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Nonoscillatory Solutions to Higher-Order Nonlinear Neutral Dynamic Equations

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Abstract: For a class of nonlinear higher-order neutral dynamic equations on a time scale, we analyze the existence and asymptotic behavior of nonoscillatory solutions on the basis of hypotheses that allow applications to equations with different integral convergence and divergence of the reciprocal of the coefficients. Two examples are presented to demonstrate the efficiency of new results.

Keywords: nonoscillatory behavior; existence; nonlinear neutral dynamic equation; higher-order; time scale

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1. Introduction

In this article, we investigate the existence and asymptotic behavior of nonoscillatory solutions to a class of dynamic equations on a time scale $\mathbb{T}$

$$R_n(t, x(t)) + f(t, x(h(t))) = 0, \tag{1}$$

where $\sup \mathbb{T} = \infty$, $t \in [t_0, \infty)_{\mathbb{T}}$ with $t_0 \in \mathbb{T}$, $n \geq 3$, and

$$R_k(t, x(t)) = \begin{cases} x(t) + p(t)x(g(t)), & k = 0, \\ r_{n-k}(t)R^{\Delta}_{k-1}(t, x(t)), & 1 \leq k \leq n-1, \\ R^{\Delta}_{n-1}(t, x(t)), & k = n. \end{cases}$$

Definition 1. *As is customary in this field, a solution of Equation (1) is termed nonoscillatory provided that x is either eventually positive or eventually negative; otherwise, it is said to be oscillatory.*

We refer the reader to [1–6], where the fundamental theory of time scales was investigated. In the last few years, the analysis of oscillatory and nonoscillatory behavior of differential and difference equations has been unified, extended, and generalized by corresponding theory of dynamic equations on time scales; see, for instance, Refs. [7–24]. Some conclusions for the existence and asymptotic behavior of nonoscillatory solutions to various classes of neutral dynamic equations have been shown in [11–13,16,19–22,24]. Zhu and Wang [24] studied a dynamic equation

$$[x(t) + p(t)x(g(t))]^{\Delta} + f(t, x(h(t))) = 0 \tag{2}$$

and established several criteria for the existence of the solutions via the Krasnoselskii's fixed point theorem. As a matter of fact, Equation (2) can be regarded as Equation (1) in the case when $n = 1$. In the special case when $n = 2$, Equation (1) reduces to a dynamic equation

$$\left[r(t)(x(t) + p(t)x(g(t)))^{\Delta}\right]^{\Delta} + f(t, x(h(t))) = 0, \tag{3}$$

which was examined by Deng and Wang [11] and Gao and Wang [13]. The different assumptions $\int_{t_0}^{\infty} 1/r(t)\Delta t = \infty$ in [11] and $\int_{t_0}^{\infty} 1/r(t)\Delta t < \infty$ in [13] cause a phenomenon that the asymptotic behavior of nonoscillatory solutions to Equation (3) is greatly different. Moreover, it is clear that the asymptotic behavior is more complicated assuming that $\int_{t_0}^{\infty} 1/r(t)\Delta t = \infty$.

To find a more general rule of the existence and asymptotic behavior of nonoscillatory solutions to Equation (1), Qiu [19] considered Equation (1) in the special case where $n = 3$, namely,

$$\left(r_1(t)\left(r_2(t)\left(x(t) + p(t)x(g(t))\right)^{\Delta}\right)^{\Delta}\right)^{\Delta} + f(t, x(h(t))) = 0 \tag{4}$$

with $\int_{t_0}^{\infty} 1/r_1(t)\Delta t = \int_{t_0}^{\infty} 1/r_2(t)\Delta t = \infty$. The author introduced two functions

$$R_1(t) = 1 + \int_{t_0}^{t} \frac{1}{r_2(s)}\Delta s \quad \text{and} \quad R_2(t) = 1 + \int_{t_0}^{t}\int_{t_0}^{s} \frac{1}{r_1(u)r_2(s)}\Delta u\Delta s$$

to divide the eventually positive solutions of Equation (4) into five groups, and presented some existence conditions of them, respectively.

Qiu and Wang [20] were concerned with Equation (1) under the conditions $\int_{t_0}^{\infty} 1/r_i(t)\Delta t < \infty$ for $i = 1, 2, \ldots, n-1$, which include Equation (4) when $n = 3$ with $\int_{t_0}^{\infty} 1/r_i(t)\Delta t < \infty$ for $i = 1, 2$. It shows that there exist only two cases that $\lim_{t\to\infty} x(t) = b > 0$ and $\lim_{t\to\infty} x(t) = 0$, where x is assumed to be an eventually positive solution of Equation (1). Furthermore, this result can be extended to [13] when $n = 2$ and [24] when $n = 1$.

When the convergence and divergence of the integrals $\int_{t_0}^{\infty} 1/r_i(t)\Delta t$ for $i = 1, 2, \ldots, n-1$ are different, for Equation (4), there exist two cases as follows:

$(B1)$ $\int_{t_0}^{\infty} 1/r_1(t)\Delta t = \infty$ and $\int_{t_0}^{\infty} 1/r_2(t)\Delta t < \infty$;
$(B2)$ $\int_{t_0}^{\infty} 1/r_1(t)\Delta t < \infty$ and $\int_{t_0}^{\infty} 1/r_2(t)\Delta t = \infty$.

Qiu et al. considered the case $(B1)$ in [22] and the case $(B2)$ in [21], successively. The conclusions complement the results in [19,20] when $n = 3$.

For Equation (1), it is significant to continue to investigate more general cases of the convergence and divergence of the integrals $\int_{t_0}^{\infty} 1/r_i(t)\Delta t$ for $i = 1, 2, \ldots, n-1$. Throughout, we assume that the following hypotheses are satisfied:

$(C1)$ $r_i \in \mathrm{C_{rd}}([t_0,\infty)_{\mathbb{T}}, (0,\infty))$, $i = 1, 2, \ldots, n-1$, and there are constants $M_i > 0$, $i = 2, 3, \ldots, n-1$ such that
$$\int_{t_0}^{\infty} \frac{\Delta t}{r_1(t)} = \infty \quad \text{and} \quad \int_{t_0}^{\infty} \frac{\Delta t}{r_i(t)} = M_i < \infty, \quad i = 2, 3, \ldots, n-1;$$
$(C2)$ $p \in \mathrm{C_{rd}}([t_0,\infty)_{\mathbb{T}}, \mathbb{R})$ and $\lim_{t\to\infty} p(t) = p_0$, where $|p_0| < 1$;
$(C3)$ $g, h \in \mathrm{C_{rd}}([t_0,\infty)_{\mathbb{T}}, \mathbb{T})$, $g(t) \le t$, $\lim_{t\to\infty} g(t) = \lim_{t\to\infty} h(t) = \infty$, and if $p_0 \in (-1, 0]$, then there exists a sequence $\{c_k\}_{k\ge 0}$ satisfying $\lim_{k\to\infty} c_k = \infty$ and $g(c_{k+1}) = c_k$;
$(C4)$ $f \in \mathrm{C}([t_0,\infty)_{\mathbb{T}} \times \mathbb{R}, \mathbb{R})$ is nondecreasing in x and $xf(t,x) > 0$ for $x \neq 0$;
$(C5)$ if
$$\int_{t_0}^{\infty}\int_{t_0}^{u_{n-1}}\int_{t_0}^{u_{n-2}}\cdots\int_{t_0}^{u_2} \frac{1}{\prod_{i=1}^{n-1} r_i(u_i)}\Delta u_1\Delta u_2\cdots\Delta u_{n-1} = \infty, \tag{5}$$

then define

$$R(t) = 1 + \int_{t_0}^{t} \int_{t_0}^{u_{n-1}} \int_{t_0}^{u_{n-2}} \cdots \int_{t_0}^{u_2} \frac{1}{\prod_{i=1}^{n-1} r_i(u_i)} \Delta u_1 \Delta u_2 \cdots \Delta u_{n-1}, \tag{6}$$

where

$$\lim_{t\to\infty} \frac{R(g(t))}{R(t)} = \eta \in (0,1]$$

is supposed to hold.

