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Existence of Positive Solutions and Its Asymptotic Behavior of $(p(x), q(x))$-Laplacian Parabolic System

Hamza Medekhel [1,2], **Salah Boulaaras** [3,4,*], **Khaled Zennir** [3] and **Ali Allahem** [5]

1 Department of Mathematics and Computer Science, Larbi Tebessi University, Tebessa 12002, Algeria; hamza-medekhel@univ-eloued.dz
2 Department of Mathematics, Exact Sciences Faculty, University of EL Oued, P.O. Box 789, El-Oued 39000, Algeria
3 Department of Mathematics, College of Sciences and Arts, Al-Rass, Qassim University, Buraydah 51452, Saudi Arabia; k.zennir@qu.edu.sa
4 Laboratory of Fundamental and Applied Mathematics of Oran (LMFAO), University of Oran 1, Ahmed Benbella, Oran 31000, Algeria
5 Department of Mathematics, College of Sciences, Qassim University, Buraydah 51452, Saudi Arabia; aallahem@qu.edu.sa
* Correspondence: saleh_boulaares@yahoo.fr or S.Boularas@qu.edu.sa

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Abstract: This paper deals with the existence of positively solution and its asymptotic behavior for parabolic system of $(p(x), q(x))$-Laplacian system of partial differential equations using a sub and super solution according to some given boundary conditions, Our result is an extension of Boulaaras's works which studied the stationary case, this idea is new for evolutionary case of this kind of problem.

Keywords: existence; laplacian parabolic system; sub-supersolution method; asymptotic behavior

JEL Classification: 35J60; 35B30; 35B40

1. Introduction

In this paper, we consider the following evolutionary problem: find $u \in L^2(0, T, H_0^1(\Omega))$ solution of

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta_{p(x)} u = \lambda^{p(x)} \left[\lambda_1 a(x) f(v) + \mu_1 c(x) h(u)\right] & \text{in} \quad Q_T = (0, T) \times \Omega, \\ \dfrac{\partial v}{\partial t} - \Delta_{q(x)} v = \lambda^{q(x)} \left[\lambda_2 b(x) g(u) + \mu_2 d(x) \tau(v)\right] & \text{in} \quad Q_T = (0, T) \times \Omega, \\ u = v = 0 \quad \text{on} \quad \partial Q_T = (0, T) \times \partial\Omega, \\ u(x, 0) = \kappa_1(x), u(x, 0) = \kappa_2(x), \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain and the functions $p(x), q(x)$ belong to $C^1(\overline{\Omega})$ and satisfying the following conditions:

$$1 < p^- := \inf_{x\in\Omega} p(x) \leq p^+ := \sup_{\Omega} p(x) < \infty, 1 < q^- := \inf_{x\in\Omega} q(x) \leq q^+ := \sup_{x\in\Omega} q(x) < \infty \tag{2}$$

and satisfy some natural growth condition at $u = \infty$.

$\Delta_{p(x)}$ is given by $\Delta_{p(x)}u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called $p(x)$-Laplacian, the parameters $\lambda, \lambda_1, \lambda_2, \mu_1$ and μ_2 are positive with a, b, c, d are regular functions. In addition we did not consider any sign condition on $f(0), g(0), h(0), \tau(0)$.

The linear and nonlinear stationary equations with operators of quasilinear homogeneous type as p-Laplace operator can be carried out according to the standard Sobolev spaces theory of $W^{m,p}$, and thus we can find the weak solutions. The last spaces consist of functions having weak derivatives which verify some conditions of integrability. Thus, we can have the nonhomogeneous case of $p(.)$-Laplace operators in this last condition. We will use Sobolev spaces of the exponential variable in our standard framework, so that $L^{p(.)}(\Omega)$ will be used instead of Lebesgue spaces $L^p(\Omega)$.

We denote new Sobolev space by $W^{m,p}(\Omega)$, if we replace $L^p(\Omega)$ by $L^{p(.)}(\Omega)$, the Sobolev spaces becomes $W^{m,p(.)}(\Omega)$. Several Sobolev spaces properties have been extended to spaces of Orlicz-Sobolev, particularly by O'Neill in the reference ([1]). The spaces $W^{m,p(.)}(\Omega)$ and $L^{p(.)}(\Omega)$ have been carefully studied by many researchers team (see the references ([2] and [3–5]).

Here, in our study we consider the boundedness condition in domain Ω, because many results under p-Laplacian theory are not usually verified for the $p(x)$-Laplacian theory; for that in ([6]) the quotient

$$\lambda_{p(x)} = \inf_{u \in W_0^{1,p(x)}(\Omega) \setminus \{0\}} \frac{\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx}{\int_\Omega \frac{1}{p(x)} |u|^{p(x)} dx} \tag{3}$$

becomes 0 generally. Then $\lambda_{p(x)}$ can be positive only for some given conditions. In fact, the first eigenvalue of $p(x)$-Laplacian and its associated eigenfunction cannot exist, the existence of the positive first eigenvalue λ_p and getting its eigenfunction are very important in the p-Laplacian problem study. Therefore, the study of existence of solutions of our problems have more meaning. Many studies of the experimental side have been studied on various materials that rely on this advanced theory, as they are important in electrical fluids, which states that viscosity relates to the electric field in a certain liquid.

Recently, in ([2,6–8]), we have proved the existence of positive solutions of many classes of $(p(x), q(x))$-Laplacian stationary problems by using the sub -super solution concept. The current results are an extension of our previous stationary study to the parabolic case, where we follow-up the same procedures mathematical proofs similar to that in ([2,7]) by using difference time scheme taking into consideration the stability analysis of the used scheme and the same conditions which have given in references mentioned earlier. Our result is an extension for our previous study in ([2,7,9]) which studied the stationary case, this idea is new for evolutionary case of this kind of problem.

