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Cosine Distance Measure between Neutrosophic Hesitant Fuzzy Linguistic Sets and Its Application in Multiple Criteria Decision Making

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Abstract: This paper proposes a neutrosophic hesitant fuzzy linguistic term set (NHFLTS) based on hesitant fuzzy linguistic term set (HFLTS) and neutrosophic set (NS), which can express the inconsistent and uncertainty information flexibly in multiple criteria decision making problems. The basic operational laws of NHFLTS based on linguistic scale function are also discussed. Then we propose the generalized neutrosophic hesitant fuzzy linguistic distance measure and discuss its properties. Furthermore, a new similarity measure of NHFLTS combines the generalized neutrosophic hesitant fuzzy linguistic distance measure and the cosine function is given. A corresponding cosine distance measure between NHFLTSs is proposed according to the relationship between the similarity measure and the distance measure, and we develop the technique for order preference by similarity to an ideal solution (TOPSIS) method to the obtained cosine distance measure. The main advantages of the proposed NHFLTS is defined on linguistic scale function, the decision makers can flexibly convert the linguistic information to semantic values, and the proposed cosine distance measure between NHFLTSs with TOPSIS method can deal with the related decision information not only from the point of view of algebra, but also from the point of view of geometry. Finally, the reasonableness and effectiveness of the proposed method is demonstrated by the illustrative example, which is also compared to the other existing methods.

Keywords: neutrosophic hesitant fuzzy linguistic set; cosine distance measure; similarity measure; linguistic scale function; TOPSIS

1. Introduction

Multiple criteria decision making (MCDM) is a very important topic in economic analysis, investment management, complex engineering, etc. In some studies [1,2], the following steps have been mentioned to determine the decision alternative: (1) Providing the alternatives and evaluation criteria; (2) Analysing the consequences and developing the criteria; (3) Modelling preferences and aggregating performances of the variants; (4) Determining the optimal alternative on the basis of the results of evaluation of variants.

Since MCDM is a very important scientific area, there are many applications of the MCDM in different disciplines [3–10]. For example, Ziemba et al. [3] proposed the integrated approach to e-commerce websites evaluation by incorporating eye tracking based measurement and analysis into the criteria set, Petrović et al. [4] proposed an approach in selection and evaluation of the criteria and attributes of criteria for selecting the air traffic protection aircraft. Mukhametzyanov et al. [5] introduced a study of estimating the variation of alternatives according to the criteria for the results of ranking alternatives. Vesković et al. [6] evaluated the railway management by using a new integrated model DELPHI-SWARA-MABAC. Ziemba et al. [7] proposed a new fuzzy multiple criteria decision making method based on the adjustment of mapping trapezoidal fuzzy numbers, and there still have many other related studies [8–10].

However, in the practical multiple criteria decision making problems, the decision information is often uncertain or imprecise due to lack of data, information processing capabilities and the decision makers limited attention. In order to deal with the uncertain information, Zadeh [11] proposed the concept of fuzzy set. Since it was proposed, it has attracted a lot of attention in many fields, and it is an effective tool to solve the corresponding multiple criteria decision making problems, fuzzy inference, pattern recognition, etc. [12–14]. However, the fuzzy set only has membership degree, it cannot describe the following decision information: a 10-member panel of experts evaluated the feasibility of a company's investment programme, six experts considered the programme feasible, three experts disapproved of it and one remained neutral. In order to describe the above information better, Atanassov [15] proposed the concept of an intuitionistic fuzzy set (IFS) E by adding the non-membership degree. So, the above decision information can be represented as $E = \{0.6, 0.3\}$, and the indeterminacy degree equals $0.1(= 1 - 0.6 - 0.3)$. However, the IFS also has some limitations in multiple criteria decision making problems. For instance, if the group of experts evaluate the expected benefits of company's investment in the automotive industry, they believe that the possibility of profit is 0.5, the possibility of loss is 0.6 and the possibility of the investment programme being profitable is 0.2. Obviously, the IFS cannot describe this type of decision information. In order to process the relevant information effectively, Smarandache [16] put forward the neutrosophic set(NS) $A = \{T_A(x), I_A(x), F_A(X) | x \in X\}$, where $T_A(x) : X \to]^-0, 1^+[$, $I_A(x) : X \to]^-0, 1^+[$ and $F_A(x) : X \to]^-0, 1^+[$ represents the truth-membership degree, the indeterminacy-membership degree and the falsity-membership degree respectively. Then the experts' evaluation about the investment in the automotive industry can be expressed as $A = \{0.5, 0.2, 0.6\}$. Since the introduction of neutrosophic set, neutrosophic numbers are widely used in pattern recognition, supplier evaluation and engineering optimisations, we can see [17–21].

It is already known that there is much uncertainty and fuzziness in multiple criteria decision making problems; people may hesitate about the membership degree of fuzzy set and need several values to describe it. In order to express the relevant decision information well, Torra [22,23] proposed the hesitant fuzzy set (HFS), which includes several possible values representing the membership degree of an element in [0,1]. Since it was proposed, some researchers have been working on this subject. For example, Xia et al. [24] proposed the aggregation operator for hesitant fuzzy information and applied it to the multiple criteria decision making problem. Farhadinia [25] defined the score function of hesitant fuzzy set and used it to sort the alternatives. Beg et al. [26] introduced the concept of intuitionistic hesitant fuzzy set and applied it to group decision making. Recently, Rodríguez et al. [27] gave a review of the HFS and put forward its possible directions for future research. However, due to the uncertainty of the multiple criteria decision making problems, it is difficult to describe the qualitative decision information by numerical value. At this time the decision maker often uses linguistic term sets to express such information. They regard the linguistic information as the values of linguistic variable, that is to say, the values of the variable are no longer represented by the exact numerical value, but by the linguistic term sets. Up to now, many researchers have studied the linguistic multiple criteria decision making problems. In 1993, Herrera et al. [28] proposed a linguistic term set for multiple criteria decision making, furthermore, Herrera et al. [29] proposed a consensus model in group decision making problems under linguistic assessment, and Herrera et al. [30] put forward the linguistic ordered weighted averaging (LOWA) operator to deal with linguistic information problems. Later, Xu [31] defined the uncertain linguistic ordered weighted geometric (LOWG) operator for solving group decision making problems. The study on linguistic multiple criteria decision making problems is developing rapidly [32–37]. However, in some practical decision making problems, due to the uncertainty and complexity in the decision process, the decision maker cannot express his/her view with a single value. In order to express the decision maker's hesitancy to the membership degree of a given linguistic term set, Rodríguez et al. [38] proposed the hesitant fuzzy linguistic term sets (HFLTS) based on HFS and linguistic term set (LTS). The element of the HFLTS is called a hesitant fuzzy linguistic number (HFLN). For example, $\langle s_1, (0.4, 0.6)\rangle$ is a HFLN, and 0.4, 0.6 are the possible membership degrees of the linguistic term s_1. Liao et al. [39] proposed a series of distance and similarity measures between two HFLTSs and applied them to multiple criteria decision making problems. The HFLTS has made great progress in processing linguistic information and it can be regarded as an innovative construction to some extent. However, in some cases, the HFLTS cannot describe some decision information. For example, a set of five linguistic terms set $S = \{s_0 = bad, s_1 = slightly\ bad, s_2 = fair, s_3 = slightly\ well, s_4 = well\}$, the experts believe that the membership degree of possibility that a profitable investment plan will be well is 0.7 or 0.8, the non-membership degree of possibility that a profitable investment plan will be well is 0.3 or 0.4 or 0.5, and the uncertain membership degree of possibility that a profitable investment plan will be well is 0.4. It is already known there is no fuzzy set to describe the above information at present. Motivated by this, we propose a neutrosophic hesitant fuzzy linguistic set (NHFLS) to process such information in the paper.

On the other hand, distance and similarity measures are also important topics in fuzzy set theory. They are widely used in some fields, such as medical diagnosis, pattern recognition and so on. One of the important similarity measures is the cosine similarity measure; it is defined as the inner product of two vectors divided by the product of their lengths. It has been widely studied in the past 10 years, we can see [39–46]. For example, Liao et al. [39], Ye [40] proposed a cosine similarity measure between hesitant fuzzy linguistic term set and intuitionistic fuzzy set, respectively. It is already known that the cosine similarity measures defined by them do not satisfy the axiom of similarity measure (the example can be seen in Section 4), if they are applied in multiple criteria decision making problems, they may cause the decision information to be distorted.

