative quantum B-algebra, since*

$$c = 1 \rightarrow c \not\leq b \;=\; (c \rightarrow d) \rightarrow (1 \rightarrow d).$$

Table 7. Cayley table of operation $\rightarrow$.

$\rightarrow$	***a***	***b***	***c***	***d***	***1***
a	1	1	1	1	1
b	d	1	1	1	1
c	d	d	1	1	1
d	b	c	d	1	1
1	b	b	c	b	1

The following example shows that element *e* may be not maximal.

Example 6. *Let $X = \{a, b, c, d, 1\}$. Define $a \leq b \leq c \leq d$, $a \leq b \leq c \leq 1$ and operation $\rightarrow$ on X as per the following Cayley Table 8. Then, X is a normal basic implication algebra, and element 1 is not maximal and is not a unit.*

Table 8. Cayley table of operation $\rightarrow$.

$\rightarrow$	*a*	*b*	*c*	*d*	**1**
a	1	1	1	1	1
b	*c*	1	1	1	1
c	*c*	*c*	1	1	1
d	*a*	*c*	*a*	1	*c*
1	*a*	*b*	*b*	*c*	1

Definition 13. *A nonempty subset F of basic implication algebra $(X, \rightarrow, \leq)$ is called a filter of X if it satisfies:*

(1) *F is an upper set of X, that is, $x \in F$ and $x \leq y \in X \Longrightarrow y \in F$;*
(2) *for all $x \in F$, $x \rightarrow x \in F$;*
(3) *$x \in F$, $y \in X$, $x \rightarrow y \in F \Longrightarrow y \in F$;*
(4) *$x \in X$, $y \rightarrow z \in F \Longrightarrow (x \rightarrow y) \rightarrow (x \rightarrow z) \in F$;*
(5) *$x \in X$, $y \rightarrow z \in F \Longrightarrow (z \rightarrow x) \rightarrow (y \rightarrow x) \in F$.*

For normal basic implication algebra $(X, \rightarrow, \leq, e)$, a filter F of X is considered to be regular, if it satisfies:

(6) *$x \in X$, $(x \rightarrow y) \rightarrow e \in F$ and $(y \rightarrow z) \rightarrow e \in F \Longrightarrow (x \rightarrow z) \rightarrow e \in F$.*

Proposition 14. *Let $(X, \rightarrow, \leq, e)$ be a normal basic implication algebra and $F \subseteq X$. Then, F is a filter of X if and only if it satisfies:*

(1) *$e \in F$;*
(2) *$x \in F$, $y \in X$, $x \rightarrow y \in F \Longrightarrow y \in F$;*
(3) *$x \in X$, $y \rightarrow z \in F \Longrightarrow (x \rightarrow y) \rightarrow (x \rightarrow z) \in F$;*
(4) *$x \in X$, $y \rightarrow z \in F \Longrightarrow (z \rightarrow x) \rightarrow (y \rightarrow x) \in F$.*

Obviously, if e is the maximal element of normal basic implication algebra $(X, \rightarrow, \leq, e)$, then any filter of X is regular.

Theorem 3. *Let X be a basic implication algebra and F a filter of X. Define binary $\approx_F$ on X as follows:*

$$x \approx_F y \Longleftrightarrow x \rightarrow y \in F \text{ and } y \rightarrow x \in F, \text{ where } x, y \in X.$$

Then

(1) *$\approx_F$ is a equivalent relation on X;*
(2) *$\approx_F$ is a congruence relation on X, that is, $x \approx_F y \Longrightarrow (z \rightarrow x) \approx_F (z \rightarrow y)$, $(x \rightarrow z) \approx_F (y \rightarrow z)$, for all $z \in X$.*

Proof (1) $\forall x \in X$, from Definition 13 (2), $x \rightarrow x \in F$, thus $x \approx_F x$. Moreover, $\forall x, y \in X$, if $x \approx_F y$, then $y \approx_F x$.

If $x \approx_F y$ and $y \approx_F z$. Then $x \rightarrow y \in F$, $y \rightarrow x \in F$, $y \rightarrow z \in F$, and $z \rightarrow y \in F$. Applying Definition 13 (4) and (5), we have

$$(x \rightarrow y) \rightarrow (x \rightarrow z) \in F,\ (z \rightarrow y) \rightarrow (z \rightarrow x) \in F.$$

From this and Definition 13 (3), we have $x \rightarrow z \in F$, $z \rightarrow x \in F$. Thus, $x \approx_F z$.

Hence, $\approx_F$ is a equivalent relation on X.