The outline of paper consists as follow: In first section we give some definitions, basic theorems and necessarily propositions in the functional analysis which will be used in our study. Then in Section 3, we prove our main result.

2. Preliminaries Results and Assumptions

In order to discuss problem (1), we need some theories on $W_0^{1,p(x)}(\Omega)$ which we call variable exponent Sobolev space. Firstly we state some basic properties of spaces $W_0^{1,p(x)}(\Omega)$ which will be used later (for details, see [10]).

Let us define

$$L^{p(x)}(\Omega) = \left\{ u : u \text{ is a measurable real-valued function such that } \int_\Omega |u(x)|^{p(x)} dx < \infty \right\}.$$

We introduce the norm on $L^{p(x)}(\Omega)$ by

$$|u(x)|_{p(x)} = \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}$$

and

$$W^{1,p(x)}(\Omega) = \left\{u \in L^{p(x)}(\Omega); |\nabla u| \in L^{p(x)}(\Omega)\right\},$$

with the norm

$$\|u\| = |u|_{p(x)} + |\nabla u|_{p(x)}, \forall u \in W^{1,p(x)}(\Omega).$$

We denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$.

We introduce in this applying for problem (2), we will assume that:

(H_1) $p, q \in C^1(\overline{\Omega})$ and $1 < p_- < p_+, 1 < q_- < q_+$;

(H_2) $f, g, h$and $\tau : [0, +\infty[\to \mathbb{R}$ are C^1,monotone functions, such that

$$\lim_{u\to+\infty} f(u_k) = +\infty \lim_{u\to+\infty} g(u_k) = +\infty, \lim_{u\to+\infty} h(u_k) = +\infty, \lim_{u\to+\infty} \tau(u_k) = +\infty,$$

(H_3) $\lim\limits_{u\to+\infty} \frac{f(M(g(u_k))^{\frac{1}{q^- -1}})}{u_k^{p^- -1}} = 0$, for all $M > 0$;

(H_4) $\lim\limits_{u\to+\infty} \frac{h(u_k)}{u_k^{p^- -1}} = 0$, and $\lim\limits_{u\to+\infty} \frac{\tau(u_k)}{u_k^{p^- -1}} = 0$;

(H_5) $a, b, c, d : \overline{\Omega} \to (0, +\infty)$ are contionous functions, such that

$$a_1 = \min_{x\in\overline{\Omega}} a(x), b_1 = \min_{x\in\overline{\Omega}} b(x), c_1 = \min_{x\in\overline{\Omega}} c(x), d_1 = \min_{x\in\overline{\Omega}} d(x),$$

$$a_2 = \max_{x\in\overline{\Omega}} a(x), b_2 = \max_{x\in\overline{\Omega}} b(x), c_2 = \max_{x\in\overline{\Omega}} c(x), d_2 = \max_{x\in\overline{\Omega}} d(x).$$

The Semi-Discrete Problem

We discrete the problem (1) by difference time scheme, we obtain the following problems

$$\begin{cases} u_k - \tau'\Delta_{p(x)}u_k = \tau'\lambda^{p(x)}\left[\lambda_1 a(x) f(v) + \mu_1 c(x) h(u_k)\right] + u_{k-1} \text{ in } \Omega, \\ v_k - \tau'\Delta_{q(x)}v = \tau'\lambda^{q(x)}\left[\lambda_2 b(x) g(u_k) + \mu_2 d(x)\tau(v)\right] + v_{k-1} \text{ in } \Omega, \\ u_k = v = 0 \text{ on } \partial\Omega, \\ u_0 = \varphi_0, \end{cases} \tag{4}$$

where $N\tau' = T, 0 < \tau' < 1$, and for $1 \le k \le N$.

We define

$$\langle L(u_k), v\rangle = \int_\Omega |\nabla u_k|^{p(x)-2} \nabla u_k \nabla v dx, \forall u_k, v \in W_0^{1,p(x)}(\Omega).$$

According to ([11] in Theorem 3.1), the bounded operator $L : W_0^{1,p(x)}(\Omega) \to \left(W_0^{1,p(x)}(\Omega)\right)^*$ is a continuous and strictly monotone, and it is a homeomorphism.

We considere mapping $A : W_0^{1,p(x)}(\Omega) \to \left(W_0^{1,p(x)}(\Omega)\right)^*$ as

$$\langle A(u_k), \varphi\rangle = \int_\Omega \left(|\nabla u_k|^{p(x)-2} \nabla u_k \nabla\varphi + h(x, u_k)\varphi\right) dx, \text{for all } u_k, v \in W_0^{1,p(x)}(\Omega),$$

where $h(x, u_k)$ is continuous on $\overline{\Omega} \times \mathbb{R}$, and $h(x, .)$ is increasing function.It is easy to verify that A is a continuous bounded mapping. By the proof ([12]).