Inspired by this, we propose a new cosine distance measure between neutrosophic hesitant fuzzy linguistic term sets, the main objects and advantages of the paper are given as follows:

(1) A neutrosophic hesitant fuzzy linguistic set (NHFLS) is defined based on NHFS and linguistic term set, its operations and comparison rules are given based on linguistic scale function, which can describe some decision information more accurately.
(2) A new method is proposed to construct a similarity measure of NHFLTS based on the generalized neutrosophic hesitant fuzzy linguistic distance measure and the cosine similarity measure, which satisfies with the axiom of similarity measure. Then the corresponding cosine distance measure between NHFLTSs is obtained according to the relationship between the similarity measure and the distance measure.
(3) The proposed cosine distance measure with TOPSIS method is developed to solve multiple criteria decision making with neutrosophic hesitant fuzzy linguistic environment, which can deal with the related decision information not only from the point of view of algebra but also from the point of view of geometry.

The rest of the paper is organized as follows. In Section 2, some basic concepts of NS, LTS, HFS, Neutrosophic hesitant fuzzy set (NHFS) and linguistic scale function are briefly reviewed. In Section 3, we first give the concept of NHFLS, then the operations of NHFLSs and their related properties are proposed. Furthermore, the score function and the comparison rules of the NHFLSs are given. In Section 4, we propose a new method to construct the cosine distance measure between NHFLS and discuss some related properties. In Section 5, the proposed cosine distance measure with TOPSIS method is developed. In Section 6, we give an application of the proposed cosine distance measure and make comparison analysis with the existing method. The conclusions are given in Section 7.

2. Preliminaries

In this section, we review and discuss some related basic concepts, including NS, LTS, HFS, NHFS and linguistic scale functions. Throughout this paper, we use $X = \{x_1, x_2, \cdots, x_n\}$ to denote the discourse set.

2.1. NS

Definition 1 (Smarandache [16]). *Given a fixed set X, the neutrosophic set A on X is defined as:*

$$A = \{< x, (T_A(x), I_A(x), F_A(x)) > |x \in X\},$$

where the truth-membership function $T_A(x) \in]0^-, 1^+[$, the indeterminacy-membership function $I_A(x) \in]0^-, 1^+[$ and the falsity-membership function $F_A(x) \in]0^-, 1^+[$. For each $x \in X$, we have $0^- \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3^+$. When the set $X = \{x_1, x_2, \cdots, x_n\}$ has only one element, the neutrosophic set A is reduced to $A = (T_A(x), I_A(x), F_A(x)) = (T_A, I_A, F_A)$. For convenience, we call $A = (T_A, I_A, F_A)$ a neutrosophic number (NN).

2.2. LTS

Sometimes it is inappropriate to use numerical values to express the decision information. When we use a linguistic term set to express the relevant information, the inconvenience would be avoided. Suppose that $S = \{s_k | k = 0, 1, \cdots, \tau\}$ is an ordered discrete linguistic term set, where s_k is the possible language variable. For example, a set of seven terms set S can be given as follows:

$$S = \{s_0 = very\ bad, s_1 = bad, s_2 = slightly\ bad, s_3 = fair, s_4 = slightly\ well, s_5 = well, s_6 = very\ well\}.$$

Xu [47] generalized the discrete linguistic term set S to the continuous linguistic term set $\widetilde{S} = \{s_k | k \in [0, t]\}$, where $t(t > \tau)$ is a positive integer. Based on the generalized linguistic term set $\widetilde{S}$, let $s_k, s_l \in \widetilde{S}$ and $\eta \in [0, 1]$, the operational laws can be given as follows (Xu [48]):

(1) $s_k \oplus s_l = s_{k+l}$;
(2) $\eta s_k = s_{\eta k}$;
(3) $s_k > s_l$ if $k > l$.

2.3. HFS

Definition 2 (Torra [22]). *Given a fixed set X, a hesitant fuzzy set(HFS) H on X can be defined as:*

$$H = \{< x, h(x) > |x \in X\},$$

where $h(x)$ is a set composed of values from 0 to 1, they represent the possible membership degree of the element $x \in X$, and they are called the hesitant fuzzy elements (HFEs). Of course we assume the cardinality of the set H is finite.

Throughout this paper, we assume that $q(h(x))$ represents the number of elements in $h(x)$. When we calculate the distance between two hesitant fuzzy sets $h_1(x)$ and $h_2(x)$, if the number of elements in two HFSs is not equal, that is $q(h_1(x)) \neq q(h_2(x))$, we should make their elements equal. At this time, we assume $l = \max\{q(h_1(x)), q(h_2(x))\}$, the rules of regulation is given by Xu et al. [49]. The set of fewer number of elements is extended by adding the maximum value, the minimum value or any value in it until it has the same number of elements with the more one. When we add the element to the set of fewer elements, the best way is to add the same value in it. In practical application, the added elements are related to the risk preference of the decision makers. The pessimists want to add the minimum value, while the optimists want to add the maximum value. The risk preferences of the decision maker directly affects the final decisions (Yu et al. [50]). In this paper, we extend the lesser element of the set by adding the minimum value in it. Ye [51] Wang et al. [52]

2.4. NHFS

Definition 3 (Ye [51]). *Given a non-empty fixed set X, a neutrosophic hesitant fuzzy set (NHFS) G on X is often expressed by:*

$$G = \{< x, (\tilde{T}_G(x), \tilde{I}_G(x), \tilde{F}_G(x)) > | x \in X\},$$

where $\tilde{T}_G(x)$, $\tilde{I}_G(x)$, $\tilde{F}_G(x)$ are three sets of some values in $]0^-, 1^+[$, which denote the possible truth-membership degrees, the possible indeterminacy-membership degrees and the possible falsity-membership degrees of the element $x \in X$ to the set G, and satisfies:

$$0^- \leq \sup \tilde{\alpha} + \sup \tilde{\beta} + \sup \tilde{\gamma} \leq 3^+,$$

where $\tilde{\alpha} \in \tilde{T}_G(x)$, $\tilde{\beta} \in \tilde{I}_G(x)$, $\tilde{\gamma} \in \tilde{F}_G(x)$, $(\forall x \in X)$.

When the set $X = \{x_1, x_2, \cdots, x_n\}$ has only one element, the neutrosophic hesitant fuzzy set G is reduced to $G = (\tilde{T}_G(x), \tilde{I}_G(x), \tilde{F}_G(x)) = (\tilde{T}_G, \tilde{I}_G, \tilde{F}_G)$. For convenience, we call $G = (\tilde{T}_G, \tilde{I}_G, \tilde{F}_G)$ a neutrosophic hesitant fuzzy number (NHFN).

2.5. Linguistic Scale Functions

In some situations, operations cannot be carried out directly to the subscript of the linguistic terms set. When fuzzy numbers are combined directly with the linguistic term set, the linguistic scale function can assign different semantic values to the linguistic term set under different circumstances. In practice, the linguistic scale functions are very flexible because they can give more deterministic results according to different semantic environments.

Definition 4 (Wang et al. [52]). *Let S be a linguistic term, and $S = \{s_i | i = 0, 1, \cdots, 2\tau\}$. If ξ_i is a numerical value between 0 and 1, then the linguistic scale function φ^* can be defined as follows:*

$$\varphi^* : s_i \rightarrow \xi_i (i = 0, 1, \cdots, 2\tau),$$

where $0 \leq \xi_0 < \xi_1 < \cdots < \xi_{2\tau} \leq 1$. The linguistic scale function is strictly monotonously increasing function with respect to s_i. In fact, the function value represents the semantics of the linguistic terms.

Next we introduce three kinds of linguistic scale functions as follows:

$$\varphi_1^*(s_i) = \xi_i = \frac{i}{2\tau} (i = 0, 1, \cdots, 2\tau). \quad (1)$$

The evaluation scale of the linguistic information expressed by $\varphi_1^*(s_i)$ is divided on average.

$$\varphi_2^*(s_i) = \xi_i = \begin{cases} \frac{a^\tau - a^{\tau - i}}{2a^\tau - 2}, & i = 0, 1, \cdots, \tau, \\ \frac{a^\tau + a^{i-\tau} - 2}{2a^\tau - 2}, & i = \tau + 1, \tau + 2, \cdots, 2\tau. \end{cases} \quad (2)$$

For $\varphi_2^*(s_i)$, the absolute deviation between adjacent linguistic term sets will increase when we extend it from the middle of a language set to both ends.

$$\varphi_3^*(s_i) = \xi_i = \begin{cases} \frac{\tau^{\alpha'} - (\tau - i)^{\alpha'}}{2\tau^{\alpha'}}, & i = 0, 1, \cdots, \tau, \\ \frac{\tau^{\beta'} - (i - \tau)^{\beta'}}{2\tau^{\beta'}}, & i = \tau + 1, \tau + 2, \cdots, 2\tau. \end{cases} \quad (3)$$

For $\varphi_3^*(s_i)$, the absolute deviation between adjacent linguistic sets will decrease when we extend it from the middle of a language set to both ends.

The above linguistic scale function can be developed to a continuous and strictly monotonically increasing function $\varphi^* : \widetilde{S} \rightarrow R^+$(where R^+ is a non-negative real number).

