Existence Results for Second Order Nonconvex Sweeping Processes in $q$-Uniformly Convex and 2-Uniformly Smooth Separable Banach Spaces

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Abstract: We prove an existence result, in the separable Banach spaces setting, for second order differential inclusions of type sweeping process. This type of differential inclusion is defined in terms of normal cones and it covers many dynamic quasi-variational inequalities. In the present paper, we prove in the nonconvex case an existence result of this type of differential inclusions when the separable Banach space is assumed to be $q$-uniformly convex and 2-uniformly smooth. In our proofs we use recent results on uniformly generalized prox-regular sets. Part of the novelty of the paper is the use of the usual Lipschitz continuity of the set-valued mapping which is very easy to verify contrarily to the ones used in the previous works. An example is stated at the end of the paper, showing the application of our existence result.

Keywords: uniformly convex spaces; uniformly smooth spaces; generalized proximal normal cone; generalized uniformly prox-regularity; Nonconvex sweeping process

1. Introduction

In [1,2], the authors considered the following suitable extension of Second Order Convex Sweeping Processes (SSP) from the setting of Hilbert spaces $H$ to the setting of uniformly convex and uniformly smooth Banach spaces $X$:

$$\begin{align*}
\text{(SSP)} \quad & \text{Find } T > 0, x : I := [0, T] \to \overline{cl}(U_0), \text{ and } u : I \to X \text{ such that} \\
& \left\{ \begin{array}{ll}
\quad u(0) = u_0 \in K(x_0), \quad x_0 \in X; \\
\quad x(t) = x_0 + \int_0^t u(s)ds, \forall t \in I; \\
\quad u(t) \in K(x(t)), \forall t \in I \text{ and } -\frac{d}{dt} J(u(t)) \in N(K(x(t)); u(t)) \text{ a.e. on } I,
\end{array} \right.
\end{align*}$$

where $U_0$ is an open neighborhood of $x_0$ in $X$ and $K : \overline{cl}(U_0) \rightrightarrows X$ is a convex-valued mapping in $X$. Here, $J : X \rightrightarrows X^*$ is the normalized duality mapping defined from $X$ into $X^*$ which is single-valued whenever the space $X$ has a smooth norm.

By taking the set-valued mapping $K$ to be nonconvex-valued and replacing the convex normal cone $N(K(x(t)); u(t))$ in (SSP) by the Clarke normal cone $N^C(K(x(t)); u(t))$, we get the following Nonconvex Second Order Sweeping Process (NSSP):
Find \( T > 0, x : I \rightarrow \text{cl}(U_0) \), and \( u : I \rightarrow X \) such that

\[
\begin{cases}
  u(0) = u_0 \in K(x_0), \ x_0 \in X; & x(t) = x_0 + \int_0^t u(s)ds, \forall t \in I; \\
  u(t) \in K(x(t)), \ \forall t \in I; & \text{and } -\frac{d}{dt}f(u(t)) \in N^C(K(x(t)); u(t)) \text{ a.e. on } I.
\end{cases}
\]

Clearly, (NSSP) corresponds to Nonconvex Second Order Sweeping Processes in the Hilbert space setting in which \( J \) is the identity mapping (see for instance \([1,3]\)). The problem of existence of solutions of (NSSP) in Hilbert spaces has been the subject of tremendous papers (see for instance Chapter 6 in \([3]\) and the references therein) and the existence of solutions in the convex case (SSP) in Banach spaces is proved in \([1,2]\). Our main aim in the present work is to prove the existence result of (NSSP) whenever the space \( X \) is assumed to be separable \( q \)-uniformly convex and \( 2 \)-uniformly smooth (Theorem 2 in Section 3). In addition, we weaken the Lipschitz assumptions on \( K \) used in the papers \([1,2]\). We will assume that \( K \) is Lipschitz continuous in the usual sense (see \((8)\)) which is easier to check compared to the ones used in \([1,2]\) (see \((27)\) and \((28)\)).

2. Preliminaries

Throughout the paper, we will denote by \( X \) a Banach space with dual space \( X^* \). By \( B \) and \( B_\varepsilon \), we will denote the closed unit balls in \( X \) and \( X^* \), respectively. We will denote by \( d_S \) the usual distance function associated with a closed set \( S \), that is, \( d_S(x) := \inf_{s \in S} \| x - s \| \). We recall the definition of the normalized duality mapping \( J : X \rightrightarrows X^* \):

\[
J(x) = \{ x^* \in X^* : (x^*, x) = \| x \|^2 \}.
\]

Similarly, we define on \( X^* \) the normalized duality mapping \( J^* : X^* \rightrightarrows X \). Let \( V : X^* \times X \rightarrow \mathbb{R} \) be a bifunction defined by

\[
V(x^*, x) = \| x^* \|^2 - 2(x^*, x) + \| x \|^2, \ \forall x^* \in X^* \text{ and } \forall x \in X.
\]

Using this bifunction \( V \) the author in \([4]\) introduced the generalized projections of points in \( X^* \) on \( S \) as follows:

**Definition 1.** For a given nonempty closed set \( S \) in \( X \) and a given \( x^* \in X^* \). We define the generalized projection of \( x^* \) on \( S \) as any point \( \bar{x} \in S \) satisfying

\[
V(x^*, \bar{x}) = d^V_S(x^*) := \inf_{x \in S} V(x^*, x).
\]

The set of all those points is denoted by \( \pi_S(x^*) \). We point out that this set may be empty when the space is not reflexive even if \( S \) is closed and convex (see Example 1.4. in \([5]\)). In addition, we notice that for nonconvex sets \( S \), the set \( \pi_S(x^*) \) may be empty for some points \( x^* \in X^* \)(see Example 4.1 in \([6]\)).

Using this concept of generalized projection on nonempty closed sets, the authors in \([7]\) introduced and studied the concept of \( V \)-proximal normal cone \( N^V(S; \bar{x}) \) (called in \([7]\) generalized proximal normal cone).