Definition 1. *An weak solution to discretized problems* (P_k) *is a sequence* $(u_k, v)_{0\leq k\leq N}$ *such that* $u_0 = \varphi_0$ *and* (u_k, v) *is defined by*

$$\begin{cases} u_k - \tau' \Delta_{p(x)} u_k = \tau' \lambda^{p(x)} \left[\lambda_1 a(x) f(v) + \mu_1 c(x) h(u_k)\right] + u_{k-1} & \text{in } \Omega, \\ v_k - \tau' \Delta_{q(x)} v = \tau' \lambda^{q(x)} \left[\lambda_2 b(x) g(u_k) + \mu_2 d(x) \tau(v)\right] + v_{k-1} & \text{in } \Omega, \\ u_k = v = 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

such that

$$\begin{cases} -\Delta_{p(x)} u_k = \lambda^{p(x)} \left[\lambda_1 a(x) f(v) + \mu_1 c(x) h(u_k)\right] - \frac{u_k - u_{k-1}}{\tau'} & \text{in } \Omega, \\ -\Delta_{q(x)} v = \lambda^{q(x)} \left[\lambda_2 b(x) g(u_k) + \mu_2 d(x) \tau(v)\right] - \frac{v_k - v_{k-1}}{\tau'} & \text{in } \Omega, \\ u_k = v = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{5}$$

We have the following:

(1) *If* $(u_k, v) \in \left(W_0^{1.p(x)}(\Omega) \times W_0^{1.q(x)}(\Omega)\right)$, (u_k, v) *is called a weak solution of (5) if it satisfies*

$$\int_\Omega |\nabla u_k|^{p(x)-2} \nabla u_k . \nabla \varphi dx = \int_\Omega \left[\lambda^{p(x)} \left[\lambda_1 a(x) f(v) + \mu_1 c(x) h(u_k)\right] - \frac{u_k - u_{k-1}}{\tau'}\right] \varphi dx,$$

$$\int_\Omega |\nabla v|^{q(x)-2} \nabla v . \nabla \psi dx = \int_\Omega \left[\lambda^{q(x)} \left[\lambda_2 b(x) g(u_k) + \mu_2 d(x) \tau(v)\right] - \frac{v_k - v_{k-1}}{\tau'}\right] \psi dx. \tag{6}$$

for all

$$(\varphi, \psi) \in \left(W_0^{1.p(.)}(\Omega) \times W_0^{1.q(.)}(\Omega)\right)$$

with $(\varphi, \psi) \geqslant 0$.

(2) *We say called a sub solution (respectively a super solution) of (1) if*

$$\int_\Omega |\nabla u_k|^{p(x)-2} \nabla u_k . \nabla \varphi dx \leq (\text{respectively} \geqslant) \int_\Omega \left[\lambda^{p(x)} \left[\lambda_1 a(x) f(v) + \mu_1 c(x) h(u_k)\right] - \frac{u_k - u_{k-1}}{\tau'}\right] \varphi dx,$$

$$\int_\Omega |\nabla v|^{q(x)-2} \nabla v . \nabla \psi dx \leq (\text{respectively } \geqslant) \int_\Omega \left[\lambda^{q(x)} \left[\lambda_2 b(x) g(u_k) + \mu_2 d(x) \tau(v)\right] - \frac{v_k - v_{k-1}}{\tau'}\right] \psi dx.$$

Lemma 1. *(Comparison principle) Let* $u_k, v \in W_0^{1,p(x)}(\Omega)$ *verify* $Au_k - Av \geqslant 0$ *in* $\left(W_0^{1,p(x)}(\Omega)\right)^*$, $\varphi(x) = \min\{u_k(x) - v(x), 0\}$ *.If* $\varphi(x) \in W_0^{1,p(x)}(\Omega)$ *(i.e.,* $u_k \geqslant v$ *on* $\partial\Omega$*), then* $u_k \geqslant v$ *a.e in* Ω.

Here, we will use the notation $d(x, \partial\Omega)$ to denote the distance of $x \in \Omega$ to denote the distance of Ω.

Denote $d(x) = d(x, \partial\Omega)$ and $\partial\Omega_\varepsilon = \{x \in \Omega : d(x, \partial\Omega) < \varepsilon\}$.

Since $\partial\Omega$ is C^2 regularly, there exists a constant $\delta \in (0,1)$ such that $d(x) \in C^2(\overline{\partial\Omega_{3\delta}})$ and $|\nabla d(x)| = 1$.

Denote also

$$v_1(x)=\begin{cases}\gamma d(x), \;\; d(x)<\delta,\\ \gamma\delta+\int_\delta^{d(x)}\gamma(\frac{2\delta-t}{\delta})^{\frac{2}{p^--1}}(\lambda_1 a_1+\mu_1 c_1)^{\frac{2}{p^--1}}dt, \;\; \delta\le d(x)\le 2\delta,\\ \gamma\delta+\int_\delta^{2\delta}\gamma(\frac{2\delta-t}{\delta})^{\frac{2}{p^--1}}(\lambda_1 b_1+\mu_1 d_1)^{\frac{2}{p^--1}}dt, \;\; 2\delta\le d(x)\end{cases}$$

and

$$v_2(x)=\begin{cases}\gamma d(x), \;\; d(x)<\delta,\\ \gamma\delta+\int_\delta^{d(x)}\gamma(\frac{2\delta-t}{\delta})^{\frac{2}{p^--1}}(\lambda_2 a_2+\mu_2 c_2)^{\frac{2}{q^--1}}dt, \;\; \delta\le d(x)\le 2\delta,\\ \gamma\delta+\int_\delta^{2\delta}\gamma(\frac{2\delta-t}{\delta})^{\frac{2}{p^--1}}(\lambda_2 b_2+\mu_2 d_2)^{\frac{2}{q^--1}}dt, \;\; 2\delta\le d(x).\end{cases}$$

Obviously,

$$0\le v_1(x), v_2(x)\in C^1(\overline{\Omega}).$$

Considering

$$\begin{cases}-\Delta_{p(x)}w(x)=\eta \;\text{ in }\Omega\\ w=0 \;\;\text{ on }\partial\Omega.\end{cases}\tag{7}$$

Lemma 2. *([13]), If positive parameter η is large enough and w is the unique solution of (7), then we have*

(i) For any $\theta\in(0,1)$ there exists a positive constant C_1, such that

$$C_1\eta^{\frac{1}{p^+-1+\theta}}\le\max_{x\in\overline{\Omega}}w(x).$$

(ii) There exists a positive constant C_2, such that

$$\max_{x\in\overline{\Omega}}w(x)\le C_2\eta^{\frac{1}{p^--1}}$$

3. Main Result

In the following, once we have no misunderstanding, we always use C_i to denote the positive constants.