Definition 5 (Wang et al. [52]). *The negation operator of HFLN α is $neg(\alpha) = \langle (\varphi^*)^{-1}(\varphi^*(s_{2\tau}) - \varphi^*(s_\alpha))$, $\bigcup_{r\in h_\alpha}\{1-r\}\rangle$, where φ^* is the linguistic scale function, the inverse of φ^* is represented as φ^{*-1}.*

3. Neutrosophic Hesitant Fuzzy Linguistic Term Set

In this section, we first propose a neutrosophic hesitant fuzzy linguistic set (NHFLS) based on NHFS and fuzzy linguistic term set, and then give the operations and comparison between neutrosophic hesitant fuzzy linguistic term sets based on linguistic scale function.

Definition 6. *Let $X = \{x_1, x_2, \cdots, x_n\}$ be a fixed set and $s_{x_i} \in S$. A NHFLS N in X is defined as follows:*

$$N = \{< (x_i, s_{x_i}), T_N(x_i), I_N(x_i), F_N(x_i) > | x_i \in X\},$$

where $T_N(x_i)$, $I_N(x_i)$ and $F_N(x_i)$ are three sets of different values in $]0^-, 1^+[$, respectively represents the possible degrees of truth-membership, the indeterminacy-membership and the falsity-membership of $x_i \in s_{x_i}$. For $\forall x_i \in X$, we have $0^- \leq \sup T_N(x_i) + \sup I_N(x_i) + \sup F_N(x_i) \leq 3^+$. Here we still assume the cardinality of $T_N(x_i)$, $I_N(x_i)$ and $F_N(x_i)$ are finite. If the set X has only one element, we call $N =< s_x, T_N(x), I_N(x), F_N(x) >=< s, T_N, I_N, F_N >$ a neutrosophic hesitant fuzzy linguistic number (NHFLN). NHFLN is a special case of NHFLS, which is flexible for information evaluation.

Example 1. *Let $X = \{x_1, x_2\}$ be a universal set, $S = \{s_0 = very\ poor, s_1 = poor, s_2 = slightly\ poor, s_3 = fair, s_4 = slightly\ good, s_5 = good, s_6 = very\ good\}$ be a linguistic term set. A NHFLS $N = \{\langle (x_1, s_3),$ $\{0.3, 0.4\}, \{0.5\}, \{0.7, 0.8\}\rangle, \langle (x_2, s_4), \{0.4, 0.6\}, \{0.2, 0.3\}, \{0.8\}\rangle$, then $\langle s_3, \{0.3, 0.4\}, \{0.5\}, \{0.7, 0.8\}\rangle$ and $\langle s_4, \{0.4, 0.6\}, \{0.2, 0.3\}, \{0.8\}\rangle$ are two NHFLNs. For x_1, $\{0.3, 0.4\}$ represents the possible truth-membership degree of x_1 belongs to s_3 (fair), $\{0.5\}$ represents the possible degree of indeterminacy-membership of x_1 belongs to s_3 (fair) and $\{0.7, 0.8\}$ represents the possible falsity-membership degree of x_1 belongs to s_3 (fair).*

In multi-criteria decision making problems, we should distinguish different semantic information under different situations. Direct operations based on linguistic subscripts cannot handle linguistic decision information effectively, but the linguistic scale function can assign different semantic values to linguistic term sets according to semantic environment. Inspired by this, we define the basic operations and comparison method of NHFLNs based on linguistic scale function as follows.

Definition 7. *Assume that $S = \{s_\alpha | \alpha = 0, 1, \cdots, 2\tau\}$ is a linguistic term set, let $N =< s_N, T_N, I_N, F_N >$, $N_1 =< s_{N_1}, T_{N_1}, I_{N_1}, F_{N_1} >$ and $N_2 =< s_{N_2}, T_{N_2}, I_{N_2}, F_{N_2} >$ be three NHFLNs and $\lambda > 0$, the basic operations of NHFLNs based on linguistic scale function φ^* are defined as:*

(1) $neg(N) =< \varphi^{*-1}(\varphi^*(s_{2\tau}) - \varphi^*(s_N)), \bigcup_{f_N \in F_N, i_N \in I_N, t_N \in T_N}(f_N, 1 - i_N, t_N) >;$
(2) $N_1 \oplus N_2 =< \varphi^{*-1}(\varphi^*(s_{N_1}) + \varphi^*(s_{N_2})), \bigcup_{t_{N_j} \in T_{N_j}, i_{N_j} \in I_{N_j}, f_{N_j} \in F_{N_j}}(t_{N_1} + t_{N_2} - t_{N_1}t_{N_2}, i_{N_1}i_{N_2}, f_{N_1}f_{N_2}) >;$ $(j = 1, 2)$
(3) $N_1 \otimes N_2 =< \varphi^{*-1}(\varphi^*(s_{N_1})\varphi^*(s_{N_2})), \bigcup_{t_{N_j} \in T_{N_j}, i_{N_j} \in I_{N_j}, f_{N_j} \in F_{N_j}}(t_{N_1}t_{N_2}, i_{N_1} + i_{N_2} - i_{N_1}i_{N_2}, f_{N_1} + f_{N_2} - f_{N_1}f_{N_2}) >; (i = 1, 2)$
(4) $\lambda N =< \varphi^{*-1}(\lambda\varphi^*(s_N)), \bigcup_{t_N \in T_N, i_N \in I_N, f_N \in F_N}(1 - (1 - t_N)^\lambda, i_N^\lambda, f_N^\lambda) >;$
(5) $N^\lambda =< \varphi^{*-1}((\varphi^*(s_N))^\lambda), \bigcup_{t_N \in T_N, i_N \in I_N, f_N \in F_N}(t_N^\lambda, 1 - (1 - i_N)^\lambda, 1 - (1 - f_N)^\lambda) >;$

Example 2. *Let $S = \{s_0, s_1, s_2, s_3, s_4, s_5, s_6\}$ be a linguistic term set, $N_1 = \langle s_1, \{0.3, 0.4\}, \{0.5\}, \{0.7, 0.8\}\rangle$ and $N_2 = \langle s_4, \{0.4, 0.6\}, \{0.2, 0.3\}, \{0.8\}\rangle$ are two NHFLNs, $\lambda = 3$, if the linguistic scale function is $\varphi^*(s_i) = \varphi_2^*(s_i)$ $(a = 1.4, \tau = 3)$, then*

(1) $neg(N_1) = \langle s_5, \{0.7, 0.8\}, \{0.5\}, \{0.3, 0.4\}\rangle;$
(2) $N_1 \oplus N_2 = \langle s_{5.3216}, \{0.56, 0.64, 0.72, 0.76\}, \{0.1, 0.15\}, \{0.56, 0.64\}\rangle;$
(3) $N_1 \otimes N_2 = \langle s_{0.5740}, \{0.12, 0.16, 0.18, 0.24\}, \{0.6, 0.65\}, \{0.94, 0.96\}\rangle;$
(4) $3N_1 = \langle s_{4.4117}, \{0.657, 0.784\}, \{0.125\}, \{0.343, 0.512\}\rangle;$
(5) $N_1^3 = \langle s_{0.0432}, \{0.027, 0.064\}, \{0.875\}, \{0.657, 0.488\}\rangle.$

One easily sees that the above results are still NHFLNs, and the following properties about NHFLNs can be proven easily.

Theorem 1. *Let* $N_1 =< s_{N_1}, T_{N_1}, I_{N_1}, F_{N_1} >$ *and* $N_2 =< s_{N_2}, T_{N_2}, I_{N_2}, F_{N_2} >$ *be any two NHFLNs, and* $\lambda, \lambda_1, \lambda_2 \geq 0$*, then we have*

(1) $N_1 \oplus N_2 = N_2 \oplus N_1$;
(2) $N_1 \otimes N_2 = N_2 \otimes N_1$;
(3) $\lambda(N_1 \oplus N_2) = \lambda N_1 \oplus \lambda N_2$;
(4) $(\lambda_1 + \lambda_2)N_1 = \lambda_1 N_1 + \lambda_2 N_1$;
(5) $(N_1 \otimes N_2)^\lambda = N_1^\lambda \otimes N_2^\lambda$;
(6) $N_1^{\lambda_1+\lambda_2} = N_1^{\lambda_1} \otimes N_2^{\lambda_2}$;