**Definition 2.** Assume that \( X \) is a reflexive smooth Banach space. We define the \( V \)-proximal normal cone of \( S \) at \( \bar{x} \) by:

\[
N^V(S; \bar{x}) = \{ x^* \in X^* : \exists \alpha > 0, \ \bar{x} \in \pi_S(J(\bar{x}) + \alpha x^*) \}.
\]

We recall also the definition of the usual proximal normal cone \( N^P(S; \bar{x}) \) and the Clarke normal cone \( N^C(S; \bar{x}) \) (see for instance \([3]\)).
**Definition 3.** Let $X$ be a reflexive smooth Banach space. Then

$$N^p(S; \bar{x}) = \{x^* \in X^* : \exists \varepsilon > 0, \langle x^*, x - \bar{x} \rangle \leq \varepsilon \|x - \bar{x}\|^2, \forall x \in S\}$$

and

$$N^c(S; \bar{x}) = \overline{\text{co}}^w N^p(S; \bar{x}),$$

where $\overline{\text{co}}^w$ symbolises the weak closure of the convex hull.

We recall from [8] that, whenever the space $X$ is 2-uniformly smooth, we have the following inclusions hold: $N^u(S; x) \subset N^p(S; \bar{x}) \subset N^c(S; \bar{x})$. For the definition of $p$-uniformly smooth Banach spaces and $q$-uniformly convex Banach spaces we refer the reader to [9]. We point out that all the spaces $L^q$, $l^q$ with $q \geq 2$ are 2-uniformly smooth and $q$-uniformly convex Banach spaces (see [9]) which is our setting in the present work. We recall also the concepts of subdifferentials for l.s.c. functions. Let $f : X \to \mathbb{R} \cup \{\infty\}$ be a l.s.c. function and $\bar{x} \in X$ with $f(\bar{x}) < \infty$. The $V$-proximal subdifferential of $f$ at $\bar{x}$ is defined by

$$\partial^P f(\bar{x}) := \{x^* \in X^* : (x^*, -1) \in N^p(\text{epi} f; (\bar{x}, f(\bar{x})))\}.$$

Here $\text{epi} f$ stands for the epigraph of $f$, that is, $\text{epi} f := \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}$. Similarly, the Clarke subdifferential of $f$ at $\bar{x}$ is defined by $\partial^C f(\bar{x}) := \{x^* \in X^* : (x^*, -1) \in N^c(\text{epi} f; (\bar{x}, f(\bar{x})))\}$. In the following proposition we recall two results on $V$-proximal subdifferentials and usual proximal normal cones which are needed in our proofs. For more properties and characterisations and their proofs we refer to [3,7].

**Proposition 1.** Assume that $S$ is a nonempty closed subset in a reflexive Banach space $X$ and let $\bar{x}$ be some point in $S$. Then we always have

1. $\partial^P d_s(\bar{x}) \subset \partial^C d_s(\bar{x})$;
2. $\partial^P d_s(\bar{x}) = N^p(S; \bar{x}) \cap B_s$.

The nonconvex concept that will be used in the present work is quoted from [8].

**Definition 4.** Let $S$ be a nonempty closed set in a reflexive smooth Banach space $X$ and let $\bar{x} \in S$. We will say that $S$ is uniformly generalized prox-regular with respect to some positive number $r > 0$, if and only if, for all $x \in S$ and for any nonzero $x^* \in N^u(S; x)$ the point $x$ is a generalized projection of $\bar{x} + r \frac{x^*}{\|x^*\|}$ on $S$, that is, $x \in \pi_S(\bar{x} + r \frac{x^*}{\|x^*\|})$.

Obviously, this concept coincides with the uniform prox-regularity introduced and studied in the Hilbert spaces settings (see for instance [10] and the references therein). The proof of the assertions 1 and 2 in the following example are given in [8].

**Example 1.**

1. Any closed convex set is uniformly generalized prox-regular w.r.t. any $r > 0$;
2. The set $S := B \cup \{x_0 + B\}$ (with $\|x_0\| > 3$) is a closed nonconvex set which is uniformly generalized prox-regular w.r.t. some positive number $r > 0$.
3. Let $S := C \cap [B \cup \{x_0 + B\}]$ with $\|x_0\| > 3$ and $C$ is a closed convex set in $X$. Using the same reasoning in Example 4.10 in [8], we can prove that $S$ is not convex but uniformly generalized prox-regular w.r.t. some positive number $r > 0$.

The following proposition establishes an important property of uniformly generalized prox-regular sets which is necessary in our proofs of the main results. For more properties and characterisations of this class of nonconvex sets we refer to [8].
Proposition 2. Let $X$ be a 2-uniformly smooth and $q$-uniformly convex Banach space. If $S$ is uniformly generalized prox-regular with respect to $r > 0$, then the following assertion holds:

For any $x^* \in U^V_S(r)$ the generalized projection $\pi_S(x^*)$ exists, where

$$U^V_S(r) := \{x^* \in X : d^V_S(x^*) < r^2\}.$$ 

The results in the following lemma are also needed in our proofs.

Lemma 1. Let $X$ be a 2-uniformly smooth and $q$-uniformly convex Banach space. For any $R > 0$ there exist $v_R > 0$, $\beta_R > 0$, $\beta^*_R > 0$, and $v^*_R > 0$ (depending on $R$ and the spaces $X$ and $X^*$) such that

1. $$V(f(x); y) \leq \beta_R \|x - y\|^2, \quad \forall x, y \in RB;$$
2. $$V(y^*; x) \geq \beta^*_R \|J(x) - y^*\|^2, \quad \forall x \in RB, \forall y^* \in R\tilde{B};$$
3. $$\|f(x) - f(y)\| \leq v_R \|x - y\|, \quad \forall x, y \in RB;$$
4. $$\|f^*(x^*) - f^*(y^*)\| \leq v^*_R \|x^* - y^*\|^\frac{1}{2}, \quad \text{for all } x^*, y^* \in R\tilde{B}.$$

Proof of Lemma 1. The first inequality follows directly from Lemma 4.1 in [7]. Since the space $X$ is 2-uniformly smooth Banach then the dual space $X^*$ is a 2-uniformly convex Banach space and hence by Lemma 4.1 in [7] we can find a positive constant $\beta_R > 0$ so that

$$V_*(f^*(x^*); y^*) \geq \beta^*_R \|x^* - y^*\|^2, \quad \forall x^*, y^* \in R\tilde{B}.$$ (1)

Here the functional $V_*$ is defined similarly to $V$ defined previously, that is, $V_* : X^{**} \times X^* \rightarrow \mathbb{R}$ is defined by

$$V_*(x^{**}, x^*) = \|x^{**}\|^2 - 2\langle x^{**}, x^* \rangle + \|x^*\|^2, \quad \forall x^{**} \in X^* \text{ and } \forall x^* \in X^{**}.$$ 