Theorem 1. *Assume that the conditions (H_1)–(H_5) are statisfied.Then, problem (1) has a positive solution when λ is large enough.*

Proof. We establish Theorem 1 by constructing a positive subsolution (ϕ_{k_1},ϕ_{k_2}) and supersolution (z_{k_1},z_{k_2}) of (1) such that $\phi_{k_1}\le z_{k_1}$ and $\phi_{k_2}\le z_{k_2}$, that is (ϕ_{k_1},ϕ_{k_2}) and (z_{k_1},z_{k_2}) satisfies

$$\begin{aligned}\int_\Omega|\nabla\phi_{k_1}|^{p(x)-2}\nabla\phi_{k_1}.\nabla\varphi dx &\le \int_\Omega\left[\lambda^{p(x)}\left[\lambda_1 a(x)f(\phi_{k_2})+\mu_1 c(x)h(\phi_{k_1})\right]-\frac{\phi_{k_1}-\phi_{k_1-1}}{\tau'}\right]\varphi dx,\\ \int_\Omega|\nabla\phi_{k_2}|^{q(x)-2}\nabla\phi_{k_2}.\nabla\psi dx &\le \int_\Omega\left[\lambda^{q(x)}\left[\lambda_2 b(x)g(\phi_{k_1})+\mu_2 d(x)\tau(\phi_{k_2})\right]-\frac{\phi_{k_1}-\phi_{k_1-1}}{\tau'}\right]\psi dx,\end{aligned}$$

and

$$\int_\Omega |\nabla z_{k_1}|^{p(x)-2}\nabla z_{k_1}.\nabla\varphi dx \geq \int_\Omega \left[\lambda^{p(x)}\left[\lambda_1 a(x)f(z_{k_2})+\mu_1 c(x)h(z_{k_1})\right]-\frac{z_{k_1}-z_{k_1-1}}{\tau'}\right]\varphi dx,$$

$$\int_\Omega |\nabla z_{k_2}|^{q(x)-2}\nabla z_{k_2}.\nabla\psi dx \geq \int_\Omega \left[\lambda^{q(x)}\left[\lambda_2 b(x)g(z_{k_1})+\mu_2 d(x)\tau(z_{k_2})\right]-\frac{z_{k_1}-z_{k_1-1}}{\tau'}\right]\psi dx,$$

for all $(\varphi,\psi)\in\left(W_0^{1.p(x)}(\Omega)\times W_0^{1.q(x)}(\Omega)\right)$ with $(\varphi,\psi)\geqslant 0$. According to the sub-super solution method for $((p(x),q(x)))$ $-$Laplacian systems see ([9,13]), the problem (1) has a positive solution.

Step 1. We will construct a subsolution of (1). Let $\sigma\in(0,\delta)$ is small enough. Denote

$$\phi_{k_1}(x)=\begin{cases} e^{kd(x)}-1, \ d(x)<\sigma, \\ e^{kd(x)}-1+\int_\delta^{d(x)} ke^{k\sigma}\left(\frac{2\delta-t}{2\delta-\sigma}\right)^{\frac{2}{p^--1}}dt, \quad \sigma\leq d(x)<2\delta, \\ e^{kd(x)}-1+\int_\sigma^{2\delta} ke^{k\sigma}\left(\frac{2\delta-t}{2\delta-\sigma}\right)^{\frac{2}{p^--1}}dt, \ 2\delta\ \leq d(x) \end{cases}$$

and

$$\phi_{k_2}(x)=\begin{cases} e^{kd(x)}-1, \ d(x)<\sigma, \\ e^{kd(x)}-1+\int_\delta^{d(x)} ke^{k\sigma}\left(\frac{2\delta-t}{2\delta-\sigma}\right)^{\frac{2}{q^--1}}dt, \quad \sigma\leq d(x)<2\delta, \\ e^{kd(x)}-1+\int_\sigma^{2\delta} ke^{k\sigma}\left(\frac{2\delta-t}{2\delta-\sigma}\right)^{\frac{2}{q^--1}}dt, \ 2\delta\ \leq d(x). \end{cases}$$

It easy to see that $\phi_{k_1},\phi_{k_2}\in C^1(\overline{\Omega})$.

Denote

$$\alpha=\min\left\{\frac{\inf p(x)-1}{4(\sup|\nabla p(x)+1|)},\frac{\inf q(x)-1}{4(\sup|\nabla q(x)+1|)},1\right\}$$

and

$$\xi=\min\left\{\lambda_1 a_1 f(0)+\mu_1 c_1 h(0),\lambda_2 b_1 g(0)+\mu_2 d_1\sigma(0),-1\right\}.$$