In view of the results established in [11,13], it is not difficult to see that the existence and asymptotic behavior of nonoscillatory solutions to Equation (1) are more complex than those in [20]. Therefore, the criteria obtained in this article develop and improve some known conclusions reported in the references. Finally, we present two examples to demonstrate the versatility of new results.

2. Auxiliary Results

To establish criteria for the existence of nonoscillatory solutions to Equation (1), we need a Banach space and Krasnoselskii's fixed point theorem as follows.

Definition 2. *Letting $\lambda = 0, 1$, define a Banach space*

$$\mathrm{BC}_\lambda[T_0,\infty)_{\mathbb{T}} = \left\{ x : x \in \mathrm{C}([T_0,\infty)_{\mathbb{T}}, \mathbb{R}) \text{ and } \sup_{t\in[T_0,\infty)_{\mathbb{T}}} \left| \frac{x(t)}{R^{2\lambda}(t)} \right| < \infty \right\}$$

with the norm

$$\|x\|_\lambda = \sup_{t\in[T_0,\infty)_{\mathbb{T}}} \left| \frac{x(t)}{R^{2\lambda}(t)} \right|,$$

where $\mathrm{C}([T_0,\infty)_{\mathbb{T}}, \mathbb{R})$ is the set containing all continuous functions mapping $[T_0,\infty)_{\mathbb{T}}$ into $\mathbb{R}$.

Lemma 1. *(Krasnoselskii's fixed point theorem) Let Ω be a bounded, convex, and closed subset of a Banach space X. Assume that there are two operators $U, S : \Omega \to X$ such that U is contractive, S is completely continuous, and $Ux + Sy \in \Omega$ for all $x, y \in \Omega$. Then, $U + S$ has a fixed point in Ω.*

Define $z(t) = x(t) + p(t)x(g(t))$. Without loss of generality, we consider only the eventually positive solutions of Equation (1). Then, we have the following lemma (see [12] (Lemma 2.3) and [22] (Lemma 2.1)).

Lemma 2. *Let x be an eventually positive solution of Equation (1) and*

$$\lim_{t\to\infty} \frac{z(t)}{R^\lambda(t)} = a, \quad \lambda = 0, 1,$$

where $\lambda = 1$ only if condition (5) holds. Suppose that a is finite. Then,

$$\lim_{t\to\infty} \frac{x(t)}{R^\lambda(t)} = \frac{a}{1 + p_0\eta^\lambda},$$

or x/R^λ is unbounded.

For the sake of simplicity, we give a classification to divide all eventually positive solutions of Equation (1) into four types.

Theorem 1. *Let x be an eventually positive solution of Equation (1). Then, there are four possible types for x:*

$(A1)$ $\lim_{t\to\infty} x(t) = 0$;
$(A2)$ $\lim_{t\to\infty} x(t) = b$ *for some constant* $b > 0$;
$(A3)$ $\lim_{t\to\infty} x(t) = \infty$ *and* $\lim_{t\to\infty} x(t)/R(t) = b$, *where* $b > 0$ *is a constant;*
$(A4)$ $\limsup_{t\to\infty} x(t) = \infty$ *and* $\lim_{t\to\infty} x(t)/R(t) = 0$.

Proof. Let x be an eventually positive solution of Equation (1). By virtue of $(C2)$ and $(C3)$, there exist a $t_1 \in [t_0,\infty)_{\mathbb{T}}$ and $|p_0| < p_1 < 1$ satisfying $x(t) > 0$, $x(g(t)) > 0$, $x(h(t)) > 0$, and $|p(t)| \le p_1$ for $t \in [t_1,\infty)_{\mathbb{T}}$. For $t \in [t_1,\infty)_{\mathbb{T}}$, we get

$$R^{\Delta}_{n-1}(t,x(t)) = -f(t,x(h(t))) < 0,$$

which implies that $R_{n-1}(t,x(t)) = r_1(t)R^{\Delta}_{n-2}(t,x(t))$ is strictly decreasing on $[t_1,\infty)_{\mathbb{T}}$. Then, we need to consider two cases.

Case 1. Suppose first that R^{Δ}_{n-2} is eventually negative. Then,

$$\lim_{t\to\infty} r_1(t)R^{\Delta}_{n-2}(t,x(t)) = L_2,$$

where $-\infty \le L_2 < 0$. Hence, there exist a constant $c_1 < 0$ and a $t_2 \in [t_1,\infty)_{\mathbb{T}}$ such that $r_1(t)R^{\Delta}_{n-2}(t,x(t)) \le c_1$ for $t \in [t_2,\infty)_{\mathbb{T}}$, which yields

$$R^{\Delta}_{n-2}(t,x(t)) \le \frac{c_1}{r_1(t)}, \quad t \in [t_2,\infty)_{\mathbb{T}}. \tag{7}$$

Integrating inequality (7) from t_2 to t, $t \in [\sigma(t_2),\infty)_{\mathbb{T}}$, we deduce that

$$R_{n-2}(t,x(t)) - R_{n-2}(t_2,x(t_2)) \le c_1 \int_{t_2}^{t} \frac{\Delta s}{r_1(s)}.$$

In view of $(C1)$, letting $t \to \infty$, we obtain $R_{n-2}(t,x(t)) = r_2(t)R^{\Delta}_{n-3}(t,x(t)) \to -\infty$, which means that R^{Δ}_{n-3} is negative and R_{n-3} is strictly decreasing for large t. When $n = 3$, z is nonoscillatory. We can declare that

$$\lim_{t\to\infty} z(t) = L_0, \tag{8}$$

where $0 \le L_0 < \infty$. Do not assume it; that is, $\lim_{t\to\infty} z(t) < 0$. Then, we have $p_0 \in (-1,0]$ and so there exists a $t_3 \in [t_2,\infty)_{\mathbb{T}}$ such that

$$x(t) < -p(t)x(g(t)) \le p_1 x(g(t)), \quad t \in [t_3,\infty)_{\mathbb{T}}.$$

It follows from $(C3)$ that there is a positive integer k_0 such that $c_k \in [t_3,\infty)_{\mathbb{T}}$ for all $k \ge k_0$. For any given $k \ge k_0 + 1$, we always arrive at

$$x(c_k) < p_1 x(g(c_k)) = p_1 x(c_{k-1}) < p_1^2 x(c_{k-2}) < \cdots < p_1^{k-k_0} x(c_{k_0}),$$

which yields $\lim_{k\to\infty} x(c_k) = 0$ and $\lim_{k\to\infty} z(c_k) = 0$. This contradicts the assumption, and so equality (8) holds.