Since $X$ is reflexive, that is, $X^{**} = X$, the functional $V_*$ can be written in the following simpler form:

$$V_*(x, x^*) = \|x\|^2 - 2\langle x, x^* \rangle + \|x^*\|^2 = V(x^*, x), \quad \forall x^* \in X^* \text{ and } \forall x \in X.$$ 

Therefore, the inequality (2) ensures for any $x \in RB$ and any $y^* \in R\tilde{B}$

$$V(y^*; x) = V_*(x; y^*) \geq \beta^*_R \|J(x) - y^*\|^2,$$

and hence the proof of (2) is complete. We turn now to the proof of the inequality (3). Fix any $x, y \in R\tilde{B}$. Combining (1) with (2), we can write

$$\beta_R \|f(x) - f(y)\|^2 \leq V(f(y), x) \leq \beta_R \|y - x\|^2.$$

This yields

$$\|f(x) - f(y)\| \leq v_R \|y - x\|, \quad \text{with } v_R := \sqrt{\frac{\beta_R}{\beta^*_R}}.$$ 

Thus completing the proof of (3).
Let us prove (4). Since the space $X$ is $q$-uniformly convex then the dual space $X^*$ is $q^*$-uniformly smooth ($\frac{1}{q^*} + \frac{1}{q} = 1$). Hence by Lemma 4.1 in [7], we can find $\beta_R^*>0$ so that

$$V_\epsilon(J^*(x^*); y^*) \leq \beta_R^* \|x^* - y^*\|^{q^*}, \forall x^*, y^* \in R\mathbb{B}_e. \quad (2)$$

Since $V_\epsilon(x, x^*) = V(x^*, x)$ the previous inequality becomes

$$V(y^*; x) = V_\epsilon(J^*(J(x)); y^*) \leq \beta_R^* \|J(x) - y^*\|^{q^*}, \forall x \in R\mathbb{B}, \forall y^* \in R\mathbb{B}_e. \quad (3)$$

On the other hand, the $q$-uniform convexity of the space and Lemma 4.1 in [7] give the inequality

$$\partial_R \|y - x\|^{q} \leq V(J(x); y), \forall x \in R\mathbb{B}, \forall y^* \in R\mathbb{B}_e. \quad (4)$$

Fix now any $x^*, y^* \in R\mathbb{B}_e$. Then we have by combining (3) and (4)

$$\partial_R \|J^*(x^*) - J^*(y^*)\|^{q} \leq V(y^*, J^*(x^*)) \leq \beta_R^* \|x^* - y^*\|^{q^*}.$$

This yields

$$\|J^*(x^*) - J^*(y^*)\| \leq \nu_R \|x^* - y^*\|^{\frac{1}{\gamma - 1}}, \text{ with } \nu_R := \left(\frac{\beta_R^*}{\partial_R}\right)^\frac{1}{\gamma - 1}.$$

Thus completing the proof of (4). □

In the next theorem we state an important property of generalized prox-regular sets and it is one of the key points in our next proofs. It has been proved in [8]. To avoid the paucity of the work, we state its proof here.

**Proposition 3.** Assume that $S$ is a closed nonempty subset in a reflexive smooth Banach space $X$ and let $\alpha > 0$ be any positive number. Fix any $x \in S$ and any nonzero $x^* \in N^\alpha(S; x)$. If $S$ is a uniformly generalized prox-regular with ratio $r > 0$, then

$$\left\langle \frac{x^*}{\|x^*\|}; y - x \right\rangle \leq \left(\frac{\|x^*\| + r + 3\alpha}{r}\right) d_S(y) + \frac{1}{r} \langle y - Jx; y - x \rangle, \forall y \in a\mathbb{B}.$$  

**Proof of Proposition 3.** Let $r > 0$ be given as in Definition 4. Fix $\alpha > 0$, $x \in S$, and $0 \neq x^* \in N^\alpha(S; x)$. By definition of uniform generalized prox-regularity, the point $x$ is the generalized projection of $Jx + r \frac{x^*}{\|x^*\|}$ on $S$, that is,

$$V(Jx + r \frac{x^*}{\|x^*\|}; x) \leq V(Jx + r \frac{x^*}{\|x^*\|}; s), \forall s \in S. \quad (5)$$

Since the functional $u \mapsto V(Jx + r \frac{x^*}{\|x^*\|}, u)$ is Lipschitz on $S \cap 3\alpha \mathbb{B}$ with constant $K := 2 (\|x\| + r + 3\alpha)$, then by Clarke penalisation in Proposition 6.3 on page 50 in [11], we have

$$V(Jx + r \frac{x^*}{\|x^*\|}, x) \leq V(Jx + r \frac{x^*}{\|x^*\|}; y) + K d_{S \cap 3\alpha \mathbb{B}}(y), \forall y \in X. \quad (6)$$

On the other side, the functional $u \mapsto V(Jx + r \frac{x^*}{\|x^*\|}, u)$ is convex differentiable on $X$ and its derivative is given by $\nabla^F V(Jx + r \frac{x^*}{\|x^*\|}, \cdot)(y) = 2 (y - Jx - r \frac{x^*}{\|x^*\|})$. Then we can write

$$\langle 2(y - Jx - r \frac{x^*}{\|x^*\|}); x - y \rangle \leq V(Jx + r \frac{x^*}{\|x^*\|}, x) - V(Jx + r \frac{x^*}{\|x^*\|}, y), \forall y \in X.$$
Thus, we obtain for any \( x \in S \) and any \( y \in X \)
\[
2\langle y - f(x) - r \frac{x^*}{\|x^*\|} \rangle x^* \| y - x \| \leq 2 \left( \|x\| + r + 3\alpha \right) d_{S \cap \mathcal{B}}(y). \tag{7}
\]

Observe that \( d_{S \cap \mathcal{B}}(y) = d_S(y) \), for any \( y \in \mathcal{B} \). Therefore, the inequality (7) becomes
\[
\langle x^* \rangle y - x \leq \left( \|x\| + r + 3\alpha \right) \frac{1}{r} d_S(y) + \frac{1}{r} \langle y - f(x) \rangle y - x, \quad \forall y \in \mathcal{B}.
\]

This completes the proof. \( \square \)

3. Main Result

In this section, we are going to prove the main results of the paper. We start by proving the following existence result of approximate solutions to (NSSP).