By some simple computations we obtain

$$-\Delta_{p(x)}\phi_{k_1}=\begin{cases} -k(e^{kd(x)})^{p(x)-1}\left[(p(x)-1)+(d(x)+\frac{\ln k}{k})\nabla p\nabla d+\frac{\Delta d}{k}\right], \ d(x)<\sigma \\ \left\{\frac{1}{2\delta-\sigma}\frac{2(p(x)-1)}{p^--1}-\left(\frac{2\delta-d}{2\delta-\sigma}\right)\left[\left(\ln ke^{k\sigma}\right)\left(\frac{2\delta-d}{2\delta-\sigma}\right)^{\frac{2}{p^--1}}\nabla p\nabla d+\Delta d\right]\right\} \\ \times\left(Ke^{k\sigma}\right)^{p(x)-1}\left(\frac{2\delta-d}{2\delta-\sigma}\right)^{\frac{2(p(x)-1)}{p^--1}-1}, \ \sigma\leq d(x)<2\delta, \\ 0, \ 2\delta\leq d(x) \end{cases}$$

and

$$-\Delta_{p(x)}\phi_{k_2} = \begin{cases} -k(e^{kd(x)})^{q(x)-1}\left[(q(x)-1)+(d(x)+\frac{\ln k}{k})\nabla q\nabla d+\frac{\Delta d}{k}\right], \; d(x)<\sigma, \\ \left\{\frac{1}{2\delta-\sigma}\frac{2(q(x)-1)}{q^{-}-1}-\left(\frac{2\delta-d}{2\delta-\sigma}\right)\left[\left(\ln ke^{k\sigma}\right)\left(\frac{2\delta-d}{2\delta-\sigma}\right)^{\frac{2}{q^{-}-1}}\nabla q\nabla d+\Delta d\right]\right\} \\ \times\left(Ke^{k\sigma}\right)^{q(x)-1}\left(\frac{2\delta-d}{2\delta-\sigma}\right)^{\dfrac{2(q(x)-1)}{q^{-}-1}-1}, \; \sigma\le d(x)<2\delta, \\ 0, \; 2\delta\le d(x). \end{cases}$$

From (H_3) there exists a positive constant $M>1$ such that

$$\begin{aligned} f(M-1) &\geqslant 1, \; g(M-1)\geqslant 1, \\ h(M-1) &\geqslant 1, \; \sigma(M-1)\geqslant 1. \end{aligned}$$

Let $\sigma=\frac{1}{k}\ln M$, then

$$\sigma k=\ln M. \tag{8}$$

If k is sufficiently large, from (8), we have

$$-\Delta_{p(x)}\phi_{k_1}\le -k^{p(x)}\alpha, \; d(x)<\sigma. \tag{9}$$

Let $\lambda\xi=k\alpha$, then

$$k^{p(x)}\alpha\geqslant -\lambda^{p(x)}\xi.$$

From (9), we have

$$\begin{cases} -\Delta_{p(x)}\phi_{k_1}\le\lambda^{p(x)}\xi\le\lambda^{p(x)}(\lambda_1 a_1 f(0)+\mu_1 c_1 h(0)) \\ \le\lambda^{p(x)}(\lambda_1 a(x)f(\phi_{k_2})+\mu_1 c(x)h(\phi_{k_1})), \quad d(x)<\sigma. \end{cases} \tag{10}$$

Since $d(x)\in C^2(\overline{\partial\Omega_{3\delta}})$, there exists a positive constant C_3, such that

$$\begin{aligned} -\Delta_{p(x)}\phi_{k_1} \le\;& \left(Ke^{k\sigma}\right)^{p(x)-1}\left(\frac{2\delta-d}{2\delta-\sigma}\right)^{\dfrac{2(p(x)-1)}{p^{-}-1}-1}(\lambda_1 a_1+\mu_1 c_1) \\ &\times\left|\left\{\frac{1}{2\delta-\sigma}\frac{2(p(x)-1)}{p^{-}-1}-\left(\frac{2\delta-d}{2\delta-\sigma}\right)\right.\right. \\ &\times\left.\left.\left[\left(\ln ke^{k\sigma}\right)\left(\frac{2\delta-d}{2\delta-\sigma}\right)^{\frac{2}{p^{-}-1}}\nabla p\nabla d+\Delta d\right]\right\}\right| \\ \le\;& C_3\left(Ke^{k\sigma}\right)^{p(x)-1}(\lambda_1 a_1+\mu_1 c_1)\ln k, \;\; \sigma\le d(x)<2\delta. \end{aligned}$$

If k is sufficiently large, let $\lambda\xi=k\alpha$, then we have

$$\begin{aligned} C_3\left(Ke^{k\sigma}\right)^{p(x)-1}(\lambda_1 a_1+\mu_1 c_1)\ln k &= C_3\,(kM)^{p(x)-1}(\lambda_1 a_1+\mu_1 c_1)\ln k \\ &\le \lambda^{p(x)}(\lambda_1 a_1+\mu_1 c_1), \end{aligned}$$

then

$$-\Delta_{p(x)}\phi_{k_1} \leq \lambda^{p(x)}(\lambda_1 a_1 + \mu_1 c_1), \quad \sigma \leq d(x) < 2\delta \tag{11}$$

Since $\phi_{k_1}(x), \phi_{k_2}(x)$ and f, h are monotone, when λ is large enough, we have

$$-\Delta_{p(x)}\phi_{k_1} \leq \lambda^{p(x)}(\lambda_1 a(x) f(\phi_{k_2}) + \mu_1 c(x) h(\phi_{k_1})), \ \sigma \leq d(x) < 2\delta$$

and

$$-\Delta_{p(x)}\phi_{k_1} = 0 \leq \lambda^{p(x)}(\lambda_1 a_1 + \mu_1 c_1) \leq \lambda^{p(x)}(\lambda_1 a(x) f(\phi_{k_2}) + \mu_1 c(x) h(\phi_{k_1})), \ 2\delta \leq d(x). \tag{12}$$

Combining (10), (12) and (13), we can deduce that

$$-\Delta_{p(x)}\phi_{k_1} \leq \lambda^{p(x)}(\lambda_1 a(x) f(\phi_{k_2}) + \mu_1 c(x) h(\phi_{k_1})), a.e. \text{ on } \Omega. \tag{13}$$