When $n \ge 4$, since $R_{n-3}(t,x(t)) = r_3(t)R^{\Delta}_{n-4}(t,x(t))$ is eventually strictly decreasing, there exists a $t_4 \in [t_2,\infty)_{\mathbb{T}}$ such that for $t \in [t_4,\infty)_{\mathbb{T}}$, we have

$$R^{\Delta}_{n-4}(t,x(t)) \le \frac{r_3(t_4)R^{\Delta}_{n-4}(t_4,x(t_4))}{r_3(t)}. \tag{9}$$

If there is a $t_5 \in [t_4,\infty)_{\mathbb{T}}$ such that $R^{\Delta}_{n-4}(t_5,x(t_5)) \le 0$, then $r_3(t)R^{\Delta}_{n-4}(t,x(t)) \le 0$ for $t \in [t_5,\infty)_{\mathbb{T}}$ and thus R^{Δ}_{n-4} is eventually negative. Otherwise, if $R^{\Delta}_{n-4}(t,x(t)) > 0$ for all $t \in [t_4,\infty)_{\mathbb{T}}$, then R^{Δ}_{n-4} is eventually positive. Hence, R_{n-4} is always eventually monotonic. Integrating inequality (9) from t_4 to t, $t \in [\sigma(t_4), \infty)_{\mathbb{T}}$, we conclude that

$$R_{n-4}(t, x(t)) - R_{n-4}(t_4, x(t_4)) \leq r_3(t_4) R^{\Delta}_{n-4}(t_4, x(t_4)) \int_{t_4}^{t} \frac{\Delta s}{r_3(s)} \leq r_3(t_4)|R^{\Delta}_{n-4}(t_4, x(t_4))| \cdot M_3,$$

which implies that R_{n-4} is upper bounded. If $n = 4$, then we see that z is eventually monotonic and upper bounded, and equality (8) holds.

When $n \geq 5$, since $R_{n-4}(t, x(t))$ (or $r_4(t) R^{\Delta}_{n-5}(t, x(t))$) is eventually monotonic, R^{Δ}_{n-5} is nonoscillatory. It follows that R_{n-5} is eventually monotonic. Noticing that R_{n-4} is upper bounded, there exist a constant c_2 and a $t_6 \in [t_4, \infty)_{\mathbb{T}}$ such that for $t \in [t_6, \infty)_{\mathbb{T}}$,

$$R_{n-4}(t, x(t)) = r_4(t) R^{\Delta}_{n-5}(t, x(t)) \leq c_2,$$

which yields

$$R^{\Delta}_{n-5}(t, x(t)) \leq \frac{c_2}{r_4(t)}, \quad t \in [t_6, \infty)_{\mathbb{T}}. \tag{10}$$

Integrating inequality (10) from t_6 to t, $t \in [\sigma(t_6), \infty)_{\mathbb{T}}$, we get

$$R_{n-5}(t, x(t)) - R_{n-5}(t_6, x(t_6)) \leq c_2 \int_{t_6}^{t} \frac{\Delta s}{r_4(s)} \leq |c_2| \cdot M_4,$$

which means that R_{n-5} is upper bounded. If $n = 5$, then we deduce that equality (8) holds similarly. Analogously, for $n \geq 3$, it follows that equality (8) always holds. Therefore, by virtue of Lemma 2, we conclude that $(A1)$ or $(A2)$ holds.

Case 2. Assume now that R^{Δ}_{n-2} is eventually positive. Then,

$$\lim_{t \to \infty} r_1(t) R^{\Delta}_{n-2}(t, x(t)) = L_2,$$

where $0 \leq L_2 < \infty$. We consider the following two cases:

$$\lim_{t \to \infty} r_1(t) R^{\Delta}_{n-2}(t, x(t)) = b > 0 \quad \text{and} \quad \lim_{t \to \infty} r_1(t) R^{\Delta}_{n-2}(t, x(t)) = 0.$$

If $\lim_{t \to \infty} r_1(t) R^{\Delta}_{n-2}(t, x(t)) = b > 0$, then there is a $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that for $t \in [t_2, \infty)_{\mathbb{T}}$,

$$\left(r_2(t) R^{\Delta}_{n-3}(t, x(t)) \right)^{\Delta} > \frac{b}{r_1(t)}. \tag{11}$$

Integrating inequality (11) from t_2 to t, $t \in [\sigma(t_2), \infty)_{\mathbb{T}}$, we arrive at

$$r_2(t) R^{\Delta}_{n-3}(t, x(t)) > r_2(t_2) R^{\Delta}_{n-3}(t_2, x(t_2)) + b \int_{t_2}^{t} \frac{\Delta s}{r_1(s)}.$$

By virtue of $(C1)$, $r_2(t) R^{\Delta}_{n-3}(t, x(t)) \to \infty$ as $t \to \infty$, which implies that R^{Δ}_{n-3} is positive and R_{n-3} is strictly increasing for large t. Thus, R_{n-3} is nonoscillatory. When $n = 3$, $R_{n-3} = z$. As before, we have

$$\lim_{t \to \infty} z(t) = L_0, \tag{12}$$

where $0 \leq L_0 \leq \infty$. When $n \geq 4$, since $R_{n-3}(t, x(t)) = r_3(t) R^{\Delta}_{n-4}(t, x(t))$, we deduce that $R^{\Delta}_{n-4}(t, x(t))$ is nonoscillatory, and R_{n-4} is eventually monotonic. If $n = 4$, then $R_{n-4} = z$, and equality (12) holds. When $n \geq 5$, it follows that R_{n-5} is eventually monotonic similarly. Analogously, for $n \geq 3$, it follows that equality (12) always holds.

If $\lim_{t\to\infty} r_1(t)R^{\Delta}_{n-2}(t,x(t)) = 0$, since $R^{\Delta}_{n-2} = \left(r_2 R^{\Delta}_{n-3}\right)^{\Delta}$ is eventually positive, then

$$\lim_{t\to\infty} r_2(t)R^{\Delta}_{n-3}(t,x(t)) = L_1,$$

where $-\infty < L_1 \le \infty$. Moreover, $r_2 R^{\Delta}_{n-3}$ is strictly increasing for large t. It follows that R^{Δ}_{n-3} is nonoscillatory. Thus, R_{n-3} is always eventually monotonic and nonoscillatory. Similarly as before, we deduce that $\lim_{t\to\infty} z(t) = L_0 \ge 0$ when $n \ge 3$.

When $L_1 = \infty$, we get $0 \le L_0 \le \infty$ similarly as before. If $-\infty < L_1 < \infty$, then there exist a constant $d_1 > 0$ and a $t_3 \in [t_1,\infty)_{\mathbb{T}}$ such that $r_2(t)R^{\Delta}_{n-3}(t,x(t)) \le d_1$ for $t \in [t_3,\infty)_{\mathbb{T}}$, which yields

$$R^{\Delta}_{n-3}(t,x(t)) \le \frac{d_1}{r_2(t)}, \quad t \in [t_3,\infty)_{\mathbb{T}}. \tag{13}$$

Integrating inequality (13) from t_3 to t, $t \in [\sigma(t_3),\infty)_{\mathbb{T}}$, we have

$$R_{n-3}(t,x(t)) - R_{n-3}(t_3,x(t_3)) \le d_1 \int_{t_3}^{t} \frac{\Delta s}{r_2(s)} \le d_1 \cdot M_2.$$

When $n = 3$, $R_{n-3} = z$, and so z is upper bounded. When $n \ge 4$, there exist a constant $d_2 > 0$ and a $t_4 \in [t_3,\infty)_{\mathbb{T}}$ such that $r_3(t)R^{\Delta}_{n-4}(t,x(t)) \le d_2$ for $t \in [t_4,\infty)_{\mathbb{T}}$. It follows that

$$R^{\Delta}_{n-4}(t,x(t)) \le \frac{d_2}{r_3(t)}, \quad t \in [t_4,\infty)_{\mathbb{T}}.$$

Similarly, we see that R_{n-4} is upper bounded. If $n = 4$, then $R_{n-4} = z$, and thus z is upper bounded. Analogously, for $n \ge 3$, we deduce that z is always upper bounded. Hence, $0 \le L_0 < \infty$.