**Theorem 1.** Let \( X \) be a \( q \)-uniformly convex and \( 2 \)-uniformly smooth separable Banach space and let \( K : cl(U_0) \to X \) be a set-valued mapping with nonempty closed values in \( X \) and verifying:
\[
|d_{K(x)}(u) - d_{K(x)}(u)| \leq \lambda(u) \|x' - x\|, \quad \forall x, x' \in cl(U_0) \text{ and } \forall u \in X.
\tag{8}
\]

Here \( \lambda : X \to [0, \infty) \) is a function which is bounded on bounded sets. Assume that \( K \) has uniformly generalized prox-regular values w.r.t some \( r > 0 \). Then for any initial points \( x_0 \in X \) and \( u_0 \in K(x_0) \), there exist sequences of mappings \( \rho_n, \theta_n : I \to I, u_n, v_n : I \to X, x_n : I \to cl(U_0) \) such that \( \rho_n(t) \to t \) and \( \theta_n(t) \to t \) uniformly on \( I \) and the following approximate differential inclusion holds almost everywhere on \( I \):
\[
\left\{ \begin{array}{l}
\frac{d}{dt}(v_n(t)) \in N^T(K(x_n(\rho_n(t)))) \text{ a.e. on } I \\
v_n(0) = u_0, x_n(0) = x_0, \text{ and } x_n(t) = x_0 + \int_0^t u_n(s)ds + v_n(t)
\end{array} \right.
\tag{9}
\]

**Proof of Theorem 1.** Choose a positive number \( \mu > 0 \) so that \( x_0 + \mu \mathcal{B} \subset U_0 \) and let \( l > 0 \) such that \( \mathcal{L} \subset l \mathcal{B}_n \). Fix \( T \in (0, \frac{l}{r}) \). For each \( n \in \mathbb{N} \), we divide the interval \( I \) into subintervals as follows: \( I_{n,i} := [t_{n,i}, t_{n,i+1}] \), for all \( i = 0, \ldots, n - 1 \), with \( t_{n,i} = i\mu_n \), \( \mu_n := \frac{T}{n} \), and \( I_{n,n} := \{ T \} \).

For every \( n \in \mathbb{N} \) we delineate by induction the sequences of mappings \( (x_n, u_n)_n \) on each interval \( I_{n,i} \) as follows
\[
\left\{ \begin{array}{l}
u_n^i(t) := f(u_{n,i}), \quad u_n(t) = f(u_n^i(t)) = u_{n,i}, \\
x_n(t) = x_0 + \int_0^t u_n(s)ds, \quad x_n^i(t) = f(x_n(t)).
\end{array} \right.
\tag{9}
\]

Here \( u_{n,0} := u_0 \in K(x_0) \) and the sequence of points \( u_{n,i+1} \) is constructed using the generalized projection as follows:
\[
u_{n,i+1} \in \pi_K(x_n(t_{n,i+1}))(f(u_{n,i})), \quad \forall i = 0, \ldots, n - 1.
\tag{10}
\]

As
\[
x_n(t_{n,i+1}) = x_0 + \int_0^{t_{n,i+1}} u_n(s)ds \subset x_0 + l t_{n,i+1} \mathcal{B}_n \subset x_0 + \mu \mathcal{B} \subset U_0,
\]
so \( K(x_n(t_{n,i+1})) \neq \emptyset \). Since the values of \( K \) are not convex, we have to prove the well-definedness of the generalized projection in (10). To do that we have to prove the following claim:

**Claim 1**
The sequence of points \( \{ u_{n,i} \} \) satisfies for \( n \) large enough the inclusion \( \{ u_{n,i} \} \in U_{K(x_n(t_{n,i+1}))}^V(r) \).
We have to flag that the sequence \( \{ u_{n,j}^* \} \) is bounded by the mentioned positive constant \( l > 0 \). Using the Lipschitz continuity of \( K \) in (8), we have for any \( n \in \mathbb{N} \)

\[
\begin{align*}
d^V_{K(x_n(t_{n,j+1}))}(u_{n,j}) & = d^V_{K(x_n(t_{n,j+1}))}(u_{n,j}) - d^V_{K(x_n(t_{n,j}))}(u_{n,j}) \\
& \leq \lambda(u_{n,j}) \| x_n(t_{n,j+1}) - x_n(t_{n,j}) \| \\
& \leq \lambda \| \int_{t_{n,j}}^{t_{n,j+1}} u_n(z)\|dz \leq \lambda t \|u_n\|
\end{align*}
\]

where \( \lambda \) is the bound of \( \lambda(\cdot) \) on \( \mathcal{B} \), i.e., \( \lambda(u) \leq \bar{\lambda}, \forall u \in \mathcal{B} \). Using now Lemma 1 and our construction of the sequence of points \( \{ u_{n,j}^* \} \), we obtain

\[
\begin{align*}
d^V_{K(x_n(t_{n,j+1}))}(f(u_{n,j})) & = \inf_{z \in K(x_n(t_{n,j+1}))} V(f(u_{n,j}), z) \\
& \leq \beta_l \inf_{z \in K(x_n(t_{n,j+1}))} \| u_{n,j} - z \|^2 \\
& \leq \beta_l d^2_{K(x_n(t_{n,j+1}))}(u_{n,j}),
\end{align*}
\]

where \( \beta_l \) is the constant found in Lemma 1 Part (1) and it depends only on \( l \) and \( X \). Hence

\[
d^V_{K(x_n(t_{n,j+1}))}(f(u_{n,j})) \leq \beta_l d^2_{K(x_n(t_{n,j+1}))}(u_{n,j}) \leq \beta_l \bar{\lambda}^2 \mu_0^2.
\]

Fix \( n_0 \in \mathbb{N} \) such that

\[
\mu_n < \frac{r^2}{\beta_l \bar{\lambda}^2}, \quad \text{for all } n \geq n_0.
\]

By the choice of \( n_0 \) we can write

\[
d^V_{K(x_n(t_{n,j+1}))}(f(u_{n,j})) \leq \beta_l d^2_{K(x_n(t_{n,j+1}))}(u_{n,j}) \leq \beta_l \bar{\lambda}^2 \mu_0^2 < r^2,
\]

that is, \( \{ f(u_{n,j}) \} \subset \bigcup_{n \geq n_0} U^V_{K(x_n(t_{n,j+1}))}(r), \quad \forall n \geq n_0 \). Using now Proposition 2 the generalized projection of \( f(u_{n,j}) \) on the set \( K(x_n(t_{n,j+1})) \) exists for any \( n \geq n_0 \) and hence (10) is well defined.