(2) Assume $x \approx_F y$. By the definition of bianary relation $\approx_F$, we have $x{\to}y{\in}F$, $y{\to}x{\in}F$. Using Definition 13 (4),

$$(z{\to}x){\to}(z{\to}y){\in}F,\ (z{\to}y){\to}(z{\to}x){\in}F.$$

This means that $(z{\to}x) \approx_F (z{\to}y)$. Moreover, using Definition 13 (5), we have

$$(y{\to}z){\to}(x{\to}z){\in}F,\ (x{\to}z){\to}(y{\to}z){\in}F.$$

Hence, $(x{\to}z) \approx_F (y{\to}z)$. □

Theorem 4. *Let $(X, \to, \leq, e)$ be a normal basic implication algebra and F a regular filter of X. Define quotient operation $\to$ and binary relation $\leq$ on $X/\approx_F$ as follows:*

$$[x]_F{\to}[y]_F = [x]_F{\to}[y]_F,\ \forall x, y{\in}X;$$
$$[x]_F{\leq}\ [y]_F \Longleftrightarrow [x]_F \to[y]_F = [e]_F,\ \forall x, y{\in}X.$$

Then, $(X/\approx_F, \to, \leq, [e]_F)$ is a normal basic implication algebra, and $(X, \to, \leq, e) \sim (X/\approx_F, \to, \leq, [e]_F)$.

Proof. Firstly, we prove that binary relation $\leq$ on $X/\approx_F$ is a partial order.

(1) $\forall x{\in}X$, obviously, $[x]_F \leq [x]_F$.

(2) Assume that $[x]_F \leq [y]_F$ and $[y]_F \leq [x]_F$, then

$$[x]_F{\to}[y]_F = [x{\to}y]_F = [e]_F,\ [y]_F{\to}[x]_F = [y{\to}x]_F = [e]_F.$$

It follows that $e{\to}(x{\to}y){\in}F$, $e{\to}(y{\to}x){\in}F$. Applying Proposition 14 (1) and (2), we get that $(x{\to}y){\in}F$ and $(y{\to}x){\in}F$. This means that $[x]_F = [y]_F$.

(3) Assume that $[x]_F \leq [y]_F$ and $[y]_F \leq [z]_F$, then

$$[x]_F{\to}[y]_F = [x{\to}y]_F = [e]_F,\ [y]_F{\to}[z]_F = [y{\to}z]_F = [e]_F.$$

Using the definition of equivalent relation $\approx_F$, we have

$$e{\to}(x{\to}y){\in}F,\ (x{\to}y){\to}e{\in}F;\ e{\to}(y{\to}z){\in}F,\ (y{\to}z){\to}e{\in}F.$$

From $e{\to}(x{\to}y){\in}F$ and $e{\to}(y{\to}z){\in}F$, applying Proposition 14 (1) and (2), $(x{\to}y){\in}F$ and $(y{\to}z){\in}F$. By Proposition 14 (4), $(x{\to}y){\to}(x{\to}z){\in}F$. It follows that $(x{\to}z){\in}F$. Hence, $(x{\to}x){\to}(x{\to}z){\in}F$, by Proposition 14 (4). Therefore,

$$e{\to}(x{\to}z) = (x{\to}x){\to}(x{\to}z){\in}F.$$

Moreover, from $(x{\to}y){\to}e{\in}F$ and $(y{\to}z){\to}e{\in}F$, applying regularity of F and Definition 13 (6), we get that $(x{\to}z){\to}e{\in}F$.

Combining the above $e{\to}(x{\to}z){\in}F$ and $(x{\to}z){\to}e{\in}F$, we have $x{\to}z \approx_F e$, that is, $[x{\to}z]_F = [e]_F$. This means that $[x]_F \leq [z]_F$. It follows that the binary relation $\leq$ on $X/\approx_F$ is a partially order.

Therefore, applying Theorem 3, we know that $(X/\approx_F \to, \leq, [e]_F)$ is a normal basic implication algebra, and $(X, \to, \leq, e) \sim (X/\approx_F \to, \leq, [e]_F)$ in the homomorphism mapping f: $X{\to}X/\approx_F$; $f(x) = [x]_F$. □

Example 7. *Let X = {a, b, c, d, 1}. Define operations $\to$ on X as per the following Cayley Table 9, and the order binary on X is defined as follows: $a \leq b \leq c \leq 1$, $b \leq d \leq 1$. Then $(X, \to, \leq, 1)$ is a normal basic implication algebra (it is not a quantum B-algebra). Denote F = {1}, then F is regular filters of X, and the quotient algebras $(X, \to, \leq, 1)$ is isomorphism to $(X/\approx_F, \to, [1]_F)$.*

Table 9. Cayley table of operation $\rightarrow$.