According to Lemma 2, if $0 \le L_0 < \infty$, then case $(A1)$ or case $(A2)$ holds; if $L_0 = \infty$, then we obtain that x is infinite. Furthermore, by virtue of L'Hôpital's rule (see [5] (Theorem 1.120)), we deduce that

$$\lim_{t\to\infty} R_n(t,x(t)) = \lim_{t\to\infty} \frac{z(t)}{R(t)} = L_2,$$

where $0 \le L_2 < \infty$. It follows that one of cases $(A3)$ and $(A4)$ holds.

The proof is complete. □

3. Main Results

We establish several criteria for the existence of various types of eventually positive solutions of Equation (1). Firstly, suppose that

$$\int_{t_0}^{\infty} \int_{t_0}^{u_{n-1}} \int_{t_0}^{u_{n-2}} \cdots \int_{t_0}^{u_2} \frac{1}{\prod_{i=1}^{n-1} r_i(u_i)} \Delta u_1 \Delta u_2 \cdots \Delta u_{n-1} < \infty, \tag{14}$$

which means that condition (5) is not satisfied.

Theorem 2. *Let condition* (14) *be fulfilled. Then, Equation* (1) *has an eventually positive solution x satisfying $\lim_{t\to\infty} x(t) = b$ iff*

$$\int_{t_0}^{\infty} \int_{t_0}^{u_{n-1}} \int_{t_0}^{u_{n-2}} \cdots \int_{t_0}^{u_1} \frac{f(u_0,K)}{\prod_{i=1}^{n-1} r_i(u_i)} \Delta u_0 \Delta u_1 \cdots \Delta u_{n-1} < \infty \tag{15}$$

for some constant $K > 0$, where $b > 0$ is a constant.

Proof. Let x be an eventually positive solution of Equation (1) that satisfies $\lim_{t\to\infty} x(t) = b > 0$. Then, $\lim_{t\to\infty} z(t) = (1+p_0)b$, and there is a $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t) > 0$, $x(g(t)) > 0$, and $x(h(t)) \geq b/2$ for $t \in [t_1,\infty)_{\mathbb{T}}$. Integrating Equation (1) from t_1 to t, $t \in [\sigma(t_1),\infty)_{\mathbb{T}}$, we arrive at

$$R^{\Delta}_{n-2}(t,x(t)) = \frac{R_{n-1}(t_1,x(t_1))}{r_1(t)} - \frac{\int_{t_1}^{t} f(u_0,x(h(u_0)))\Delta u_0}{r_1(t)}. \tag{16}$$

Integrating equality (16) from t_1 to t, $t \in [\sigma(t_1),\infty)_{\mathbb{T}}$, we get

$$\begin{aligned} R_{n-2}(t,x(t)) - R_{n-2}(t_1,x(t_1)) &= \int_{t_1}^{t} R^{\Delta}_{n-2}(u_1,x(u_1))\Delta u_1 \\ &= R_{n-1}(t_1,x(t_1))\int_{t_1}^{t}\frac{\Delta u_1}{r_1(u_1)} - \int_{t_1}^{t}\int_{t_1}^{u_1}\frac{f(u_0,x(h(u_0)))}{r_1(u_1)}\Delta u_0\Delta u_1. \end{aligned}$$

Similarly, for $n \geq 3$, we conclude that

$$\begin{aligned} z(t) - z(t_1) \;=\;& R_1(t_1,x(t_1))\int_{t_1}^{t}\frac{1}{r_{n-1}(u_{n-1})}\Delta u_{n-1} \\ &+\sum_{k=2}^{n-1} R_k(t_1,x(t_1))\int_{t_1}^{t}\int_{t_1}^{u_{n-1}}\int_{t_1}^{u_{n-2}}\cdots\int_{t_1}^{u_{n-k+1}}\frac{1}{\prod_{i=n-k}^{n-1} r_i(u_i)}\Delta u_{n-k}\Delta u_{n-k+1}\cdots\Delta u_{n-1} \\ &-\int_{t_1}^{t}\int_{t_1}^{u_{n-1}}\int_{t_1}^{u_{n-2}}\cdots\int_{t_1}^{u_1}\frac{f(u_0,x(h(u_0)))}{\prod_{i=1}^{n-1} r_i(u_i)}\Delta u_0\Delta u_1\cdots\Delta u_{n-1}. \end{aligned} \tag{17}$$

Letting $t \to \infty$, condition (14) holds if $n \geq 3$, and, when $n \geq 4$, for all $2 \leq k \leq n-2$, by virtue of $(C1)$, we deduce that

$$\int_{t_1}^{\infty}\int_{t_1}^{u_{n-1}}\int_{t_1}^{u_{n-2}}\cdots\int_{t_1}^{u_{n-k+1}}\frac{1}{\prod_{i=n-k}^{n-1} r_i(u_i)}\Delta u_{n-k}\Delta u_{n-k+1}\cdots\Delta u_{n-1} \leq \prod_{i=n-k}^{n-1} M_i < \infty.$$

Hence,

$$\int_{t_1}^{\infty}\int_{t_1}^{u_{n-1}}\int_{t_1}^{u_{n-2}}\cdots\int_{t_1}^{u_1}\frac{f(u_0,x(h(u_0)))}{\prod_{i=1}^{n-1} r_i(u_i)}\Delta u_0\Delta u_1\cdots\Delta u_{n-1} < \infty.$$

Since $x(h(t)) \geq b/2$, from $(C4)$, it is obvious that

$$\int_{t_1}^{\infty}\int_{t_1}^{u_{n-1}}\int_{t_1}^{u_{n-2}}\cdots\int_{t_1}^{u_1}\frac{f(u_0,b/2)}{\prod_{i=1}^{n-1} r_i(u_i)}\Delta u_0\Delta u_1\cdots\Delta u_{n-1} < \infty,$$

which implies that condition (15) holds.

Suppose that there exists some constant $K > 0$ satisfying condition (15). Then, we will analyze two cases: (i) $0 \leq p_0 < 1$ and (ii) $-1 < p_0 < 0$, respectively.

Case (i). $0 \leq p_0 < 1$. Take a constant p_1 such that $p_0 < p_1 < (1+4p_0)/5 < 1$. When $p_0 > 0$, by virtue of $(C2)$ and condition (15), there is a $T_0 \in [t_0,\infty)_{\mathbb{T}}$ such that for $t \in [T_0,\infty)_{\mathbb{T}}$,

$$p(t) > 0, \quad \frac{5p_1 - 1}{4} \leq p(t) \leq p_1 < 1,$$

$$\int_{T_0}^{\infty}\int_{T_0}^{u_{n-1}}\int_{T_0}^{u_{n-2}}\cdots\int_{T_0}^{u_1}\frac{f(u_0,K)}{\prod_{i=1}^{n-1} r_i(u_i)}\Delta u_0\Delta u_1\cdots\Delta u_{n-1} \leq \frac{(1-p_1)K}{8}.$$

When $p_0 = 0$, choose a constant p_1 such that $|p(t)| \leq p_1 \leq 1/13$ for $t \in [T_0,\infty)_{\mathbb{T}}$. By virtue of $(C3)$, there is a $T_1 \in (T_0,\infty)_{\mathbb{T}}$ such that $g(t) \geq T_0$ and $h(t) \geq T_0$ for $t \in [T_1,\infty)_{\mathbb{T}}$.