Let us stick \( \theta_n(t) := t_{n,j} \) and \( \rho_n(t) := t_{n,j+1} \) if \( t \in I_{n,j} \). The definition of the mappings \( x_n(\cdot) \) and \( u_n(\cdot) \) give

\[
u_n(t) \in K(x_n(\theta_n(t))) \subset \mathcal{B}, \quad \text{for all } t \in I.
\]

This shows that the family of the mappings \( x_n(\cdot) \) is \( l \)-equi-Lipschitz and equibounded, and it satisfies \( \| x_n \|_\infty \leq \| x_0 \| + IT \). By Part (3) in Lemma 1, we have \( x_n^*(\cdot) \) are Lipschitz continuous with ratio \( l\nu_l \) where \( \nu_l \) is the constant in Part (3) in Lemma 1. In addition, we have \( \| x_n^* \|_\infty \leq \| x_0 \| + IT \).

Observe that

\[
x_n(t) \in U_0, \quad \forall n \geq n_0, \forall t \in I.
\]

In effect, the definition of the mappings \( x_n \) and \( u_n \) yield

\[
x_n(t) = x_0 + \int_0^t u_n(s)ds \in x_0 + lt \mathcal{B} \subset x_0 + \mu \mathcal{B} \subset U_0, \quad \forall t \in I.
\]

This shows the well-definedness of \( K(x_n(t)) \) for all \( t \in I \).

Now we define a new sequence of mappings from \( I \) to \( X^* \) by

\[
v_n^*(t) := f(u_{n,j}) + \mu_n^{-1} (t - t_{n,j})(f(u_{n,j+1}) - f(u_{n,j})), \quad \text{if } t \in I_{n,j}.
\]
Associate to this sequence one more sequence of mappings \( v_n \) as follows:

\[
v_n(t) = J^*(v_n^*(t)), \quad \text{for all } t \in I. \tag{14}
\]

Remark that \( v_n^*(\theta_n(t)) = J(u_{n,i}) \) and \( v_n(\theta_n(t)) = u_{n,i} \forall i = 0, \ldots, n \) and so by (10), (12), and (13) we have

\[
v_n(\theta_n(t)) \in K(x_n(\theta_n(t))) \subset \mathbb{I}_B. \tag{15}
\]

Now, we turn to the verification of the equi-Lipschitz property of the family of mappings \( \{v_n^*\} \). To do that, we are looking for an upper bound estimate of \( \|J(u_{n,i+1}) - J(u_{n,i})\| \). Since \( X \) is 2-uniformly smooth, then by Lemma 1 we can pick two positive constants \( \beta_1 > 0 \) and \( \beta^*_1 > 0 \) so that

\[
\beta^*_1 \|J(x) - J(y)\|^2 \leq V(J(x), y) \leq \beta_1 \|y - x\|^2, \forall x, y \in \mathbb{I}_B. \tag{16}
\]

Applying this inequality with \( \{u_{n,i}\} \) (bounded by \( I \)), we gain

\[
\beta^*_1 \|J(u_{n,i+1}) - J(u_{n,i})\|^2 \leq V(J(u_{n,i}), u_{n,i+1}) \leq \beta_1 \|u_{n,i+1} - u_{n,i}\|^2.
\]

This ensures by (16) that

\[
\beta^*_1 \|J(u_{n,i+1}) - J(u_{n,i})\|^2 \leq V(J(u_{n,i}), u_{n,i+1}) \leq \beta_1 \|u_{n,i+1} - u_{n,i}\|^2.
\]

This gives the estimate

\[
\|J(u_{n,i+1}) - J(u_{n,i})\| \leq \beta \lambda \mu_n. \tag{17}
\]

Consequently, we obtain

\[
\|v_n^*(t') - v_n^*(t)\| = \mu^{-1}_n |t' - t| \|J(u_{n,i+1}) - J(u_{n,i})\| \leq \beta \lambda |t' - t|, \quad \forall t, t' \in I_{n,i}.
\]

This inequality, with the continuity of \( v_n^* \) at \( (t_{n,i})_i \), means that the family \( \{v_n^*\} \) is \( \delta \)-equi-Lipschitz on all \( I \) where \( \delta := \lambda \beta \lambda \). Using the fact that \( J^* \) is uniformly continuous on bounded sets, we can affirm that the family of mappings \( \{v_n := J^*(v_n^*)\} \) is uniformly continuous on \( I \). Therefore,

\[
\|v_n^*(t) - u_n^*(t)\| \leq \mu^{-1}_n |t - t_{n,i}||J(u_{n,i+1}) - J(u_{n,i})| \leq \delta \mu_n,
\]

and hence

\[
\lim_{n \to \infty} \|v_n^* - u_n^*\|_\infty = 0. \tag{18}
\]
The definition of $v_n^*(\cdot)$ yields
\[ v_n^*(t) = \mu_n^{-1}(f(u_{n,i+1}) - f(u_{n,i})). \tag{19} \]

Using now the definition of the $V^*$-proximal normal cone in Definition 2, we can write
\[ -\partial v_n^*(t) \in N^\pi(K(x_n(\rho_n(t))):v_n(\rho_n(t))), \text{ a.e. I}. \]

Indeed, by construction we have
\[ u_{n,i+1} \in \pi_{K(x_n(t_{n,i+1}))}(f(u_{n,i})) \]
\[ = \pi_{K(x_n(t_{n,i}))}(f(u_{n,i+1}) - [f(u_{n,i+1}) - f(u_{n,i})]) \]
\[ \Leftrightarrow f(u_{n,i+1}) - f(u_{n,i}) \in -N^\pi(K(x_n(t_{n,i+1})); u_{n,i+1}) \]
\[ \Leftrightarrow \mu_n^{-1}(f(u_{n,i+1}) - f(u_{n,i})) \in -N^\pi(K(x_n(\rho_n(t))):v_n(\rho_n(t))) \]
and so
\[ -\partial v_n^*(t) \in N^\pi(K(x_n(\rho_n(t))):v_n(\rho_n(t))). \]

This ends the proof of Theorem 1.