Proof. According to Definition 7, it is observed that (1) and (2) obviously exist. The proof of (3) is presented as follows:
$\lambda(N_1 \oplus N_2) = \lambda < \varphi^{*-1}(\varphi^*(s_{N_1}) + \varphi^*(s_{N_2})), \bigcup_{t_{N_j}\in T_{N_j}, i_{N_j}\in I_{N_j}, f_{N_j}\in F_{N_j}} (t_{N_1} + t_{N_2} - t_{N_1}t_{N_2}, i_{N_1}i_{N_2}, f_{N_1}f_{N_2}) >$
$=< \varphi^{*-1}(\lambda(\varphi^*(\varphi^{*-1}(\varphi^*(s_{N_1}) + \varphi^*(s_{N_2}))))), \bigcup_{t_{N_j}\in T_{N_j}, i_{N_j}\in I_{N_j}, f_{N_j}\in F_{N_j}} (1 - (1 - t_{N_1} - t_{N_2} + t_{N_1}t_{N_2})^\lambda, (i_{N_1}i_{N_2})^\lambda,$
$(f_{N_1}f_{N_2})^\lambda) >$
$=< \varphi^{*-1}(\lambda(\varphi^*(s_{N_1}) + \varphi^*(s_{N_2}))), \bigcup_{t_{N_j}\in T_{N_j}, i_{N_j}\in I_{N_j}, f_{N_j}\in F_{N_j}} (1 - (1 - t_{N_1})^\lambda (1 - t_{N_2})^\lambda, (i_{N_1}i_{N_2})^\lambda, (f_{N_1}f_{N_2})^\lambda) >$
$=< \varphi^{*-1}(\lambda(\varphi^*(s_{N_1}) + \varphi^*(s_{N_2}))), \bigcup_{t_{N_j}\in T_{N_j}, i_{N_j}\in I_{N_j}, f_{N_j}\in F_{N_j}} (1 - (1 - t_{N_1})^\lambda + 1 - (1 - t_{N_2})^\lambda - (1 - (1 -$
$t_{N_1})^\lambda)(1 - (1 - t_{N_2})^\lambda)), (i_{N_1})^\lambda (i_{N_2})^\lambda, (f_{N_1})^\lambda (f_{N_2})^\lambda) >$
$= \lambda N_1 \oplus \lambda N_2$.
(4)–(6) can be proved analogously, we omit the proof here. □

In order to compare the order of NHFLNs, we propose the comparison method based on the following score function and the accuracy function.

Definition 8. *Let* $N =< s_N, T_N, I_N, F_N >$ *be a NHFLN, the score function* $S(N)$ *and the accuracy function* $H(N)$ *for N can be defined as*

$$S(N) = \frac{\varphi^*(s_N)}{3}\left(\frac{1}{l_1}\sum_{i=1}^{l_1}\alpha_i + \frac{1}{l_2}\sum_{i=1}^{l_2}(1-\beta_i) + \frac{1}{l_3}\sum_{i=1}^{l_3}(1-\gamma_i)\right),$$

$$H(N) = \frac{\varphi^*(s_N)}{3}\left(\frac{1}{l_1}\sum_{i=1}^{l_1}\alpha_i - \frac{1}{l_2}\sum_{i=1}^{l_2}\beta_i - \frac{1}{l_3}\sum_{i=1}^{l_3}\gamma_i\right),$$

where $\alpha_i \in T_N$*,* $\beta_i \in I_N$*,* $\gamma_i \in F_N$*,* l_1, l_2, l_3 *are the number of values* T_N*,* I_N*,* F_N*, respectively.*

If $N_1 =< s_{N_1}, T_{N_1}, I_{N_1}, F_{N_1} >$ *and* $N_2 =< s_{N_2}, T_{N_2}, I_{N_2}, F_{N_2} >$ *are any two NHFLNs, the order relationship between* N_1 *and* N_2 *can be given as follows:*

(1) *If* $S(N_1) > S(N_2)$*, then* $N_1 \succ N_2$;
(2) *If* $S(N_1) < S(N_2)$*, then* $N_1 \prec N_2$;
(3) *If* $S(N_1) = S(N_2)$*, then*

if $H(N_1) > H(N_2)$*,* $N_1 \succ N_2$;
if $H(N_1) = H(N_2)$*,* $N_1 \sim N_2$.

Example 3. *Let* $S = \{s_0, s_1, \cdots, s_{2\tau}\} = \{s_0, s_1, s_2, s_3, s_4, s_5, s_6\}$ *be a linguistic term set,* $N_1 = \langle s_1, \{0.3, 0.4\}, \{0.5\}, \{0.7, 0.8\}\rangle$ *and* $N_2 = \langle s_4, \{0.4, 0.6\}, \{0.2, 0.3\}, \{0.8\}\rangle$ *are two NHFLNs, if the linguistic scale function* $\varphi^*(s_i) = \varphi_2^*(s_i)$ $(a = 1.4, \tau = 3)$*, then*

$$S(N_1) = \frac{1.4^3 - 1.4^2}{3 * (2 * 1.4^3 - 2)} * (0.35 + 0.5 + 0.25) = 0.1497;$$

$$S(N_2) = \frac{1.4^3 + 1.4 - 2}{3 * (2 * 1.4^3 - 2)} * (0.5 + 0.75 + 0.2) = 0.2971;$$

According to Definition 8, we have $N_1 \prec N_2$.

4. Cosine Distance and Similarity Measures between NHFLSs

In order to introduce a new cosine distance measure between NHFLSs, we first review the relationship between the similarity measure and the distance measure as follows.

If a similarity measure $\rho(\alpha, \beta)$ between any two fuzzy sets α and β satisfies the following properties:

(p1) $0 \le \rho(\alpha, \beta) \le 1$;
(p2) $\rho(\alpha, \beta) = 1$ if and only if $\alpha = \beta$;
(p3) $\rho(\alpha, \beta) = \rho(\beta, \alpha)$,

then the similarity measure $\rho(\alpha, \beta)$ is a genuine similarity, and the relationship between the distance measure $d(\alpha, \beta)$ and similarity measure $\rho(\alpha, \beta)$ is:

$$d(\alpha, \beta) = 1 - \rho(\alpha, \beta). \quad (4)$$

That is to say, if the similarity measure $\rho(\alpha, \beta)$ satisfies the above three conditions (p1), (p2) and (p3), the corresponding distance measure can be obtained by Equation (4).

If we use the method in [39,40] to define the cosine similarity measure between NHFLSs, we can see the similarity measure is not a genuine similarity measure, which can be seen from the following example.

Example 4. *Let $S = \{s_0, s_1, \cdots, s_6\}(\tau = 3)$ be the linguistic term set, and assume $\alpha = \langle s_2, \{0.1, 0.2, 0.3\}\rangle$ and $\beta = \langle s_4, \{0.2, 0.4, 0.6\}\rangle$ are two NHFLNs. If the linguistic scale function $\varphi^*(s_i) = \frac{i}{2\tau}$, using the method of defining cosine similarity measure $Cos(\alpha, \beta)$ in [24,25], we have*

$$Cos(\alpha, \beta) = \frac{\frac{2}{2*3} * \frac{4}{2*3} * (\frac{0.1*0.2+0.2*0.4+0.3*0.6}{7*7})}{(\frac{0.1^2+0.2^2+0.3^2}{9*7*7})^{\frac{1}{2}} (\frac{4*(0.2^2+0.4^2+0.6^2)}{9*7*7})^{\frac{1}{2}}} = 1.$$

Obviously, the two NHFLNs α and β are not equal, that is to say the condition (p2) is not satisfied, so $Cos(\alpha, \beta)$ is not a genuine similarity measure. That is to say, the similarity measure defined by their methods is not suitable here. So we propose a new method to construct a new similarity measure which satisfies (p1)–(p3), which is based on the widely distance measures such as the Hamming distance, Euclidean distance and generalized distance [43–45], and we can obtain the corresponding distance measure by Equation(4). In order to introduce the new similarity measure, we first propose the generalized neutrosophic hesitant fuzzy linguistic weighted distance measure $d_{gnhlw}(N_1, N_2)$ as follows.

4.1. Distance Measures between NHFLSs

Definition 9. *When the universe of discourse $X = \{x_1, x_2, \cdots, x_m\}$, let $N_1 =< (x_i, s_{N_1}(x_i)), T_{N_1}(x_i), I_{N_1}(x_i), F_{N_1}(x_i)|x_i \in X >$ and $N_2 =< (x_i, s_{N_2}(x_i)), T_{N_2}(x_i), I_{N_2}(x_i), F_{N_2}(x_i)|x_i \in X >$ be any two NHFLSs in $X = \{x_1, x_2, \cdots, x_m\}$, the associated weighting vector $\omega = (\omega_1, \omega_2, \cdots, \omega_m)^T$ satisfies $\sum_{i=1}^m \omega_i = 1(\omega_i \ge 0)$, and φ^* be a linguistic scale function, here we assume the cardinality of $T_{N_k}(x_i)$, $I_{N_k}(x_i)$ and $F_{N_k}(x_i)(k = 1, 2)$ are finite. Then a generalized neutrosophic hesitant fuzzy linguistic weighted distance measure $d_{gnhlw}(N_1, N_2)$ can be defined as:*

$$d_{gnhlw}(N_1, N_2) = [\sum_{i=1}^m \omega_i (\frac{1}{3}(\frac{1}{l_1}\sum_{j=1}^{l_1} |\varphi^*(s_{N_1}(x_i))\alpha_{1j}(x_i) - \varphi^*(s_{N_2}(x_i))\alpha_{2j}(x_i)|^\lambda + \frac{1}{l_2}\sum_{j=1}^{l_2} |\varphi^*(s_{N_1}(x_i)) \beta_{1j}(x_i) - \varphi^*(s_{N_2}(x_i))\beta_{2j}(x_i)|^\lambda + \frac{1}{l_3}\sum_{j=1}^{l_3} |\varphi^*(s_{N_1}(x_i))\gamma_{1j}(x_i) - \varphi^*(s_{N_2}(x_i))\gamma_{2j}(x_i)|^\lambda))]^{\frac{1}{\lambda}},$$

where $\alpha_{kj}(x_i) \in T_{N_k}(x_i)$, $\beta_{kj}(x_i) \in I_{N_k}(x_i)$, $\gamma_{kj}(x_i) \in F_{N_k}(x_i)(k=1,2)$, l_1, l_2, l_3 are the maximum number of elements in $T_{N_k}(x_i), I_{N_k}(x_i), F_{N_k}(x_i)(i=1,2,\cdots,m)$ respectively.