Let

$$\Omega_1 = \left\{x \in \mathrm{BC}_0[T_0,\infty)_{\mathbb{T}} : \frac{K}{2} \leq x(t) \leq K\right\}. \tag{18}$$

Clearly, Ω_1 is a bounded, convex, and closed subset of $\mathrm{BC}_0[T_0,\infty)_{\mathbb{T}}$. Define now two operators $U_1, S_1\colon \Omega_1 \to \mathrm{BC}_0[T_0,\infty)_{\mathbb{T}}$ by

$$(U_1x)(t) = \begin{cases} 3Kp_1/4 - p(t)x(g(t)), & t \in [T_1,\infty)_{\mathbb{T}}, \\ (U_1x)(T_1), & t \in [T_0,T_1)_{\mathbb{T}}, \end{cases}$$

$$(S_1x)(t) = \begin{cases} 3K/4 & \\ +\int_t^{\infty}\int_{T_1}^{u_{n-1}}\int_{T_1}^{u_{n-2}}\cdots\int_{T_1}^{u_1}\frac{f(u_0,x(h(u_0)))}{\prod_{i=1}^{n-1}r_i(u_i)}\Delta u_0\Delta u_1\cdots\Delta u_{n-1}, & t \in [T_1,\infty)_{\mathbb{T}}, \\ (S_1x)(T_1), & t \in [T_0,T_1)_{\mathbb{T}}. \end{cases} \quad (19)$$

The fact that U_1 and S_1 satisfy the conditions in Lemma 1 can be proved (see the proofs of [19] (Theorem 3.1) and [20] (Theorem 3.1)), and so is omitted. By virtue of Lemma 1, there is an $x \in \Omega_1$ such that $(U_1+S_1)x = x$. For $t \in [T_1,\infty)_{\mathbb{T}}$,

$$x(t) = \frac{3(1+p_1)K}{4} - p(t)x(g(t)) + \int_t^{\infty}\int_{T_1}^{u_{n-1}}\int_{T_1}^{u_{n-2}}\cdots\int_{T_1}^{u_1}\frac{f(u_0,x(h(u_0)))}{\prod_{i=1}^{n-1}r_i(u_i)}\Delta u_0\Delta u_1\cdots\Delta u_{n-1}.$$

Since $f(u_0,x(h(u_0))) \le f(u_0,K)$ for $u_0 \in [T_1,\infty)_{\mathbb{T}}$, and

$$\lim_{t\to\infty}\int_t^{\infty}\int_{T_1}^{u_{n-1}}\int_{T_1}^{u_{n-2}}\cdots\int_{T_1}^{u_1}\frac{f(u_0,K)}{\prod_{i=1}^{n-1}r_i(u_i)}\Delta u_0\Delta u_1\cdots\Delta u_{n-1} = 0,$$

we conclude that

$$\lim_{t\to\infty} z(t) = \frac{3(1+p_1)K}{4} \quad \text{and} \quad \lim_{t\to\infty} x(t) = \frac{3(1+p_1)K}{4(1+p_0)} > 0.$$

Case (ii). $-1 < p_0 < 0$. Choose a constant p_1 satisfying $-p_0 < p_1 < (1-4p_0)/5 < 1$. By (C2) and condition (15), there is a $T_0 \in [t_0,\infty)_{\mathbb{T}}$ such that $(5p_1-1)/4 \le -p(t) \le p_1 < 1$ for $t \in [T_0,\infty)_{\mathbb{T}}$. There also exists a $T_1 \in (T_0,\infty)_{\mathbb{T}}$ such that $g(t) \ge T_0$ and $h(t) \ge T_0$ for $t \in [T_1,\infty)_{\mathbb{T}}$. Let $\mathrm{BC}_0[T_0,\infty)_{\mathbb{T}}$ and its subset Ω_1 be as in (18). Define S_1 by (19) and U_1' on Ω_1 by

$$(U_1'x)(t) = \begin{cases} -3Kp_1/4 - p(t)x(g(t)), & t \in [T_1,\infty)_{\mathbb{T}}, \\ (U_1'x)(T_1), & t \in [T_0,T_1)_{\mathbb{T}}. \end{cases}$$

Similarly, as in the proofs of [19] (Theorem 3.1) and [20] (Theorem 3.1), we can prove that U_1' and S_1 satisfy the assumptions in Lemma 1. Hence, there is an $x \in \Omega_1$ such that $(U_1'+S_1)x = x$. For $t \in [T_1,\infty)_{\mathbb{T}}$,

$$x(t) = \frac{3(1-p_1)K}{4} - p(t)x(g(t)) + \int_t^{\infty}\int_{T_1}^{u_{n-1}}\int_{T_1}^{u_{n-2}}\cdots\int_{T_1}^{u_1}\frac{f(u_0,x(h(u_0)))}{\prod_{i=1}^{n-1}r_i(u_i)}\Delta u_0\Delta u_1\cdots\Delta u_{n-1}.$$

Letting $t \to \infty$, we deduce that

$$\lim_{t\to\infty} z(t) = \frac{3(1-p_1)K}{4} \quad \text{and} \quad \lim_{t\to\infty} x(t) = \frac{3(1-p_1)K}{4(1+p_0)} > 0.$$

This completes the proof. □

Remark 1. *Actually, the assumption* (14) *in Theorem* 2 *is not needed in the sufficiency of its proof. Thus, we obtain a corollary as follows.*

Corollary 1. *Assume that condition* (15) *is fulfilled for some constant* $K > 0$*. Then, Equation* (1) *has an eventually positive solution* x *satisfying* $\lim_{t\to\infty} x(t) = b$*, where* $b > 0$ *is a constant.*

Now, we let

$$A(\alpha) = \left\{ x \in S : \lim_{t\to\infty} x(t) = \infty,\ \lim_{t\to\infty} \frac{x(t)}{R(t)} = \alpha \right\},$$

where S stands for the set containing all eventually positive solutions of Equation (1). Then, a lemma is presented as follows.

Lemma 3. *Let x be an eventually positive solution of Equation (1) such that $\lim_{t\to\infty} x(t) = \infty$. Then, condition (5) is satisfied, and $x \in A(0)$ or $x \in A(b)$, where $b > 0$ is a constant.*

Proof. Let x be an eventually positive solution of Equation (1) that satisfies $\lim_{t\to\infty} x(t) = \infty$. Suppose that $\lim_{t\to\infty} z(t) < \infty$. Then, by Lemma 2, $\lim_{t\to\infty} x(t) < \infty$, which causes a contradiction. Therefore, $\lim_{t\to\infty} z(t) = \infty$. In view of equality (17), letting $t \to \infty$, it follows that condition (5) is fulfilled. Define R by (6). It follows from Theorem 1 that $x \in A(0)$ or $x \in A(b)$, where $b > 0$ is a constant. The proof is complete. □

Theorem 3. *Equation (1) has an eventually positive solution which is in $A(b)$ iff*

$$\int_{t_0}^{\infty} f(t, KR(h(t)))\Delta t < \infty \tag{20}$$

for some constant $K > 0$, where $b > 0$ is a constant.

Proof. Let $x \in A(b)$ be an eventually positive solution of Equation (1), where $b > 0$ is a constant. By virtue of Lemma 2 and Theorem 1, we deduce that

$$\lim_{t\to\infty} z(t) = \infty, \quad \lim_{t\to\infty} R_{n-1}(t, x(t)) = \lim_{t\to\infty} \frac{z(t)}{R(t)} = (1 + p_0\eta)b.$$

There is a $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t) > 0$, $x(g(t)) > 0$, and $x(h(t)) \geq bR(h(t))/2$ for $t \in [t_1, \infty)_{\mathbb{T}}$. Integration of Equation (1) from t_1 to s ($s \in [\sigma(T_1), \infty)_{\mathbb{T}}$) yields

$$R_{n-1}(s, x(s)) - R_{n-1}(t_1, x(t_1)) = -\int_{t_1}^{s} f(t, x(h(t)))\Delta t.$$

Letting $s \to \infty$, it follows that

$$\int_{t_1}^{\infty} f(t, x(h(t)))\Delta t < \infty.$$

Since $x(h(t)) \geq bR(h(t))/2$ for $t \in [t_1, \infty)_{\mathbb{T}}$, by (C4), we conclude that

$$\int_{t_1}^{\infty} f\left(t, \frac{b}{2}R(h(t))\right)\Delta t \leq \int_{t_1}^{\infty} f(t, x(h(t)))\Delta t < \infty,$$

which means that condition (20) holds.