Similarly

$$-\Delta_{q(x)}\phi_{k_2} \leq \lambda^{q(x)}(\lambda_2 b(x) g(\phi_{k_1}) + \mu_2 d(x) \tau(\phi_{k_2})), a.e. \text{ on } \Omega \tag{14}$$

From (13) and (14), we can see that (ϕ_{k_1}, ϕ_{k_2}) is a subsolution of problem (1).
Step 2. We will construct a supersolution of problem (1), we consider

$$\begin{cases} -\Delta_{p(x)} z_{k_1} = \lambda^{p^+}(\lambda_1 a_2 + \mu_1 c_2)\mu \ \text{ in } \Omega, \\ -\Delta_{q(x)} z_{k_2} = \lambda^{q^+}(\lambda_1 b_2 + \mu_1 d_2) g(\beta(\lambda^{p^+}(\lambda_1 a_2 + \mu_1 c_2)\mu)) \ \text{ in } \Omega, \\ z_{k_1} = z_{k_2} = 0 \ \text{ on } \partial\Omega, \end{cases}$$

where

$$\beta = \beta(\lambda^{p^+}(\lambda_1 a_2 + \mu_1 c_2)\mu) = \max_{x \in \overline{\Omega}} z_{k_1}(x).$$

We shall prove that (z_{k_1}, z_{k_2}) is a supersolution of problem (1).
From Lemma 2, we have

$$\max_{x \in \overline{\Omega}} z_{k_1}(x) \leq C_2 \left[\lambda^{p^+}(\lambda_1 a_2 + \mu_1 c_2)\mu\right]^{\frac{1}{p^- - 1}}$$

and

$$\max_{x \in \overline{\Omega}} z_{k_2}(x) \leq C_2 \left[\lambda^{q^+}(\lambda_2 b_2 + \mu_2 d_2) g(\beta(\lambda^{p^+}(\lambda_1 a_2 + \mu_1 c_2)\mu))\right]^{\frac{1}{q^- - 1}}.$$

For $\psi \in W_0^{1,q(x)}(\Omega)$ with $\psi \geqslant 0$, it is easy to see that

$$\int_\Omega |\nabla z_{k_2}|^{q(x)-2} \nabla z_{k_2}.\nabla \psi dx = \int_\Omega \lambda^{q^+}(\lambda_2 b_2 + \mu_2 d_2) g(\beta(\lambda^{p^+}(\lambda_1 a_2 + \mu_1 c_2)\mu)) \psi dx \geqslant$$

$$\int_\Omega \lambda^{q^+}\lambda_2 b(x) g(z_{k_1}) \psi dx + \int_\Omega \lambda^{q^+}\mu_2 d(x) g(\beta(\lambda^{p^+}(\lambda_1 a_2 + \mu_1 c_2)\mu)) \psi dx.$$

By (H_4), for μ a large enough, using Lemma 2, we have

$$
\begin{aligned}
& g(\beta(\lambda^{p+}(\lambda_1 a_2 + \mu_1 c_2)\mu)) \\
& \geqslant \tau(C_2\,[\lambda^{q+}(\lambda_2 b_2 + \mu_2 d_2) g(\beta(\lambda^{p+}(\lambda_1 a_2 + \mu_1 c_2)\mu))]^{\frac{1}{q^- -1}}) \\
& \geqslant \tau(z_{k_2}).
\end{aligned} \tag{15}
$$

Hence

$$
\int_\Omega |\nabla z_{k_2}|^{q(x)-2} \nabla z_{k_2}.\nabla\psi dx \geqslant \int_\Omega \lambda^{q+}\lambda_2 b(x) g(z_{k_1})\psi dx + \int_\Omega \lambda^{q+}\mu_2 d(x)\tau(z_{k_2})\psi dx. \tag{16}
$$

Also, for $\varphi \in W^{1,p(x)}(\Omega)$ with $\varphi \geq 0$, it is easy to see that

$$
\int_\Omega |\nabla z_{k_1}|^{p(x)-2} \nabla z_{k_1}.\nabla\varphi dx = \int_\Omega \lambda^{p+}(\lambda_1 a_2 + \mu_1 c_2)\mu\varphi dx.
$$

By (H_3), (H_4) and Lemma 2, when μ is sufficiently large, we have

$$
\begin{aligned}
(\lambda_1 a_2 + \mu_1 c_2)\mu \;\; & \geqslant \;\; \frac{1}{\lambda^{p+}}\left[\frac{1}{C_2}\beta(\lambda^{p+}(\lambda_1 a_2 + \mu_1 c_2)\mu)\right]^{p^- -1} \\
& \geqslant \;\; \mu_1 h(\beta(\lambda^{p+}(\lambda_1 a_2 + \mu_1 c_2)\mu)) \\
& \quad\;\; + \lambda_1 f\left(C_2\left[\lambda^{q+}(\lambda_2 b_2 + \mu_2 d_2) g(\beta(\lambda^{p+}(\lambda_1 a_2 + \mu_1 c_2)\mu))\right]^{\frac{1}{q^- -1}}\right).
\end{aligned}
$$

Then

$$
\int_\Omega |\nabla z_{k_1}|^{p(x)-2} \nabla z_{k_1}.\nabla\varphi dx \geqslant \int_\Omega \lambda^{p+}\lambda_1 a(x) f(z_{k_2})\varphi dx + \int_\Omega \lambda^{p+}\mu_1 c(x) h(z_{k_1})\varphi dx. \tag{17}
$$

According to (16) and (17), we can conclude that (z_{k_1}, z_{k_2}) is a supersolution of problem (1). It only remains to prove that $\phi_{k_1} \leq z_{k_1}$ and $\phi_{k_2} \leq z_{k_2}$.