**Proposition 4.** Let $X$ be a $q$-uniformly convex and 2-uniformly smooth separable Banach space and let $K : cl(U_0) \rightrightarrows X$ be a set-valued mapping with nonempty closed values in $X$ and verifying (8). Assume that $K$ has uniformly generalized prox-regular values w.r.t some $r > 0$. In addition, assume that for some convex compact set $\mathcal{L}$ in $X^*$, we have $J(K(x)) \subseteq \mathcal{L}$. Then for any initial points $x_0 \in X$ and $u_0 \in K(x_0)$, the sequences of approximate solutions $(x_n)_n$ and $(v_n)_n$ obtained in Theorem 1 are uniformly convergent.

**Proof of Proposition 4.** We use the same construction and notations introduced in Theorem 1. Observe that $\mu_n^{-1}(t - t_{n,i}) \leq 1, \forall t \in I_{n,i}$ and hence by the convexity of $L$ we obtain
\[ v_n^*(t) = \left(1 - \frac{t - t_{n,i}}{\mu_n}\right) f(u_{n,i}) + \frac{t_{n,i}}{\mu_n} f(u_{n,i+1}) \in \mathcal{L}. \]

Since $\mathcal{L}$ is assumed to be compact, then the set $\{v_n^*(t) : n \in \mathbb{N}\}$ is relatively compact in $X^*$ for any $t \in I$. In addition, we have obtained previously the inclusion
\[ v_n^*(t) \in \partial B_+. \tag{20} \]

Consequently, this inclusion together with Arzela-Ascoli theorem [12] produce a Lipschitz mapping $u^* : I \to X^*$ such that: $(v_n^*)$ converges uniformly to $u^*$ on $I$. The weak convergence in $L^\infty(I,X^*)$ of the sequence $(v_n^*)_n$ to some limit $\zeta \in L^\infty(I,X^*)$ is also ensured by the inclusion (20). Then, for any $t \in I$ and any $y \in L^1(I,X)$ we have
\[ \langle v_n^* - \zeta, y \rangle_{L^\infty(I,X^*)} \to 0, \text{ that is } \int_0^T \langle v_n^*(s) - \zeta(s), y(s) \rangle_{X^*,X} ds \to 0. \]
Here $\langle \cdot, \cdot \rangle_{L^\infty(I, X^*), L^1(I, X)}$ stands for the dual pairing between the spaces $L^1(I, X)$ and its dual $L^\infty(I, X^*)$, and $\langle \cdot, \cdot \rangle_{X^*, X}$ stands for the dual pairing between the spaces $X$ and $X^*$. Let $t \in I$ and set $z_m : I \to X$ by $z_m \equiv \psi_{[0,t]}(\cdot) \cdot e_m$, where $(e_m) \subset X$ is a sequence separating points in $X^*$. Thus, we have

$$\lim_{n \to \infty} \int_0^t \tilde{v}_n^*(s) ds, e_m)_{X^*, X} = \langle \int_0^t \zeta(s) ds, e_m \rangle_{X^*, X}, \quad \forall m \in \mathbb{N}.$$ 

This procures

$$\lim_{n} \int_0^t \tilde{v}_n^*(s) ds = \int_0^t \zeta(s) ds.$$ 

Consequently,

$$u^*(t) = \lim_{n} v_n^*(t) = \lim_{n} [J(u_0) + \int_0^t \tilde{v}_n^*(s) ds] = J(u_0) + \int_0^t \zeta(s) ds.$$ 

The fact that $u^*$ is absolutely continuous on $I$ and the last equality ensures that $\zeta = \dot{u}^*$ a.e. on $I$. Set

$$u(t) := J(u^*(t)), \quad \forall t \in I. \quad (21)$$

Obviously, the the uniform continuity of $J^*$ on bounded sets assure the uniform convergence of the sequence $(v_n)$ to the mapping $u$ just defined in (21).

Using this mapping $u$, we define the mapping $x : I \to X$ by

$$x(t) = x_0 + \int_0^t u(s) ds, \quad \forall t \in I, \quad (22)$$

and we set

$$x^*(t) := J(x(t)), \quad \forall t \in I. \quad (23)$$

We mention that both mappings $x$ and $x^*$ are Lipschitz continuous on $I$. Thus, we have

$$\|x_n(t) - x(t)\| = \| \int_0^t u_n(s) - u(s) ds \| \leq T \| u_n - u \|_\infty$$

$$\leq T \| u_n - v_n \|_\infty + T \| v_n - u \|_\infty$$

$$\leq T \| J^*(u_n^*) - J^*(v_n^*) \|_\infty + T \| v_n - u \|_\infty$$

$$\leq T \nu_0 \| u_n^* - v_n^* \|_\infty^{-1} + T \| v_n - u \|_\infty.$$ 

So,

$$\|x_n - x\|_\infty \leq T \nu_0 \| u_n^* - v_n^* \|_\infty^{-1} + T \| v_n - u \|_\infty.$$ 

Consequently, the equality (18) ensures

$$\lim_{n \to \infty} \|x_n - x\|_\infty = 0. \quad (24)$$

This means that $(x_n)_n$ is uniformly convergent to $x$ on $I$. Thus completing the proof of Theorem 4. \qed

**Proposition 5.** Assume that all the assumptions of Proposition 4 are fulfilled. Then the limits $x$ and $u$ of the approximate solutions $x_n$ and $v_n$ constructed in the proof of Theorem 1 satisfies (NSSP).
Proof of Proposition 5. We start by remarking that both \((x_n \circ \rho_n)\) and \((v_n \circ \rho_n)\) are uniformly converging on \(I\) to \(x\) and \(u\), respectively. Recall that \(v_n(\rho_n(t)) \in K(x_n(\rho_n(t))), \forall t \in I\) and \(\forall n \geq n_0\). Then, our hypothesis ensure the following estimates:

\[
d_{K(x(t))}(v_n(\rho_n(t))) = d_{K(x(t))}(v_n(\rho_n(t))) - d_{K(x_n(\rho_n(t)))}(v_n(\rho_n(t)))
\leq \frac{\lambda}{\delta} \|x(t) - x_n(\rho_n(t))\|
\leq \frac{\lambda}{\delta} \|x(t) - x_n(t)\| + \lambda \|x_n(t) - x_n(\rho_n(t))\|
\leq \frac{\lambda}{\delta} \|x_n - x\|_\infty + \lambda \|x_n - x\|_\infty.
\]

Hence, by the fact that \(\|x_n - x\|_\infty \to 0\) we obtain

\[
\lim_{n \to \infty} d_{K(x(t))}(v_n(\rho_n(t))) = 0.
\]

On the other side, we have

\[
d_{K(x(t))}(u(t)) \leq d_{K(x(t))}(v_n(\rho_n(t))) + \|v_n(\rho_n(t)) - u(t)\|
\leq d_{K(x(t))}(v_n(\rho_n(t))) + \|v_n(\rho_n(t)) - \bar{v}_n(t)\| + \|\bar{v}_n(t) - u(t)\|.
\]

Therefore, we obtain by taking the limit in the last inequality: \(d_{K(x(t))}(u(t)) = 0\). This ensures by the fact that the values of \(K\) are closed that \(u(t) \in K(x(t)), \forall t \in I\).