Then, suppose that condition (20) holds for some constant $K > 0$.

Case (i). $0 \leq p_0 < 1$. Choose a constant p_1 as in the proof of Theorem 2. When $p_0 > 0$, there is a $T_0 \in [t_0, \infty)_{\mathbb{T}}$ such that for $t \in [T_0, \infty)_{\mathbb{T}}$,

$$p(t) > 0, \quad \frac{5p_1 - 1}{4} \leq p(t) \leq p_1 < 1, \quad p(t)\frac{R(g(t))}{R(t)} \geq \frac{5p_1 - 1}{4}\eta,$$

$$\int_{T_0}^{\infty} f(t, KR(h(t)))\Delta t \leq \frac{(1 - p_1\eta)K}{8}.$$

When $p_0 = 0$, take a constant p_1 satisfying $|p(t)| \le p_1 \le 1/13$ for $t \in [T_0, \infty)_{\mathbb{T}}$. There also exists a $T_1 \in (T_0, \infty)_{\mathbb{T}}$ such that $g(t) \ge T_0$ and $h(t) \ge T_0$ for $t \in [T_1, \infty)_{\mathbb{T}}$.

Let

$$\Omega_2 = \left\{ x \in \mathrm{BC}_1[T_0, \infty)_{\mathbb{T}} : \frac{K}{2} R(t) \le x(t) \le KR(t) \right\}. \tag{21}$$

Then, Ω_2 is also a bounded, convex, and closed subset of $\mathrm{BC}_1[T_0, \infty)_{\mathbb{T}}$. Define now two operators U_2, S_2: $\Omega_2 \to \mathrm{BC}_1[T_0, \infty)_{\mathbb{T}}$ by

$$(U_2 x)(t) = \begin{cases} 3Kp_1\eta R(t)/4 - p(T_1)x(g(T_1))R(t)/R(T_1), & t \in [T_0, T_1)_{\mathbb{T}}, \\ 3Kp_1\eta R(t)/4 - p(t)x(g(t)), & t \in [T_1, \infty)_{\mathbb{T}}, \end{cases}$$

$$(S_2 x)(t) = \begin{cases} 3KR(t)/4, & t \in [T_0, T_1)_{\mathbb{T}}, \\ 3KR(t)/4 \\ + \int_{T_1}^{t} \int_{T_1}^{u_{n-1}} \int_{T_1}^{u_{n-2}} \cdots \int_{T_1}^{u_2} \int_{u_1}^{\infty} \frac{f(u_0, x(h(u_0)))}{\prod_{i=1}^{n-1} r_i(u_i)} \Delta u_0 \Delta u_1 \cdots \Delta u_{n-1}, & t \in [T_1, \infty)_{\mathbb{T}}. \end{cases} \tag{22}$$

The proof that U_2 and S_2 satisfy the conditions in Lemma 1 is also omitted. Similarly, there is an $x \in \Omega_2$ such that $(U_2 + S_2)x = x$. For $t \in [T_1, \infty)_{\mathbb{T}}$,

$$\begin{aligned} x(t) =& \frac{3(1 + p_1\eta)K}{4} R(t) - p(t)x(g(t)) \\ &+ \int_{T_1}^{t} \int_{T_1}^{u_{n-1}} \int_{T_1}^{u_{n-2}} \cdots \int_{T_1}^{u_2} \int_{u_1}^{\infty} \frac{f(u_0, x(h(u_0)))}{\prod_{i=1}^{n-1} r_i(u_i)} \Delta u_0 \Delta u_1 \cdots \Delta u_{n-1}. \end{aligned}$$

Letting $t \to \infty$, we conclude that

$$\lim_{t \to \infty} \frac{z(t)}{R(t)} = \frac{3(1 + p_1\eta)K}{4} \quad \text{and} \quad \lim_{t \to \infty} \frac{x(t)}{R(t)} = \frac{3(1 + p_1\eta)K}{4(1 + p_0\eta)} > 0,$$

which yields $\lim_{t \to \infty} x(t) = \infty$.

Case (ii). $-1 < p_0 < 0$. Introduce $\mathrm{BC}_1[T_0, \infty)_{\mathbb{T}}$ and its subset Ω_2 as in (21). Define S_2 by (22) and U_2' on Ω_2 by

$$(U_2' x)(t) = \begin{cases} -3Kp_1\eta R(t)/4 - p(T_1)x(g(T_1))R(t)/R(T_1), & t \in [T_0, T_1)_{\mathbb{T}}, \\ -3Kp_1\eta R(t)/4 - p(t)x(g(t)), & t \in [T_1, \infty)_{\mathbb{T}}. \end{cases}$$

Similarly, U_2' and S_2 also satisfy the assumptions in Lemma 1. There exists an $x \in \Omega_2$ such that $(U_2' + S_2)x = x$. For $t \in [T_1, \infty)_{\mathbb{T}}$,

$$\begin{aligned} x(t) =& \frac{3(1 - p_1\eta)K}{4} R(t) - p(t)x(g(t)) \\ &+ \int_{T_1}^{t} \int_{T_1}^{u_{n-1}} \int_{T_1}^{u_{n-2}} \cdots \int_{T_1}^{u_2} \int_{u_1}^{\infty} \frac{f(u_0, x(h(u_0)))}{\prod_{i=1}^{n-1} r_i(u_i)} \Delta u_0 \Delta u_1 \cdots \Delta u_{n-1}. \end{aligned}$$

Then, we deduce that

$$\lim_{t \to \infty} \frac{z(t)}{R(t)} = \frac{3(1 - p_1\eta)K}{4} \quad \text{and} \quad \lim_{t \to \infty} \frac{x(t)}{R(t)} = \frac{3(1 - p_1\eta)K}{4(1 + p_0\eta)} > 0.$$

It follows that $\lim_{t \to \infty} x(t) = \infty$. This completes the proof. □

Theorem 4. *Assume that Equation* (1) *has an eventually positive solution which is in* $A(0)$. *Then,*

$$\int_{t_0}^{\infty} f\left(t, \frac{3}{4}\right) \Delta t < \infty \tag{23}$$

and

$$\int_{t_0}^{\infty}\int_{t_0}^{u_{n-1}}\int_{t_0}^{u_{n-2}}\cdots\int_{t_0}^{u_2}\int_{u_1}^{\infty}\frac{f(u_0,R(h(u_0)))}{\prod_{i=1}^{n-1}r_i(u_i)}\Delta u_0\Delta u_1\cdots\Delta u_{n-1}=\infty. \tag{24}$$

Suppose that $|p(t)R(t)|\leq M$ *for some constant* $M>0$ *and for* $t\in[t_0,\infty)_{\mathbb{T}}$,

$$\int_{t_0}^{\infty}f(t,R(h(t)))\Delta t<\infty, \tag{25}$$

and

$$\int_{t_0}^{\infty}\int_{t_0}^{u_{n-1}}\int_{t_0}^{u_{n-2}}\cdots\int_{t_0}^{u_2}\int_{u_1}^{\infty}\frac{f(u_0,M+3/4)}{\prod_{i=1}^{n-1}r_i(u_i)}\Delta u_0\Delta u_1\cdots\Delta u_{n-1}=\infty. \tag{26}$$

Then, Equation (1) *has an eventually positive solution which is in* $A(0)$.