$\rightarrow$	**a**	**b**	**c**	**d**	**1**
a	1	1	1	1	1
b	d	1	1	1	1
c	b	d	1	d	1
d	a	c	c	1	1
1	a	b	c	d	1

Example 8. *Denote X = {a, b, c, d, 1}. Define operations $\rightarrow$ on X as per the following Cayley Table 10, and the order binary on X is defined as follows: $a \le b \le c \le 1$, $b \le d \le 1$. Then (X, $\rightarrow$, $\le$, 1) is a normal basic implication algebra (it is not a quantum B-algebra). Let F = {1, d}, then F is a regular filters of X, and the quotient algebras ($X/\approx_F$, $\rightarrow$, $[1]_F$) is presented as the following Table 11, where $X/\approx_F$ = {{a}, {b, c}, $[1]_F$ = {1, d}}. Moreover, (X, $\rightarrow$, $\le$, 1) $\sim$ ($X/\approx_F$ $\rightarrow$, $[1]_F$).*

Table 10. Cayley table of operation $\rightarrow$.

$\rightarrow$	**a**	**b**	**c**	**d**	**1**
a	1	1	1	1	1
b	c	1	1	1	1
c	b	d	1	d	1
d	a	c	c	1	1
1	a	b	c	d	1

Table 11. Quotient algebra ($X/\approx_F$, $\rightarrow$, $[1]_F$).

$\rightarrow$	**{a}**	**{b,c}**	**$[1]_F$**
{a}	$[1]_F$	$[1]_F$	$[1]_F$
{b,c}	{b,c}	$[1]_F$	$[1]_F$
$[1]_F$	{a}	{b,c}	$[1]_F$

5. Conclusions

In this paper, we introduced the notion of a q-filter in quantum B-algebras and investigated quotient structures; by using q-filters as a corollary, we obtained quotient pseudo-BCI algebras by their filters. Moreover, we pointed out that the concept of quantum B-algebra does not apply to non-associative fuzzy logics. From this fact, we proposed the new concept of basic implication algebra, and established the corresponding filter theory and quotient algebra. In the future, we will study in depth the structural characteristics of basic implication algebras and the relationship between other algebraic structures and uncertainty theories (see References [33–36]). Moreover, we will consider the applications of q-filters for Gentzel's sequel calculus.

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References

1. Hájek, P. *Metamathematics of Fuzzy Logic*; Kluwer: Dordrecht, The Netherlands, 1998.
2. Klement, E.P.; Mesiar, R.; Pap, E. *Triangular Norm*; Springer: Berlin, Germany, 2000.
3. Esteva, F.; Godo, L. Monoidal t-norm based logic: Towards a logic for left-continuous t-norms. *Fuzzy Sets Syst.* **2001**, *124*, 271–288. [CrossRef]
4. Flaminio, T. Strong non-standard completeness for fuzzy logics. *Soft Comput.* **2008**, *12*, 321–333. [CrossRef]