Now, we proceed to prove that \(x\) is a solution of (NSSP). By Proposition 3 we have

\[
\langle - \frac{\partial^*_n(t)}{\delta}; w - \bar{v}_n(\rho_n(t)) \rangle \leq \frac{\delta + r + 3l}{\rho} d_{K(x_n(\rho_n(t)))}(w)
+ \frac{1}{r} (J w - J(\bar{v}_n(\rho_n(t))); w - \bar{v}_n(\rho_n(t))),
\]

for a.e. \(t \in I\) and for any \(w \in \mathbb{I}^B\). Let any \(t \in I\) for which \(\partial^*_n(t)\) and \(\bar{u}^*(t)\) exist and let \(z \in K(x(t)) \subset \mathbb{I}^B\). Then we have

\[
d_{K(x_n(\rho_n(t)))}(z) = d_{K(x_n(\rho_n(t)))}(z) - d_{K(x(t))}(z)
\leq \lambda(z) \|x_n(\rho_n(t)) - x(t)\|
\leq \lambda \|x_n \circ \rho_n - x\|_\infty
\leq \lambda \|x_n \circ \rho_n - x\|_\infty + \lambda \|x_n - x\|_\infty.
\]

Put \(\epsilon_n := \lambda \|x_n \circ \rho_n - x\|_\infty + \lambda \|x_n - x\|_\infty\). Clearly, the uniform convergence of both \((x_n)\) and \((x_n \circ \rho_n)\) to \(x\) ensure that \(\epsilon_n \downarrow 0\) as \(n \to \infty\). Thus \(z \in K(x_n(\rho_n(t))) + \epsilon_n \mathbb{I}^B\), that is, there exists \(y_n(t) \in K(x_n(\rho_n(t)))\) such that \(\|z - y_n(t)\| \leq \epsilon_n\). Hence (25) yields

\[
\langle - \frac{\bar{u}^*(t)}{\delta}; z - u(t) \rangle = \langle \frac{\partial^*_n(t) - \bar{u}^*(t)}{\delta}; z - u(t) \rangle + \langle - \frac{\partial^*_n(t)}{\delta}; z - u(t) \rangle
= \langle \frac{\partial^*_n(t) - \bar{u}^*(t)}{\delta}; z - u(t) \rangle + \langle - \frac{\partial^*_n(t)}{\delta}; v_n(\rho_n(t)) - u(t) \rangle
+ \langle - \frac{\partial^*_n(t)}{\delta}; v_n(\rho_n(t)) - u(t) \rangle
\leq \langle \frac{\partial^*_n(t) - \bar{u}^*(t)}{\delta}; z - u(t) \rangle + \langle - \frac{\partial^*_n(t)}{\delta}; v_n(\rho_n(t)) - u(t) \rangle
+ \langle \frac{\delta + r + 3l}{\rho} d_{K(x_n(\rho_n(t)))}(z) + \frac{1}{r} (J z - J(\bar{v}_n(\rho_n(t))); z - \bar{v}_n(\rho_n(t)))
\]

Therefore, \(\epsilon_n \downarrow 0\) as \(n \to \infty\). Hence (25) yields

\[
\langle - \frac{\bar{u}^*(t)}{\delta}; z - u(t) \rangle = \langle \frac{\partial^*_n(t) - \bar{u}^*(t)}{\delta}; z - u(t) \rangle + \langle - \frac{\partial^*_n(t)}{\delta}; z - u(t) \rangle
\leq \langle \frac{\partial^*_n(t) - \bar{u}^*(t)}{\delta}; z - u(t) \rangle + \langle - \frac{\partial^*_n(t)}{\delta}; v_n(\rho_n(t)) - u(t) \rangle
+ \langle \frac{\delta + r + 3l}{\rho} d_{K(x_n(\rho_n(t)))}(z) + \frac{1}{r} (J z - J(\bar{v}_n(\rho_n(t))); z - \bar{v}_n(\rho_n(t)))
\]

Thus, we have shown that \(\partial^*_n(x_n(t)) \to \partial^*_n(x(t))\) as \(n \to \infty\) which completes the proof of Proposition 5.
\[
\begin{align*}
\leq & \quad \langle \bar{v}_n^*(t) - \bar{u}^*(t) \rangle_{\delta} (z - u(t)) + \|v_n(\rho_n(t)) - u(t)\| \\
+ & \quad \frac{(\delta + r + 3l)}{r} d_{K(x_n(\rho_n(t)))}(z) + \frac{v_l}{r} \|z - v_n(\rho_n(t))\|^2,
\end{align*}
\]
where \( v_l \) is the constant given in Lemma 1 Part (1).

By our construction of the approximate solutions in Theorem 1, we have

\[ -\bar{v}_n^*(t) \in N^\sigma(K(x_n(\rho_n(t)));
\quad \bar{v}_n(\rho_n(t))) \subset N^P(K(x_n(\rho_n(t)));
\quad v_n(\rho_n(t))). \tag{26} \]

The last inclusion follows from the fact that \( X \) is 2-uniformly smooth.