In the definition of $v_1(x)$, let

$$
\gamma = \frac{2}{\delta}\left(\max_{\overline{\Omega}} \phi_{k_1}(x) + \max_{\overline{\Omega}} |\nabla\phi_{k_1}|(x)\right).
$$

We claim that

$$
\phi_{k_1}(x) \leq v_1(x), \;\; \forall x \in \Omega. \tag{18}
$$

From the definition of v_1, it is easy to see that

$$
\phi_{k_1}(x) \leq 2\max_{\overline{\Omega}} \phi_{k_1}(x) \leq v_1(x), \text{ when } d(x) = \delta,
$$

$$
\phi_{k_1}(x) \leq 2\max_{\overline{\Omega}} \phi_{k_1}(x) \leq v_1(x), \text{ when } \; d(x) \geqslant \delta
$$

and

$$
\phi_{k_1}(x) \leq v_1(x) \quad \text{when } d(x) < \delta.
$$

Since $v_1 - \phi_{k_1} \in C^1(\overline{\partial\Omega_\delta})$, there exists a point $x_0 \in \overline{\partial\Omega_\delta}$, such that

$$v_1(x_0) - \phi_{k_1}(x_0) = \min_{x_0 \in \overline{\partial\Omega_\delta}} (v_1(x_0) - \phi_{k_1}(x_0)).$$

If $v_1(x_0) - \phi_{k_1}(x_0) < 0$, It is easy to see that $0 < d(x) < \delta$ and then

$$\nabla v_1(x_0) - \nabla\phi_{k_1}(x_0) = 0.$$

From the definition of v_1,we have

$$|\nabla v_1(x_0)| = \gamma = \frac{2}{\delta}\left(\max_{\overline{\Omega}} \phi_{k_1}(x_0) + \max_{\overline{\Omega}} \left|\nabla\phi_{k_1}\right|(x_0)\right) > \left|\nabla\phi_{k_1}\right|(x_0).$$

It is a contradiction to

$$\nabla v_1(x_0) - \nabla\phi_{k_1}(x_0) = 0.$$

Thus, (18) is valid.

Obviously, there exists a positive constants C_3, such that $\gamma \leq C_3\lambda$.

Since $d(x) \in C^2(\overline{\partial\Omega_{3\delta}})$, according to the proof of Lemma 2, there exists a positive constant C_4, such that

$$-\Delta_{p(x)} v_1(x) \leq C_* \gamma^{p(x)-1+\theta} \leq C_4 \lambda^{p(x)-1+\theta} \text{ a.e } \Omega, \text{ where } \theta \in (0,1).$$

Since $\eta \geqslant \lambda^{p+}$ is large enough, we have $-\Delta_{p(x)} v_1(x) \leq \eta$.

Under the comparaison principle, we have

$$v_1(x) \leq w(x), \text{ for all } x \in \Omega. \tag{19}$$

From (18) and (19), when $\eta \geqslant \lambda^{p+}$ and $\lambda \geqslant 1$ is sufficiently large, we have

$$\phi_{k_1}(x) \leq v_1(x) \leq w(x), \text{for all } x \in \Omega. \tag{20}$$

According to the comparaison principle, when μ is large enough, we have

$$v_1(x) \leq w(x) \leq z_{k_1}(x), \text{ for all } x \in \Omega.$$

Combining the definition of $v_1(x)$ and (20), it is easy to see that

$$\phi_{k_1}(x) \leq v_1(x) \leq w(x) \leq z_{k_1}(x), \text{ for all } x \in \Omega.$$

When $\mu \geqslant 1$ and λ is a large enough, from Lemma 2, we can note that $\beta(\lambda^{p+}(\lambda_1 a_2 + \mu_1 c_2)\mu$ is large enough, then

$$\lambda^{q+}(\lambda_2 b_2 + \mu_2 d_2) g(\beta(\lambda^{p+}(\lambda_1 a_2 + \mu_1 c_2)\mu))$$

is a large enough. Similarly, we have $\phi_{k_2}(x) \leq z_{k_2}(x)$. This completes the proof. □

Asymptotic Behavior of Solutions

Definition 2. *A measurable funtion $u : \Omega_T \to \mathbb{R}$ is an weak solution to hyperbolic systems involving of $(p(x), q(x))-$Laplacien (1) in Ω_T if $u(.,0) = u_0$ in Ω,*

$$u \in C(0,T; L^2(\Omega)) \cap L^p(0,T; H_0^1(\Omega)),$$

$$\frac{\partial u}{\partial t} \in L^2(\Omega_T), \nabla u \in \left(L^2(\Omega_T)\right)^N$$

and for all $\varphi \in C^1(\Omega_T)$ *and* $\psi \in C^1(\Omega_T)$, *we have*

$$\int_0^T \int_\Omega \frac{\partial u}{\partial t}\varphi dxdt + \int_0^T \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla \varphi dxdt + \int_0^T \int_\Omega (-\lambda^{p(x)}\mu_1 c(x)h(u))\varphi dxdt$$
$$= \int_0^T \int_\Omega \lambda^{p(x)}\lambda_1 a(x) f(v)\varphi dxdt \tag{21}$$

Lemma 3.

$$\int_0^T \int_\Omega \frac{\partial v}{\partial t}\psi dxdt + \int_0^T \int_\Omega |\nabla v|^{q(x)-2}\nabla v \nabla \psi dxdt + \int_0^T \int_\Omega (-\lambda^{q(x)}\lambda_2 b(x)g(u))\psi dxdt$$
$$= \int_0^T \int_\Omega \lambda^{q(x)}\mu_2 d(x)\sigma(v)\psi dxdt$$