If $\lambda = 1$, $d_{gnhlw}(N_1, N_2)$ reduces to the neutrosophic hesitant fuzzy linguistic weighted Hamming distance d_{hnhlw}.

$$d_{hnhlw}(N_1,N_2) = \sum_{i=1}^{m}\omega_i(\frac{1}{3}(\frac{1}{l_1}\sum_{j=1}^{l_1}|\varphi^*(s_{N_1}(x_i))\alpha_{1j}(x_i) - \varphi^*(s_{N_2}(x_i))\alpha_{2j}(x_i)| + \frac{1}{l_2}\sum_{j=1}^{l_2}|\varphi^*(s_{N_1}(x_i))$$

$$\beta_{1j}(x_i) - \varphi^*(s_{N_2}(x_i))\beta_{2j}(x_i)| + \frac{1}{l_3}\sum_{j=1}^{l_3}|\varphi^*(s_{N_1}(x_i))\gamma_{1j}(x_i) - \varphi^*(s_{N_2}(x_i))\gamma_{2j}(x_i)|)).$$

If $\lambda = 2$, $d_{gnhlw}(N_1, N_2)$ reduces to the neutrosophic hesitant fuzzy linguistic weighted Euclidean distance d_{enhlw}.

$$d_{enhlw}(N_1,N_2) = [\sum_{i=1}^{m}\omega_i(\frac{1}{3}(\frac{1}{l_1}\sum_{j=1}^{l_1}|\varphi^*(s_{N_1}(x_i))\alpha_{1j}(x_i) - \varphi^*(s_{N_2}(x_i))\alpha_{2j}(x_i)|^2 + \frac{1}{l_2}\sum_{j=1}^{l_2}|\varphi^*(s_{N_1}(x_i))$$

$$\beta_{1j}(x_i) - \varphi^*(s_{N_2}(x_i))\beta_{2j}(x_i)|^2 + \frac{1}{l_3}\sum_{j=1}^{l_3}|\varphi^*(s_{N_1}(x_i))\gamma_{1j}(x_i) - \varphi^*(s_{N_2}(x_i))\gamma_{2j}(x_i)|^2))]^{\frac{1}{2}}.$$

Example 5. *Let $X = \{x_1, x_2\}$, $S = \{s_0, s_1, \cdots, s_{2\tau}\} = \{s_0, s_1, s_2, s_3, s_4, s_5, s_6\}$ be a linguistic term set, $N_1 = \{\langle(x_1, s_2), \{0.1, 0.2, 0.3\}, \{0.3, 0.5\}, \{0.3, 0.4\}\rangle, \langle(x_2, s_3), \{0.2, 0.3, 0.4\}, \{0.4, 0.5\}, \{0.3, 0.5\}\rangle\}$ and $N_2 = \{\langle(x_1, s_4), \{0.2, 0.3, 0.4\}, \{0.5, 0.7\}, \{0.2, 0.3\}\rangle, \{(x_2, s_5), \{0.1, 0.3, 0.4\}, \{0.6, 0.7\}, \{0.3, 0.6\}\rangle\}$ are two NHFLSs, if the linguistic scale function $\varphi^*(s_i) = \frac{i}{2\tau}$, if $\lambda = 2$, the weight vector of $\{x_1, x_2\}$ is $\omega = (0.6, 0.4)$, then the distance measure $d_{enhlw}(N_1, N_2) = 0.1922$.*

The above generalized neutrosophic hesitant fuzzy linguistic weighted distance measure d_{gnhlw} can be used to define the new similarity measure.

Remark 1. *If the associated weighting vector $\omega = (\omega_1, \omega_2, \cdots, \omega_m)^T = (\frac{1}{m}, \frac{1}{m}, \cdots, \frac{1}{m})^T$, the generalized neutrosophic hesitant fuzzy linguistic weighted distance measure $d_{gnhlw}(N_1, N_2)$ is reduced to the generalized neutrosophic hesitant fuzzy linguistic distance measure $d_{gnhl}(N_1, N_2)$:*

$$d_{gnhl}(N_1,N_2) = [\sum_{i=1}^{m}(\frac{1}{m}(\frac{1}{l_1}\sum_{j=1}^{l_1}|\varphi^*(s_{N_1}(x_i))\alpha_{1j}(x_i) - \varphi^*(s_{N_2}(x_i))\alpha_{2j}(x_i)|^\lambda + \frac{1}{l_2}\sum_{j=1}^{l_2}|\varphi^*(s_{N_1}(x_i))\beta_{1j}(x_i)$$

$$-\varphi^*(s_{N_2}(x_i))\beta_{2j}(x_i)|^\lambda + \frac{1}{l_3}\sum_{j=1}^{l_3}|\varphi^*(s_{N_1}(x_i))\gamma_{1j}(x_i) - \varphi^*(s_{N_2}(x_i))\gamma_{2j}(x_i)|^\lambda))]^{\frac{1}{\lambda}},$$

where $\alpha_{kj}(x_i) \in T_{N_k}(x_i)$, $\beta_{kj}(x_i) \in I_{N_k}(x_i)$, $\gamma_{kj}(x_i) \in F_{N_k}(x_i)(k=1,2)$, l_1, l_2, l_3 are the maximum number of elements in $T_{N_k}(x_i), I_{N_k}(x_i), F_{N_k}(x_i)(i=1,2,\cdots,m)$ respectively.

Next we propose the properties of the distance $d_{gnhlw}(N_1, N_2)$ between any two NHFLSs.

Theorem 2. *Let $N_1 =< (x_i, s_{N_1}(x_i)), T_{N_1}(x_i), I_{N_1}(x_i), F_{N_1}(x_i)|x_i \in X >$ and $N_2 =< (x_i, s_{N_2}(x_i)), T_{N_2}(x_i), I_{N_2}(x_i), F_{N_2}(x_i)|x_i \in X >$ be any two NHFLSs in $X = \{x_1, x_2, \cdots, x_m\}$, and let φ^* be a linguistic scale function, then the distance measure between $d_{gnhlw}(N_1, N_2)$ satisfies the following properties:*

(1) $0 \leq d_{gnhlw}(N_1, N_2) \leq 1$;
(2) $d_{gnhlw}(N_1, N_2) = 0$ if and only if $N_1 = N_2$;
(3) $d_{gnhlw}(N_1, N_2) = d_{g\omega nhl}(N_2, N_1)$;

Proof. Obviously, according to Definition 9, (1), (2) and (3) are correct, we omit the proof here. □

4.2. Similarity Measures between NHFLSs

It is already known that the cosine similarity measures constructed by Liao et al. [39] and Ye [40] do not satisfy the condition of (p2), they are not genuine similarity measures. Therefore, their method of constructing similar measures is not applicable here. In this subsection we propose a new method to construct a similarity measure between NHFLSs, which not only satisfies the axiom of similarity measure, but also includes the linguistic scale function φ^*. Furthermore, we can obtain the distance measure according to the constructed similarity measure. The similarity measure between any two NHFLSs can be defined as follows:

Definition 10. *Let $S = \{s_0, s_1, \cdots, s_{2\tau}\}$ be the linguistic term set, $N_1 =< (x_i, s_{N_1}(x_i)), T_{N_1}(x_i), I_{N_1}(x_i), F_{N_1}(x_i)|x_i \in X >$ and $N_2 =< (x_i, s_{N_2}(x_i)), T_{N_2}(x_i), I_{N_2}(x_i), F_{N_2}(x_i)|x_i \in X >$ be any two NHFLSs in $X = \{x_1, x_2, \cdots, x_m\}$, the associated weighting vector $\omega = (\omega_1, \omega_2, \cdots, \omega_m)^T$, satisfies $\sum_{i=1}^m \omega_i = 1$ ($\omega_i \geq 0$), and let φ^* be a linguistic scale function. Then a new weighted similarity measure S_{nhlw} between two NHFLSs N_1 and N_2 can be defined as:*

$$S_{nhlw}(N_1, N_2) = \frac{Cos(N_1, N_2) + 1 - d_{gnhlw}(N_1, N_2)}{2},$$

where $Cos(N_1, N_2) = \dfrac{\sum_{i=1}^m \omega_i \varphi^*(s_{N_1}(x_i))\varphi^*(s_{N_2}(x_i)) \frac{\sum_{j=1}^{l_1} \alpha_{1j}(x_i)\alpha_{2j}(x_i) + \sum_{j=1}^{l_2} \beta_{1j}(x_i)\beta_{2j}(x_i) + \sum_{j=1}^{l_3} \gamma_{1j}(x_i)\gamma_{2j}(x_i)}{(2\tau+1)(2\tau+1)}}{H_1 \bullet H_2}$,

$$H_1 = \left(\sum_{i=1}^m \omega_i (\varphi^*(s_{N_1}(x_i)))^2 * \frac{\sum_{j=1}^{l_1}(\alpha_{1j}(x_i))^2 + \sum_{j=1}^{l_2}(\beta_{1j}(x_i))^2 + \sum_{j=1}^{l_3}(\gamma_{1j}(x_i))^2}{(2\tau+1)^2}\right)^{\frac{1}{2}},$$

$$H_2 = \left(\sum_{i=1}^m \omega_i (\varphi^*(s_{N_2}(x_i)))^2 * \frac{\sum_{j=1}^{l_1}(\alpha_{2j}(x_i))^2 + \sum_{j=1}^{l_2}(\beta_{2j}(x_i))^2 + \sum_{j=1}^{l_3}(\gamma_{2j}(x_i))^2}{(2\tau+1)^2}\right)^{\frac{1}{2}};$$

$\alpha_{kj}(x_i) \in T_{N_k}(x_i)$, $\beta_{kj}(x_i) \in I_{N_k}(x_i)$, $\gamma_{kj}(x_i) \in F_{N_k}(x_i) (k = 1, 2)$, l_1, l_2, l_3 are the maximum number of elements in $T_{N_k}(x_i), I_{N_k}(x_i), F_{N_k}(x_i) (i = 1, 2, \cdots, m)$, respectively.

Next we give a concrete example to calculate the similarity measure $S_{nhlw}(N_1, N_2)$ between NHFLSs.

Example 6. *Let $X = \{x_1, x_2\}$, $S = \{s_0, s_1, \cdots, s_{2\tau}\} = \{s_0, s_1, s_2, s_3, s_4, s_5, s_6\}$ be a linguistic term set, $N_1 = \{\langle(x_1, s_2), \{0.1, 0.2, 0.3\}, \{0.3, 0.5\}, \{0.3, 0.4\}\rangle, \langle(x_2, s_3), \{0.2, 0.3, 0.4\}, \{0.4, 0.5\}, \{0.3, 0.5\}\rangle\}$ and $N_2 = \{\langle(x_1, s_4), \{0.2, 0.3, 0.4\}, \{0.5, 0.7\}, \{0.2, 0.3\}\rangle, \{(x_2, s_5), \{0.1, 0.3, 0.4\}, \{0.6, 0.7\}, \{0.3, 0.6\}\rangle\}$ are two NHFLSs, if the linguistic scale function $\varphi^*(s_i) = \frac{i}{2\tau}$, if $\lambda = 2$, the weight vector of $\{x_1, x_2\}$ is $\omega = (0.6, 0.4)$, then the new weighted similarity measure $S_{nhlw}(N_1, N_2) = 0.4730$.*

Furthermore we can prove that the similarity measure satisfies the axiom of similarity measure, so we can obtain a corresponding distance measure by the relationship of distance measure and similarity measure.

Theorem 3. *The similarity measure S_{nhlw} between any two NHFLSs N_1 and N_2 satisfies the following properties:*

(1) $0 \leq S_{nhlw}(N_1, N_2) \leq 1$;
(2) $S_{nhlw}(N_1, N_2) = 1$ *if and only if* $N_1 = N_2$;
(3) $S_{nhlw}(N_1, N_2) = S_{nhlw}(N_2, N_1)$.

Proof. (1) $Cos(N_1, N_2)$ is the cosine value, so $0 \leq Cos(N_1, N_2) \leq 1$. According to Theorem 2, one easily sees $0 \leq 1 - d_{gnhlw}(N_1, N_2) \leq 1$, then we have $0 \leq S_{nhlw}(N_1, N_2) \leq 1$.

(2) If $N_1 = N_2$, $d_{gnhlw}(N_1, N_2) = 0$ and $Cos(N_1, N_2) = 1$ are exist, we can get $S_{nhlw}(N_1, N_2) = \frac{1+1-0}{2} = 1$. On the other hand, since $0 \leq Cos(N_1, N_2) \leq 1$ and $0 \leq 1 - d_{gnhlw}(N_1, N_2) \leq 1$, if $S_{nhlw}(N_1, N_2) = 1$, $Cos(N_1, N_2) = 1$ and $d_{gnhlw}(N_1, N_2) = 0$ must hold simultaneously. By Theorem 2, we can obtain $N_1 = N_2$.

(3) According to Definition 10, $S_{nhlw}(N_1, N_2) = S_{nhlw}(N_2, N_1)$ can be obtained directly.

The constructed similarity measure $S_{nhlw}(N_1, N_2)$ satisfies the properties in Theorem 3, it is a genuine similarity measure. According to Equation (4), the neutrosophic hesitant fuzzy linguistic cosine distance measure $d^*_{nhlw}(N_1, N_2)$ between any two NHFLSs N_1 and N_2 can be given as

$$d^*_{nhlw}(N_1, N_2) = \frac{1 - Cos(N_1, N_2) + d_{gnhlw}(N_1, N_2)}{2}.$$

One easily sees the neutrosophic hesitant fuzzy linguistic cosine distance measure $d^*_{nhlw}(N_1, N_2)$ satisfies the properties in Theorem 2, that is to say, it satisfies the axiom of distance measure. □

5. Multiple Criteria Decision Making With Distance Measure Based on TOPSIS Approach

In this section, we apply the proposed distance measure d^*_{nhlw} between NHFLSs to develop a multiple criteria decision making (MCDM) method. Suppose the decision makers decide to choose the best one from k alternatives according to m criteria. Let $A = \{A_1, A_2, \cdots, A_k\}$ be a set of alternatives and $C = \{C_1, C_2, \cdots, C_m\}$ be a set of criteria, a number of decision makers provide their evaluation values with NHFLSs $N_{ji} = \langle < (x_i, s_{N_j}(x_i)), T_{N_j}(x_i), I_{N_j}(x_i), F_{N_j}(x_i) | x_i \in C \rangle (j = 1, 2, \cdots, k; i = 1, 2, \cdots, m)$ for alternative A_j with respect to C_i. The weight of criteria C_i is ω_i, satisfying $\omega_i \geq 0 (i = 1, 2, \cdots, m)$ and $\sum_{i=1}^{m} \omega_i = 1$, the decision matrix D with NHFLSs is formed as follows:

$$D = \begin{pmatrix} N_{11} & N_{12} & N_{13} & \cdots & N_{1m} \\ N_{21} & N_{22} & N_{23} & \cdots & N_{2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ N_{k1} & N_{k2} & N_{k3} & \cdots & N_{km} \end{pmatrix},$$

where $N_{ji} (j = 1, 2, \cdots, k; i = 1, 2, \cdots, m)$ are NHFLSs.

In the following, we extend the TOPSIS method to the proposed distance measure. The multiple criteria decision making method involves the following steps:

Step 1 When the number of elements of NHFLSs about the truth-membership degree, the indeterminacy-membership degree and the falsity-membership degree are different, we can extend the fewer one by adding the minimum value respectively until they have the same number as the greater one; the new decision matrix D' is obtained.

Step 2 Normalize the decision matrix D'.

For the benefit-type criterion, we do not do anything. For the cost-type criterion, we should use the negation operator in Definition 7 to make NHFLNs normalized.

Step 3 For each alternative under different criteria, using the comparison rule defined in Definition 8, the positive ideal solution α^+ and the negative ideal solution α^- can be obtained as follows:

$$\alpha^+ = \{N_1^+, N_2^+, \cdots, N_m^+\}, N_i^+ = max\{S(N_{1i}), S(N_{2i}), \cdots, S(N_{ki})\}, \quad (i = 1, 2, \cdots, m).$$

$$\alpha^- = \{N_1^-, N_2^-, \cdots, N_m^-\}, N_i^- = min\{S(N_{1i}), S(N_{2i}), \cdots, S(N_{ki})\}, \quad (i = 1, 2, \cdots, m),$$

where $S(\cdot)$ is the score function of NHFLN defined by Definition 8.