Proof. Let $x\in A(0)$ be an eventually positive solution of Equation (1). Similarly, as in the proof of Theorem 3, we arrive at

$$\lim_{t\to\infty}z(t)=\infty,\quad \lim_{t\to\infty}R_{n-1}(t,x(t))=\lim_{t\to\infty}\frac{z(t)}{R(t)}=0.$$

There exist a $t_1\in[t_0,\infty)_{\mathbb{T}}$ and a $t_2\in(t_1,\infty)_{\mathbb{T}}$ such that $3/4\leq x(t)\leq R(t)$ for $t\in[t_1,\infty)_{\mathbb{T}}$, and $g(t)\geq t_1$, $h(t)\geq t_1$ for $t\in[t_2,\infty)_{\mathbb{T}}$. Integration of Equation (1) from t_2 to s ($s\in[\sigma(t_2),\infty)_{\mathbb{T}}$) yields

$$R_{n-1}(s,x(s))-R_{n-1}(t_2,x(t_2))=-\int_{t_2}^{s}f(t,x(h(t)))\Delta t.$$

Letting $s\to\infty$, we obtain

$$R_{n-1}(t_2,x(t_2))=\int_{t_2}^{\infty}f(t,x(h(t)))\Delta t. \tag{27}$$

Since $x(t)\geq 3/4$ for $t\in[t_2,\infty)_{\mathbb{T}}$, by (C4), we get

$$\int_{t_2}^{\infty}f\left(t,\frac{3}{4}\right)\Delta t<\infty,$$

which implies that inequality (23) holds. Then, replacing t with u_0, and t_2 with u_1 in equality (27), it follows that

$$R^{\Delta}_{n-2}(u_1,x(u_1))=\frac{\int_{u_1}^{\infty}f(u_0,x(h(u_0)))\Delta u_0}{r_1(u_1)}. \tag{28}$$

Integrating equality (28) from t_2 to u_2, $u_2\in[\sigma(t_2),\infty)_{\mathbb{T}}$, we have

$$R^{\Delta}_{n-3}(u_2,x(u_2))=\frac{r_2(t_2)R^{\Delta}_{n-3}(t_2,x(t_2))}{r_2(u_2)}+\frac{1}{r_2(u_2)}\int_{t_2}^{u_2}\int_{u_1}^{\infty}\frac{f(u_0,x(h(u_0)))}{r_1(u_1)}\Delta u_0\Delta u_1.$$

Analogously, for $n\geq 3$, we conclude that

$$\begin{aligned}z(t)-z(t_2)=&R_1(t_2,x(t_2))\int_{t_2}^{t}\frac{1}{r_{n-1}(u_{n-1})}\Delta u_{n-1}\\&+\sum_{k=2}^{n-1}R_k(t_2,x(t_2))\int_{t_2}^{t}\int_{t_2}^{u_{n-1}}\int_{t_2}^{u_{n-2}}\cdots\int_{t_2}^{u_{n-k+1}}\frac{1}{\prod_{i=n-k}^{n-1}r_i(u_i)}\Delta u_{n-k}\Delta u_{n-k+1}\cdots\Delta u_{n-1}\\&+\int_{t_2}^{t}\int_{t_2}^{u_{n-1}}\int_{t_2}^{u_{n-2}}\cdots\int_{t_2}^{u_2}\int_{u_1}^{\infty}\frac{f(u_0,x(h(u_0)))}{\prod_{i=1}^{n-1}r_i(u_i)}\Delta u_0\Delta u_1\cdots\Delta u_{n-1}.\end{aligned}$$

Letting $t \to \infty$, similarly as the proof in Theorem 2, we deduce that

$$\int_{t_2}^{\infty} \int_{t_2}^{u_{n-1}} \int_{t_2}^{u_{n-2}} \cdots \int_{t_2}^{u_2} \int_{u_1}^{\infty} \frac{f(u_0, x(h(u_0)))}{\prod_{i=1}^{n-1} r_i(u_i)} \Delta u_0 \Delta u_1 \cdots \Delta u_{n-1} = \infty.$$

Since $x(t) \leq R(t)$ for $t \in [t_2, \infty)_{\mathbb{T}}$, by virtue of $(C4)$,

$$\int_{t_2}^{\infty} \int_{t_2}^{u_{n-1}} \int_{t_2}^{u_{n-2}} \cdots \int_{t_2}^{u_2} \int_{u_1}^{\infty} \frac{f(u_0, R(h(u_0)))}{\prod_{i=1}^{n-1} r_i(u_i)} \Delta u_0 \Delta u_1 \cdots \Delta u_{n-1} = \infty,$$

which means that equality (24) holds.

Assume that $|p(t)R(t)| \leq M$ for some constant $M > 0$ and for $t \in [t_0, \infty)_{\mathbb{T}}$, conditions (25) and (26) hold. Then, $\lim_{t\to\infty} p(t) = p_0 = 0$. Choose a $T_0 \in [t_0, \infty)_{\mathbb{T}}$ and $0 < p_1 < 1$ such that for $t \in [T_0, \infty)_{\mathbb{T}}$,

$$|p(t)| \leq p_1 < 1, \quad 2M + \frac{3}{2} \leq \frac{1}{4} R(t), \quad \int_{T_0}^{\infty} f(t, R(h(t))) \Delta t \leq \frac{1 - p_1}{8}.$$

There exists a $T_1 \in (T_0, \infty)_{\mathbb{T}}$ such that $g(t) \geq T_0$ and $h(t) \geq T_0$ hold for $t \in [T_1, \infty)_{\mathbb{T}}$.
Let

$$\Omega_3 = \left\{ x \in \mathrm{BC}_1[T_0, \infty)_{\mathbb{T}} : M + \frac{3}{4} \leq x(t) \leq R(t) \right\}.$$

Then, Ω_3 is a bounded, convex, and closed subset of $\mathrm{BC}_1[T_0, \infty)_{\mathbb{T}}$. Define now two operators U_3, S_3: $\Omega_3 \to \mathrm{BC}_1[T_0, \infty)_{\mathbb{T}}$ by

$$(U_3 x)(t) = \begin{cases} M + 3/4 - p(T_1) x(g(T_1)) R(t)/R(T_1), & t \in [T_0, T_1)_{\mathbb{T}}, \\ M + 3/4 - p(t) x(g(t)), & t \in [T_1, \infty)_{\mathbb{T}}, \end{cases}$$

$$(S_3 x)(t) = \begin{cases} M + 3/4, & t \in [T_0, T_1)_{\mathbb{T}}, \\ M + 3/4 & \\ + \int_{T_1}^{t} \int_{T_1}^{u_{n-1}} \int_{T_1}^{u_{n-2}} \cdots \int_{T_1}^{u_2} \int_{u_1}^{\infty} \frac{f(u_0, x(h(u_0)))}{\prod_{i=1}^{n-1} r_i(u_i)} \Delta u_0 \Delta u_1 \cdots \Delta u_{n-1}, & t \in [T_1, \infty)_{\mathbb{T}}. \end{cases}$$

The proof that U_3 and S_3 satisfy the assumptions in Lemma 1 is also omitted. Then, there is an $x \in \Omega_3$ such that $(U_3 + S_3)x = x$. For $t \in [T_1, \infty)_{\mathbb{T}}$,

$$x(t) = 2M + \frac{3}{2} - p(t) x(g(t)) + \int_{T_1}^{t} \int_{T_1}^{u_{n-1}} \int_{T_1}^{u_{n-2}} \cdots \int_{T_1}^{u_2} \int_{u_1}^{\infty} \frac{f(u_0, x(h(u_0)))}{\prod_{i=1}^{n-1} r_i(u_i)} \Delta u_0 \Delta u_1 \cdots \Delta u_{n-1}.$$