5. Zhang, X.H. *Fuzzy Logics and Algebraic Analysis*; Science in China Press: Beijing, China, 2008.
6. Zhang, X.H.; Dudek, W.A. BIK+-logic and non-commutative fuzzy logics. *Fuzzy Syst. Math.* **2009**, *23*, 8–20.
7. Iorgulescu, A. *Implicative-Groups vs. Groups and Generalizations*; Matrix Room: Bucuresti, Romania, 2018.
8. Fodor, J.C. Contrapositive symmetry of fuzzy implications. *Fuzzy Sets Syst.* **1995**, *69*, 141–156. [CrossRef]
9. Ruiz-Aguilera, D.; Torrens, J. Distributivity of residual implications over conjunctive and disjunctive uninorms. *Fuzzy Sets Syst.* **2007**, *158*, 23–37. [CrossRef]
10. Yao, O. On fuzzy implications determined by aggregation operators. *Inf. Sci.* **2012**, *193*, 153–162.
11. Baczyński, M.; Beliakov, G.; Bustince Sola, H.; Pradera, A. *Advances in Fuzzy Implication Functions*; Springer: Berlin/Heidelberg, Germany, 2013.
12. Vemuri, N.; Jayaram, B. The $\otimes$-composition of fuzzy implications: Closures with respect to properties, powers and families. *Fuzzy Sets Syst.* **2015**, *275*, 58–87. [CrossRef]
13. Li, D.C.; Qin, S.J. The quintuple implication principle of fuzzy reasoning based on interval-valued S-implication. *J. Log. Algebraic Meth. Program.* **2018**, *100*, 185–194. [CrossRef]
14. Su, Y.; Liu, H.W.; Pedrycz, W. A method to construct fuzzy implications–rotation construction. *Int. J. Appr. Reason.* **2018**, *92*, 20–31. [CrossRef]
15. Wang, C.Y.; Wan, L.J. Type-2 fuzzy implications and fuzzy-valued approximation reasoning. *Int. J. Appr. Reason.* **2018**, *102*, 108–122. [CrossRef]
16. Luo, M.X.; Zhou, K.Y. Logical foundation of the quintuple implication inference methods. *Int. J. Appr. Reason.* **2018**, *101*, 1–9. [CrossRef]
17. Georgescu, G.; Iorgulescu, A. Pseudo-BCK algebras. In *Combinatorics, Computability and Logic*; Springer: London, UK, 2001; pp. 97–114.
18. Dudek, W.A.; Jun, Y.B. Pseudo-BCI algebras. *East Asian Math. J.* **2008**, *24*, 187–190.
19. Dymek, G. Atoms and ideals of pseudo-BCI-algebras. *Commen. Math.* **2012**, *52*, 73–90.
20. Zhang, X.H.; Jun, Y.B. Anti-grouped pseudo-BCI algebras and anti-grouped filters. *Fuzzy Syst. Math.* **2014**, *28*, 21–33.
21. Zhang, X.H. Fuzzy anti-grouped filters and fuzzy normal filters in pseudo-BCI algebras. *J. Intell. Fuzzy Syst.* **2017**, *33*, 1767–1774. [CrossRef]
22. Emanovský, P.; Kühr, J. Some properties of pseudo-BCK- and pseudo-BCI-algebras. *Fuzzy Sets Syst.* **2018**, *339*, 1–16. [CrossRef]
23. Zhang, X.H.; Park, C.; Wu, S.P. Soft set theoretical approach to pseudo-BCI algebras. *J. Intell. Fuzzy Syst.* **2018**, *34*, 559–568. [CrossRef]
24. Rump, W.; Yang, Y. Non-commutative logical algebras and algebraic quantales. *Ann. Pure Appl. Log.* **2014**, *165*, 759–785. [CrossRef]
25. Rump, W. Quantum B-algebras. *Cent. Eur. J. Math.* **2013**, *11*, 1881–1899. [CrossRef]
26. Han, S.W.; Xu, X.T.; Qin, F. The unitality of quantum B-algebras. *Int. J. Theor. Phys.* **2018**, *57*, 1582–1590. [CrossRef]
27. Han, S.W.; Wang, R.R.; Xu, X.T. On the injective hulls of quantum B-algebras. *Fuzzy Sets Syst.* **2018**. [CrossRef]
28. Rump, W. Quantum B-algebras: Their omnipresence in algebraic logic and beyond. *Soft Comput.* **2017**, *21*, 2521–2529. [CrossRef]
29. Rump, W. L-algebras, self-similarity, and l-groups. *J. Algebra* **2008**, *320*, 2328–2348. [CrossRef]
30. Botur, M.; Paseka, J. Filters on some classes of quantum B-algebras. *Int. J. Theor. Phys.* **2015**, *54*, 4397–4409. [CrossRef]
31. Botur, M.; Halaš, R. Commutative basic algebras and non-associative fuzzy logics. *Arch. Math. Log.* **2009**, *48*, 243–255. [CrossRef]
32. Botur, M. A non-associative generalization of Hájek's BL-algebras. *Fuzzy Sets Syst.* **2011**, *178*, 24–37. [CrossRef]
33. Zhang, X.H.; Smarandache, F.; Liang, X.L. Neutrosophic duplet semi-group and cancellable neutrosophic triplet groups. *Symmetry* **2017**, *9*, 275. [CrossRef]
34. Zhang, X.H.; Bo, C.X.; Smarandache, F.; Park, C. New operations of totally dependent- neutrosophic sets and totally dependent-neutrosophic soft sets. *Symmetry* **2018**, *10*, 187. [CrossRef]

35. Zhang, X.H.; Bo, C.X.; Smarandache, F.; Dai, J.H. New inclusion relation of neutrosophic sets with applications and related lattice structure. *Int. J. Mach. Learn. Cyber.* **2018**, *9*, 1753–1763. [CrossRef]
36. Caponetto, R.; Fortuna, L.; Graziani, S.; Xibilia, M.G. Genetic algorithms and applications in system engineering: A survey. *Trans. Inst. Meas. Control* **1993**, *15*, 143–156. [CrossRef]