Step 4 Utilizing the proposed distance measure d^*_{nhlw} to calculate the separation of each alternative between the positive ideal solution and the negative ideal solution. Here the separation measure between $A_j (j = 1, 2, \cdots, k)$ and α^+ is: $d_j(A_j, \alpha^+) = \sum_{i=1}^{m} \omega_i d^*_{nhlw}(N_{ji}, \alpha^+)$; The separation measure between $A_j (j = 1, 2, \cdots, k)$ and α^- is: $d_j(A_j, \alpha^-) = \sum_{i=1}^{m} \omega_i d^*_{nhlw}(N_{ji}, \alpha^-)$. Then we calculate the closeness coefficient of alternative R_j, that is

$$R_j = \frac{d_j(A_j, \alpha^+)}{d_j(A_j, \alpha^-) + d_j(A_j, \alpha^+)}, \qquad (j = 1, 2, \cdots, k).$$

Step 5 Rank all the alternatives according to the closeness coefficient R_j, the smaller closeness coefficient $R_j(j = 1, 2, \cdots, k)$, the better $A_j(j = 1, 2, \cdots, k)$ will be.

6. Illustrative Example

6.1. Background

The following background is adapted from Liu [53].

An investment company wants to invest its money in the best option. After a careful investigation, four alternatives $\{A_1, A_2, A_3, A_4\}$ are considered. The panel evaluates the alternatives from the following three factors C_1: risk, C_2: growth, C_3: environmental impact. Suppose the weight vector of $C_i(i = 1, 2, 3)$ is $\omega = \{\omega_1, \omega_2, \omega_3\} = (0.35, 0.25, 0.4)$. The panel makes evaluation by the linguistic term set $S = \{s_0 = very\ poor, s_1 = poor, s_2 = slightly\ poor, s_3 = fair, s_4 = slightly\ good, s_5 = good, s_6 = very\ good\}$. The neutrosophic hesitant fuzzy linguistic evaluation information N_{ji} is given in Table 1.

Table 1. The neutrosophic hesitant fuzzy linguistic decision matrix.

	C_1	C_2	C_3
A_1	$\langle s_4, \{0.4, 0.5\}, \{0.2\}, \{0.3\}\rangle$	$\langle s_5, \{0.4\}, \{0.2, 0.3\}, \{0.3\}\rangle$	$\langle s_3, \{0.2\}, \{0.2\}, \{0.5\}\rangle$
A_2	$\langle s_5, \{0.6\}, \{0.1, 0.2\}, \{0.2\}\rangle$	$\langle s_5, \{0.6\}, \{0.1\}, \{0.2\}\rangle$	$\langle s_5, \{0.5\}, \{0.2\}, \{0.1, 0.2\}\rangle$
A_3	$\langle s_4, \{0.3, 0.4\}, \{0.2\}, \{0.3\}\rangle$	$\langle s_5, \{0.5\}, \{0.2\}, \{0.3\}\rangle$	$\langle s_4, \{0.5\}, \{0.2, 0.3\}, \{0.2\}\rangle$
A_4	$\langle s_5, \{0.7\}, \{0.1, 0.2\}, \{0.1\}\rangle$	$\langle s_4, \{0.6\}, \{0.1\}, \{0.2\}\rangle$	$\langle s_4, \{0.6\}, \{0.3\}, \{0.2\}\rangle$

Where $N_{ji}(j = 1, 2, 3, 4; i = 1, 2, 3)$ is the characteristic information of the alternative A_j with respect to the criteria C_i.

6.2. An Illustration of the Proposed Method

At first, we assume the linguistic scale function $\varphi^* = \varphi_1(s_i) = \frac{i}{2\tau}(\tau = 3)$, the method proposed in Section 4 will be used to identify the best alternative.

Step 1 We extend the fewer elements of NHFLSs about the truth-membership degree, the indeterminacy-membership degree and the falsity-membership degree by adding the minimum value until they have the same number with the more one, respectively. For example, we can extend $\langle s_4, \{0.4, 0.5\}, \{0.2\}, \{0.3\}\rangle$ to $\langle s_4, \{0.4, 0.5\}, \{0.2, 0.2\}, \{0.3, 0.3\}\rangle$ by adding the minimum value 0.2, 0.3 respectively. Then Table 2 is transformed as follows:

Table 2. The transformation of neutrosophic hesitant fuzzy linguistic decision matrix.

	C_1	C_2	C_3
A_1	$\langle s_4, \{0.4, 0.5\}, \{0.2, 0.2\}, \{0.3, 0.3\}\rangle$	$\langle s_5, \{0.4, 0.4\}, \{0.2, 0.3\}, \{0.3, 0.3\}\rangle$	$\langle s_3, \{0.2, 0.2\}, \{0.2, 0.2\}, \{0.5, 0.5\}\rangle$
A_2	$\langle s_5, \{0.6, 0.6\}, \{0.1, 0.2\}, \{0.2, 0.2\}\rangle$	$\langle s_5, \{0.6, 0.6\}, \{0.1, 0.1\}, \{0.2, 0.2\}\rangle$	$\langle s_1, \{0.5, 0.5\}, \{0.2, 0.2\}, \{0.1, 0.2\}\rangle$
A_3	$\langle s_4, \{0.3, 0.4\}, \{0.2, 0.2\}, \{0.3, 0.3\}\rangle$	$\langle s_5, \{0.5, 0.5\}, \{0.2, 0.2\}, \{0.3, 0.3\}\rangle$	$\langle s_2, \{0.5, 0.5\}, \{0.2, 0.3\}, \{0.2, 0.2\}\rangle$
A_4	$\langle s_5, \{0.7, 0.7\}, \{0.1, 0.2\}, \{0.1, 0.1\}\rangle$	$\langle s_2, \{0.6, 0.6\}, \{0.1, 0.1\}, \{0.2, 0.2\}\rangle$	$\langle s_4, \{0.6, 0.6\}, \{0.3, 0.3\}, \{0.2, 0.2\}\rangle$

Step 2 Normalize the transformation of neutrosophic hesitant fuzzy linguistic decision matrix.

Because the criterion C_3 is the cost-type criterion, we use the negation operator in Definition 7 to make the transformation of neutrosophic hesitant fuzzy linguistic decision matrix normalized, which is shown in Table 3:

Table 3. The normalized neutrosophic hesitant fuzzy linguistic decision matrix.

	C_1	C_2	C_3
A_1	$\langle s_4, \{0.4, 0.5\}, \{0.2, 0.2\}, \{0.3, 0.3\}\rangle$	$\langle s_5, \{0.4, 0.4\}, \{0.2, 0.3\}, \{0.3, 0.3\}\rangle$	$\langle s_3, \{0.5, 0.5\}, \{0.8, 0.8\}, \{0.2, 0.2\}\rangle$
A_2	$\langle s_5, \{0.6, 0.6\}, \{0.1, 0.2\}, \{0.2, 0.2\}\rangle$	$\langle s_5, \{0.6, 0.6\}, \{0.1, 0.1\}, \{0.2, 0.2\}\rangle$	$\langle s_1, \{0.1, 0.2\}, \{0.8, 0.8\}, \{0.5, 0.5\}\rangle$
A_3	$\langle s_4, \{0.3, 0.4\}, \{0.2, 0.2\}, \{0.3, 0.3\}\rangle$	$\langle s_5, \{0.5, 0.5\}, \{0.2, 0.2\}, \{0.3, 0.3\}\rangle$	$\langle s_2, \{0.2, 0.2\}, \{0.7, 0.8\}, \{0.5, 0.5\}\rangle$
A_4	$\langle s_5, \{0.7, 0.7\}, \{0.1, 0.2\}, \{0.1, 0.1\}\rangle$	$\langle s_4, \{0.6, 0.6\}, \{0.1, 0.1\}, \{0.2, 0.2\}\rangle$	$\langle s_2, \{0.2, 0.2\}, \{0.7, 0.7\}, \{0.6, 0.6\}\rangle$

Step 3 Determine the positive ideal solution α^+ and negative ideal solution α^- by Definition 8, we have:

$$\alpha^+ = \{\langle s_5, \{0.7, 0.7\}, \{0.1, 0.2\}, \{0.1, 0.1\}\rangle, \langle s_5, \{0.6, 0.6\}, \{0.1, 0.1\}, \{0.2, 0.2\}\rangle, \langle s_3, \{0.2, 0.2\}, \{0.2, 0.2\}, \{0.5, 0.5\}\rangle\},$$

$$\alpha^- = \{\langle s_4, \{0.3, 0.4\}, \{0.2, 0.2\}, \{0.3, 0.3\}\rangle, \langle s_4, \{0.6, 0.6\}, \{0.1, 0.1\}, \{0.2, 0.2\}\rangle, \langle s_5, \{0.5, 0.5\}, \{0.2, 0.2\}, \{0.1, 0.2\}\rangle\}.$$

Step 4 It is already known the weight vector of $C_i (i = 1, 2, 3)$ is $\omega = \{\omega_1, \omega_2, \omega_3\} = (0.35, 0.25, 0.4)$. For different λ, we utilize the cosine distance measure d^*_{nhlw} to calculate the separation distance measure between A_i and $\alpha^+ (i = 1, 2, 3, 4)$, and calculate the closeness coefficient of alternative A_j. Then the closeness coefficient R_j is given in Table 4:

Table 4. Decision results based on closeness coefficient R_j for $f_1(s_i)$.