Combining the inclusion (26) with the estimation (20), we can write by Part (2) in Proposition 1

\[-\bar{v}_n^*(t) \in \delta d_{K(x_n(\rho_n(t)))}(v_n(\rho_n(t))) \subset \delta \partial K(x_n(\rho_n(t)))(v_n(\rho_n(t))). \]

Proposition 1.17 in [13] applied together with the Lipschitz continuity \( K \) procures that \( (x, y) \mapsto \partial K(x)(y) \) is scalarly upper semicontinuous, that is, for any \( z \in X \) the function \( (x, y) \mapsto \sigma^*(z, \partial K(x)(y)) \) is u.s.c. on \( X \times X \). So, for any sequence \( (h_n, k_n) \to (k, h) \) with \( k_n \in K(h_n) \) and any \( z \in X \) the following inequality holds

\[ \lim sup_n \sigma^*(z, -\delta \partial K(h_n)(k_n)) \leq \sigma^*(z, -\delta \partial K(h)(k)). \]

Then for any measurable subset \( A \) in \( I \) and every \( m \in \mathbb{N} \) we have

\[
\int_A \langle \bar{u}^*(t), e_m \rangle_{X^*, X} d\tau = \int_0^T \langle \bar{u}^*(\tau), e_m \cdot \chi_A(\tau) \rangle_{X^*, X} d\tau
\]
\[
= \lim_n \int_0^T \langle \bar{v}_n^*(\tau), e_m \cdot \chi_A(\tau) \rangle_{X^*, X} d\tau
\]
\[
= \lim sup_n \int_A \sigma^*(e_m, -\delta \partial K(x_n(\rho_n(t)))(v_n(\rho_n(t)))) d\tau
\]
\[
\leq \int_A \lim sup_n \sigma^*(e_m, -\delta \partial K(x_n(\rho_n(t)))(v_n(\rho_n(t)))) d\tau
\]
\[
\leq \int_A \sigma^*(e_m, -\delta \partial K(x(\tau))(u(\tau)))) d\tau.
\]

It follows that

\[ \langle \bar{u}^*(t), e_m \rangle_{X^*, X} \leq \sigma^*(e_m, -\delta \partial K(x(\tau))(u(\tau)))) \], a.e. \( t \in I \) and \( \forall m \in \mathbb{N} \).

Recall from Proposition III 35 in [13] that the set-valued mapping \( (x, y) \mapsto \partial K(x)(y) \) is measurable and convex weak star compact valued in \( X^* \). This yields

\[ \bar{u}^*(t) \in -\delta \partial K(x(\tau))(u(\tau)) \], a.e. \( t \in I \).

Finally, since \( u(t) \in K(x(t)) \) for all \( t \in I \), we deduce that

\[ \bar{u}^*(t) \in -N^C(K(x(t)); u(t)) \], a.e. \( t \in I \).

Hence the proof is complete. \( \square \)
Let $X$ be a $q$-uniformly convex and 2-uniformly smooth separable Banach space and let $K : cl(U_0) \rightrightarrows X$ be a set-valued mapping with nonempty closed values in $X$ and verifying (8). Assume that $K$ has uniformly generalized prox-regular values w.r.t some $r > 0$. In addition, assume that for some convex compact set $L$ in $X^*$, we have $J(K(x)) \subset L$. Then for any $x_0 \in X$ and any $u_0 \in K(x_0)$ the problem (NSSP) has at least one Lipschitz solution.

Remark 1.

- We present a simple example showing the novelty and importance of our previous results. Assume that $X = L^q(0, 1, \mathbb{R})$ with $q \geq 2$ and fix any point $x_0 \in X$ with $\|x_0\| > 3$ and let $K : X \rightrightarrows X$ be defined as: $K(x) = C \cap [B \cup (x_0 + B)]$, $\forall x \in X$, where $C$ is a convex compact set in $X$. Then, obviously, $X$ is a $q$-uniformly convex and 2-uniformly smooth separable Banach space and $K$ is Lipschitz continuous in the sense of (8) and for any $x \in X$ we have $J(K(x)) \subset L$, with $L := \overline{co}(J(C))$ (the closed convex hull of $J(C)$), which is a convex compact set in $X^*$. By Example 1 the set-valued mapping $K$ has uniformly generalized prox-regular values in $X$. Therefore, all the assumptions of our main result in Theorem 2 are fulfilled and hence there exists a Lipschitz solution of (NSSP) associated with this $K$. We have to point out that this existence of solutions of (NSSP) cannot be derived from any existing result proved in other works.

- We can consider the cases of set-valued mappings $K$ of the form $K(x) := S + g(x)$ and $K(x) := h(x).S$ with $g : X \rightarrow X$ is a bounded Lipschitz single-valued mapping and $h : X \rightarrow [0, \infty)$ is a bounded real-valued function, and $S$ is the set used in the above example. These set-valued mappings satisfy the hypothesis of Theorem 2 but their checks are very long and need more tools from nonsmooth analysis.

4. Conclusions

The main results in the present paper can be summarized as follows: In the framework of separable Banach spaces which are 2-uniformly smooth and $q$-uniformly convex, we proved: The existence of approximate solutions for generalized prox-regular set-valued mappings which are Lipschitz in the sense of (8).

If in addition, the image by $J$ of the values of the set-valued mapping are contained in a convex compact set in $X^*$, then the approximate solutions converge uniformly to a solution of (NSSP).

The Lipschitz assumption (8) is very easy to check relatively to the Lipschitz conditions used in the previous papers [1,2]. In [2], instead of (8) the authors used the following assumption: $\forall x, x' \in cl(U_0)$

$$
\|(d^{\gamma}_{K(x'_0)})^{\frac{1}{2}}(u_1) - (d^{\gamma}_{K(x)})^{\frac{1}{2}}(u_2)\| \leq \gamma_1 \|J(x') - J(x)\| + \gamma_2 \|u_1 - u_2\|, \forall u_1, u_2 \in X^*,
$$

where $\gamma_1, \gamma_2 > 0$. In [1], the authors used the condition:

$$
\|(d^{\gamma}_{K(x'_0)}(u) - d^{\gamma}_{K(x)}(u)) \| \leq \lambda(u) \|x' - x\|, \forall x, x' \in cl(U_0) \text{ and } \forall u \in X.
$$

Obviously, all the conditions (8), (27), and (28) coincide in Hilbert spaces. However, in Banach spaces the condition (27) is very hard to check even for simple forms of $K$. The difficulty comes from the definition of the function $d^{\gamma}_{K}$ (see Definition 1) and the fact that the function $d^{\gamma}_{K}$ does not preserve all the nice properties of the usual distance function $d$. To compare (8) and (28), we take for example $X = L^q(0, 1, R)$ and $K(x) = B + g(x)$, with $g : X \rightarrow X$ is a Lipschitz single-valued mapping. Obviously, the condition (8) is satisfied and it can be verified easily. The condition (28) is not satisfied since the expression $\|u - g(x)\|$ cannot be bounded from below by a positive number for any $u \in X$ and any $x, x' \in cl(U_0)$.

As future works and perspectives we are investigating the case of $p$-uniformly smooth with any $p \neq 2$. 

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