Lemma 4. *Let* $\underline{u}, \overline{u}$ *be the solutions of* (1) *with* $\underline{u}\,(x,0) = \varphi_1, \overline{u}\,(x,0) = \varphi_2$*Than* $\underline{u}(x,t)$ *is nondercreasing in* t , $\overline{u}\,(x,t)$ *is nonincreasing and* $\overline{u} > \underline{u}$ *for all* $t \geq 0, x \in \Omega$

Theorem 2. *Let hypotheses* $(H_1), (H_2)$ *and* (H_3) *be satisfied. and let* $u\,(x,t)$ *the solution of a new class of hyperbolic systems* (1) *with* $\Psi \in S^*$ *than*

$$\lim_{t\to\infty} u\,(x,t) = \begin{cases} \underline{u}_s\,(x) & if\ \widehat{u}_s \leq \Psi \leq \underline{u}_s \\ \overline{u}_s\,(x) & if\, \overline{u}_s \leq \Psi \leq \widetilde{u}_s \end{cases}$$

.

Proof. The pair $(\underline{u}_s, \widehat{u}_s)$ and the pair $(\widetilde{u}_s, \overline{u}_s)$ are both sub-super solutions of (4), the maximale and minimale property of $\overline{u}_s$ and $\underline{u}_s$ in S^* ensures that:

$\underline{u}_s$ is the unique solution in $[\widehat{u}_s, \underline{u}_s]$ and $\overline{u}_s$ is the unique solution in $[\overline{u}_s, \widetilde{u}_s]$. □

4. Conclusions

Our result is an extension for our previous study in ([2,7,8]) which studied the stationary case, this idea is new for evolutionary case of this kind of problem, This paper deals with the existence of positively solution and its asymptotic behavior for parabolic system of $(p(x), q(x))$-Laplacian system of partial differential equations using a sub and super solution according to some given boundary conditions, which is familiar in physics, since it appears clearly natural in inflation cosmology and supersymmetric filed theories, quantum mechanics, and nuclear physics (see [3,14]). This sort of problem has many applications in several branches of physics such as nuclear physics, optics, and geophysics (see [7,15]). In future work, we will try to extend this study for the hyperbolic case of the presented problem, but by using the semigroup theory.

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References

1. O'Neill, R. Fractional integration in Orlicz spaces. *Trans. Am. Math. Soc.* **1965**, *115*, 300–328. [CrossRef]
2. Boulaaras, S.; Guefaifia, R.; Kabli, S. An asymptotic behavior of positive solutions for a new class of elliptic systems involving of $(p(x), q(x))$-Laplacian systems. *Boletín de la Sociedad Matemática Mexicana* **2019**, *25*, 145–162. [CrossRef]
3. Edmunds, D.E.; Lang, J.; Nekvinda, A. On $L^{p(x)}$ norms. *Proc. R. Soc. Lond. Ser. A* **1999**, *455*, 219–225. [CrossRef]
4. Edmunds, D.E.; Rakosnk, J. Sobolev embedding with variable Exponent. *Studia Math.* **2000**, *143*, 267–293. [CrossRef]
5. Hai, D.D.; Shivaji, R. An existence result on positive solutions for a class of p-Laplacian systems. *Nonlinear Anal.* **2004**, *56*, 1007–1010. [CrossRef]
6. Boulaaras, S; Guefaifia, R.; Bouali, T. Existence of positive solutions for a class of quasilinear singular elliptic systems involving Caffarelli-Kohn-Nirenberg exponent with sign-changing weight functions. *Indian J. Pure Appl. Math.* **2018**, *49*, 705–715. [CrossRef]
7. Boulaaras, S.; Guefaifia, R. Existence of positive weak solutions for a class of Kirrchoff elliptic systems with multiple parameters. *Math. Meth. Appl. Sci.* **2018**, *41*, 5203–5210. [CrossRef]
8. Guefaifia, R.; Boulaaras, S. Existence of positive solution for a class of $(p(x), q(x))$-Laplacian systems. *Rendiconti del Circolo Matematico di Palermo Series II* **2018**, *67*, 93–103.
9. Guefaifia, R.; Boulaaras, S. Existence of positive radial solutions for $(p(x), q(x))$-Laplacian systems. *Appl. Math. E-Notes* **2018**, *18*, 209–218
10. Ricceri, B. On an elliptic Kirchhoff -type problem depending on two parameters. *J. Glob. Optim.* **2010**, *46*, 543–549. [CrossRef]
11. Boulaaras, S.; Allahem, A. Existence of positive solutions of nonlocal $p(x)$-Kirchhoff evolutionary systems via Sub-Super Solutions Concept. *Symmetry* **2019**, *11*, 253. [CrossRef]
12. Zhang, Q.H. A Strong maximun principle for differential equations with nonstandard $p(x)$-growth conditions. *J. Math. Anal.* **2005**, *312*, 24–32. [CrossRef]
13. Fan, X.L. On subsupersolution method for $p(x)$-Laplacian equation. *J. Math. Anal. Appl.* **2007**, *330*, 665–682. [CrossRef]
14. Alves, C.O.; Correa, F.J.S.A. On existence of solutions for a class of problem involving a nonlinear operator. *Commun. Appl. Nonlinear Anal.* **2001**, *8*, 43–56.
15. Azouz, N.; Bensedik, A. Existence result for an elliptic equation of Kirchhoff type with changing sign data. *Funkcialaj Ekvacioj* **2012**, *55*, 55–66. [CrossRef]