λ	R_1	R_2	R_3	R_4	Ranking
1	0.6492	0.2028	0.5017	0.3694	$A_2 \succ A_4 \succ A_3 \succ A_1$
2	0.6182	0.2523	0.4945	0.3134	$A_2 \succ A_4 \succ A_3 \succ A_1$
3	0.6043	0.2615	0.4896	0.2958	$A_2 \succ A_4 \succ A_3 \succ A_1$
5	0.5941	0.2698	0.4920	0.2864	$A_2 \succ A_4 \succ A_3 \succ A_1$
7	0.5902	0.2747	0.4959	0.2842	$A_2 \succ A_4 \succ A_3 \succ A_1$
9	0.5879	0.2775	0.4989	0.2836	$A_2 \succ A_4 \succ A_3 \succ A_1$

Step 5 Rank the alternatives according to $R_j (j = 1, 2, \cdots, 4)$.

For different λ, we can see that the ranking order of alternatives is $A_2 \succ A_4 \succ A_3 \succ A_1$, and the best alternative is A_2.

To illustrate the influence of the linguistic scale function φ^* on the decision making method, we utilize different linguistic scale functions in the distance measure d^*_{nhlw}.

Let

$$\varphi^* = \varphi_2(s_i) = \begin{cases} \frac{a^\tau - a^{\tau-i}}{2a^\tau - 2}, & i = 0, 1, \cdots, t, \\ \frac{a^\tau + a^{i-\tau} - 2}{2a^\tau - 2}, & i = \tau + 1, \tau + 2, \cdots, 2\tau, \end{cases}$$

and $a = 1.37, \tau = 3$, the ranking results are listed in Table 5.

Let

$$\varphi^* = \varphi_3(s_i) = \begin{cases} \frac{\tau^{\alpha'} - (\tau - i)^{\alpha'}}{2\tau^{\alpha'}}, & i = 0, 1, \cdots, \tau, \\ \frac{\tau^{\beta'} - (i - \tau)^{\beta'}}{2\tau^{\beta'}}, & i = \tau + 1, \tau + 2, \cdots, 2\tau, \end{cases}$$

and $\alpha' = \beta' = 0.88, \tau = 3$, then the ranking results are listed in Table 6.

From Tables 5 and 6, we can see that the most desirable alternative is A_2, the ranking results are stable.

Table 5. Decision results based on closeness coefficient R_j for $\varphi_2(s_i)$.

λ	R_1	R_2	R_3	R_4	Ranking
1	0.6553	0.2146	0.5029	0.3824	$A_2 \succ A_4 \succ A_3 \succ A_1$
2	0.6361	0.2182	0.4925	0.3809	$A_2 \succ A_4 \succ A_3 \succ A_1$
3	0.6264	0.2162	0.4920	0.3857	$A_2 \succ A_4 \succ A_3 \succ A_1$
5	0.6163	0.2168	0.4941	0.3815	$A_2 \succ A_4 \succ A_3 \succ A_1$
7	0.6113	0.2193	0.4965	0.3814	$A_2 \succ A_4 \succ A_3 \succ A_1$
9	0.6082	0.2214	0.4986	0.3813	$A_2 \succ A_4 \succ A_3 \succ A_1$

Table 6. Decision results based on closeness coefficient R_j for $\varphi_3(s_i)$.

λ	R_1	R_2	R_3	R_4	Ranking
1	0.6467	0.2014	0.5001	0.3647	$A_2 \succ A_4 \succ A_3 \succ A_1$
2	0.6290	0.2157	0.4764	0.3423	$A_2 \succ A_4 \succ A_3 \succ A_1$
3	0.6222	0.2206	0.4709	0.3308	$A_2 \succ A_4 \succ A_3 \succ A_1$
5	0.6169	0.2277	0.4732	0.3213	$A_2 \succ A_4 \succ A_3 \succ A_1$
7	0.6148	0.2324	0.4766	0.3177	$A_2 \succ A_4 \succ A_3 \succ A_1$
9	0.6136	0.2353	0.4792	0.3160	$A_2 \succ A_4 \succ A_3 \succ A_1$

It can be seen from the decision matrix of Table 1, the alternative A_2 is evaluated by "good" from the attributes of growth and environmental aspect, and the degree of possibility is 0.5, but the degree of impossibility and hesitation is relative low. Compared with the other three alternatives, although some alternative is evaluated as "very good" in a certain attribute, the possibility of membership is relatively low. The results calculated by the proposed method are consistent with the subjectiveness of the decison makers, which also explains the effectiveness of the proposed method to some extent.

6.3. Comparison Analysis with Existing Method

In order to illustrative the effectiveness of the proposed method, we make a comparison analysis with the existing methods based on the same illustrative example in this subsection.

In Liu et al. [53], the best alternative is A_4; it is slightly different from the results obtained in this paper. The reason is that the score function defined by Liu et al. [53] only focuses on the degree of neutrosophic hesitant fuzzy numbers, which does not consider the relationship of different criteria. This inevitably leads to information loss in the decision making process.

In the method proposed by Wang et al. [54], the best alternative is A_4. In the TODIM method used by Wang, the decision maker should select the preference parameters, which have a certain influence on the ranking result.

Furthermore, in the method proposed by Ye [55] we apply the similarity defined by him to calculate the numerical example, the best alternative A_2 is obtained. It is the same as the proposed method in this paper, because the similarity measure defined by Ye [55] is also based on the distance measure, but he considers it only from the point of view of algebra.

Compared with the above methods, the proposed method about neutrosophic hesitant fuzzy linguistic set can express decision information more widely, which is more meaningful in representing practical examples. Furthermore, the advantages of the approach can be given in the following:

(1) It not only considers the relationship of different criteria, but also provides a new way for processing the inconsistent and indeterminate linguistic decision information.
(2) The proposed cosine distance measure between NHFLSs is defined based on linguistic scale function φ^*, the decision makers can flexibly select the linguistic scale function φ^* on the basis of their preferences. This avoids the information distort in the decision making process.
(3) The proposed cosine distance measure between NHFLSs is constructed based on the generalized distance measure and cosine similarity measure, which can solve the decision information with NHFLSs not only from the point view of algebra but also from the point view of geometry.

6.4. Managerial Implications

According to the results of the practical example, some managerial implications are derived. It is worth noting that the decision makers mostly pay attention to the attribute of environmental impact when considering investment decisions. Therefore, in order to promote the formation of investment plans, the managers of the alternative should stress its environmental performance, because it affects the sustainable development of the alternative. Besides, The criteria of risk and growth of the alternative are almost as important to the decision makers.

Moreover, managers should conduct certain environmental strategies in the process of alternative development, such as reducing costs in product manufacturing and reducing pollution sources in the production process. The major objective in such cases is to carry out green production. Of course, if managers expect the alternative to reach high performance, they should also pay attention to controlling the risk of the alternative to maximize its production efficiency.

7. Conclusions

In this paper, we introduce the neutrosophic hesitant fuzzy linguistic set and consider its application of multiple criteria decision making. Some basic operational laws of neutrosophic hesitant fuzzy linguistic sets are defined based on linguistic scale function, and a new method was proposed for constructing a similarity measure between neutrosophic hesitant fuzzy linguistic sets based on the cosine similarity measure and generalized Euclidean distance measure. Furthermore, we extend the TOPSIS method to the corresponding cosine distance measure between neutrosophic hesitant fuzzy linguistic set for decision making problems.

The main contributions of the proposed neutrosophic hesitant fuzzy linguistic set is defined on linguistic scale function; the decision makers can flexibly convert the linguistic information to semantic values, and the proposed cosine distance measure between neutrosophic hesitant fuzzy linguistic sets with TOPSIS method can deal with the related decision information not only from the point of view of algebra, but also from the point of view of geometry.

In future research, we will continue studying the interval neutrosophic hesitant fuzzy set and its application in some fields. For example, measuring the environmental effects of transportation modes may be a complex process because of the different criteria of the subject from different aspects. Under certain conditions, there are many uncertainties in evaluating the environmental effects of transportation modes based on these criteria. The neutrosophic hesitant fuzzy linguistic set is ideal for solving these kinds of problems. Of course, it can also be applied to supply chain, pattern recognition and other decision making problems.

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