In view of condition (26), we get

$$\lim_{t\to\infty} z(t) = \infty \quad \text{and} \quad \lim_{t\to\infty} \frac{z(t)}{R(t)} = 0.$$

Since $|p(t)x(g(t))| \leq |p(t)R(t)| \leq M$, by virtue of Lemma 2, we conclude that

$$\lim_{t\to\infty} x(t) = \infty \quad \text{and} \quad \lim_{t\to\infty} \frac{x(t)}{R(t)} = 0.$$

The proof is complete. □

Remark 2. *It is not easy to establish the sufficient and necessary conditions which guarantee that Equation* (1) *has an eventually positive solution x satisfying* $\lim_{t\to\infty} x(t) = 0$. *We refer the reader to [20] (Theorems 3.2 and 3.3) for sufficient conditions to ensure it.*

Remark 3. *When $n = 3$, it is obvious that Theorems 2–4, Corollary 1, and Lemma 3 cover the results in [22]. Furthermore, even when $n = 2$, the conclusions above are also consistent with those in [12,13].*

4. Examples

The following two examples are presented to illustrate theoretical results obtained in this article.

Example 1. *Let $\mathbb{T} = \bigcup_{n=1}^{\infty}[(4n-3)c, 4nc]$, where $c > 0$. For $t \in [5c, \infty)_{\mathbb{T}}$, consider*

$$R_n(t, x(t)) + t^{\gamma} x(t) = 0, \tag{29}$$

where $n \geq 3$, $\gamma \in \mathbb{R}$, and

$$R_k(t, x(t)) = \begin{cases} x(t) - (t-c)/(2t)x(t-4c), & k = 0, \\ t^{n-k} R_{k-1}^{\Delta}(t, x(t)), & 1 \leq k \leq n-1, \\ R_{n-1}^{\Delta}(t, x(t)), & k = n. \end{cases}$$

We can see that $r_i(t) = t^i$, $i = 1, 2, \ldots, n-1$, $p(t) = -(t-c)/(2t)$, $g(t) = t - 4c$, $h(t) = t$, $f(t, x) = t^{\gamma} x$, $t_0 = 5c$, and $p_0 = -1/2$. Obviously, conditions (C1)–(C4) and (14) are satisfied. Taking $K = 1$, we conclude that

$$\begin{aligned} &\int_{t_0}^{\infty} \int_{t_0}^{u_{n-1}} \int_{t_0}^{u_{n-2}} \cdots \int_{t_0}^{u_1} \frac{f(u_0, 1)}{\prod_{i=1}^{n-1} r_i(u_i)} \Delta u_0 \Delta u_1 \cdots \Delta u_{n-1} \\ = &\int_{5c}^{\infty} \int_{5c}^{u_{n-1}} \int_{5c}^{u_{n-2}} \cdots \int_{5c}^{u_1} \frac{u_0^{\gamma}}{\prod_{i=1}^{n-1} u_i^i} \Delta u_0 \Delta u_1 \cdots \Delta u_{n-1} \\ = &\int_{5c}^{\infty} O\left(u_{n-1}^{\gamma-(n-1)(n-2)/2}\right) \Delta u_{n-1}. \end{aligned}$$

When $\gamma < n(n-3)/2$, which means that $\gamma - (n-1)(n-2)/2 < -1$, condition (15) holds. By virtue of Theorem 2 (or Corollary 1), we deduce that Equation (29) has an eventually positive solution x that satisfies $\lim_{t\to\infty} x(t) = b$, where $b > 0$ is a constant. Moreover, Equation (29) has no eventually positive solutions x satisfying $\lim_{t\to\infty} x(t) = b > 0$ provided that $\gamma \geq n(n-3)/2$.

Example 2. *Let $\mathbb{T} = \bigcup_{n=0}^{\infty}[2 \cdot 3^n, 3^{n+1}]$. For $t \in [6, \infty)_{\mathbb{T}}$, consider*

$$R_n(t, x(t)) + t^{\gamma} x^3(3t) = 0, \tag{30}$$

where $n \geq 3$, $\gamma \in \mathbb{R}$, and

$$R_k(t, x(t)) = \begin{cases} x(t) + 1/t \cdot x(t/3), & k = 0, \\ t^2 R_{k-1}^{\Delta}(t, x(t)), & 1 \leq k \leq n-2, \\ 1/t^{n-2} \cdot R_{n-2}^{\Delta}(t, x(t)), & k = n-1, \\ R_{n-1}^{\Delta}(t, x(t)), & k = n. \end{cases}$$

We get $r_1(t) = 1/t^{n-2}$, $r_i(t) = t^2$, $i = 2, 3, \ldots, n-1$, $p(t) = 1/t$, $g(t) = t/3$, $h(t) = 3t$, $f(t, x) = t^{\gamma} x^3$, $t_0 = 6$, and $p_0 = 0$. It is not difficult to see that the assumptions (C1)–(C4) are fulfilled. From (C5), we have

$$\begin{aligned} R(t) &= 1 + \int_{t_0}^{t} \int_{t_0}^{u_{n-1}} \int_{t_0}^{u_{n-2}} \cdots \int_{t_0}^{u_2} \frac{1}{\prod_{i=1}^{n-1} r_i(u_i)} \Delta u_1 \Delta u_2 \cdots \Delta u_{n-1} \\ &= 1 + \int_{6}^{t} \int_{6}^{u_{n-1}} \int_{6}^{u_{n-2}} \cdots \int_{6}^{u_2} \frac{u_1^{n-2}}{\prod_{i=2}^{n-1} u_i^2} \Delta u_1 \Delta u_2 \cdots \Delta u_{n-1} = O(t), \end{aligned}$$

which implies that

$$\eta = \lim_{t\to\infty} \frac{R(g(t))}{R(t)} = \lim_{t\to\infty} \frac{O(t/3)}{O(t)} = \frac{1}{3} \in (0,1].$$

Hence, $(C5)$ holds, and we arrive at

$$\int_{t_0}^{\infty} f(t, KR(h(t)))\Delta t = \int_6^{\infty} t^{\gamma}(O(t))^3 \Delta t = \int_6^{\infty} O(t^{\gamma+3})\Delta t.$$

Due to Theorem 3, condition (20) holds when $\gamma < -4$, and we conclude that Equation (30) has an eventually positive solution $x \in A(b)$ for some constant $b > 0$. However, Equation (30) has no eventually positive solutions $x \in A(b)$ provided that $\gamma \geq -4$.

On the other hand, when $\gamma < -5$, we obtain

$$\begin{aligned}
&\int_{t_0}^{\infty}\int_{t_0}^{u_{n-1}}\int_{t_0}^{u_{n-2}}\cdots\int_{t_0}^{u_2}\int_{u_1}^{\infty} \frac{f(u_0, R(h(u_0)))}{\prod_{i=1}^{n-1} r_i(u_i)} \Delta u_0 \Delta u_1 \cdots \Delta u_{n-1} \\
=& \int_6^{\infty}\int_6^{u_{n-1}}\int_6^{u_{n-2}}\cdots\int_6^{u_2}\int_{u_1}^{\infty} \frac{O(u_0^{\gamma+3}) \cdot u_1^{n-2}}{\prod_{i=2}^{n-1} u_i^2} \Delta u_0 \Delta u_1 \cdots \Delta u_{n-1} \\
=& \int_6^{\infty} u_{n-1}^{\gamma+4} \Delta u_{n-1} < \infty.
\end{aligned}$$

That is, inequality (23) does not hold. It follows from Theorem 4 that Equation (30) has no eventually positive solutions $x \in A(0)$.

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