symmetry

MDPI

Article

Approximation to Logarithmic-Cauchy Type Singular Integrals with Highly Oscillatory Kernels

SAIRA and **Shuhuang Xiang** *

School of Mathematics and Statistics, Central South University, Changsha 410083, China, sairahameed@csu.edu.cn
* Correspondence: xiangsh@csu.edu.cn; Tel.: +86-139-7314-3907

Received: 28 April 2019; Accepted: 22 May 2019; Published: 28 May 2019

Abstract: In this paper, a fast and accurate numerical Clenshaw-Curtis quadrature is proposed for the approximation of highly oscillatory integrals with Cauchy and logarithmic singularities, $⨍_{-1}^{1} \frac{f(x)\log(x-\alpha)e^{ikx}}{x-t}dx$, $t \notin (-1,1), \alpha \in [-1,1]$ for a smooth function $f(x)$. This method consists of evaluation of the modified moments by stable recurrence relation and Cauchy kernel is solved by steepest descent method that transforms the oscillatory integral into the sum of line integrals. Later theoretical analysis and high accuracy of the method is illustrated by some examples.

Keywords: Clenshaw-Curtis quadrature; steepest descent method; logarithmic singularities; Cauchy singularity; highly oscillatory integrals

1. Introduction

Boundary element method and finite element method are intensively eminent numerical approaches to evaluate partial differential equations (PDEs), which appear in variety of disciplines from engineering to astronomy and quantum mechanics [1–5]. Although these methods lead PDEs to Fredholm integral equations or Voltera integral equations, but these kind of integral equations posses integrals of oscillatory, Cauchy-singular, logarithmic singular, weak singular kernel functions. However, these classical methods are failed to approximate the integrals constitute kernel functions of highly oscillation and logarithmic singularity.

This paper aims at approximation of the integral

$$I^{\alpha}[f] = ⨍_{-1}^{1} \frac{f(x)\log(x-\alpha)e^{ikx}}{x-t}dx, \tag{1}$$

where $t \in (-1,1), k \gg 1, \alpha \in [-1,1]$, $f(x)$ is relatively smooth function. For integral (1) the developed strategy for logarithmic singularity $\log(x-\alpha)$ is valid for $\alpha \in [-1,1]$. In particular, the highly oscillatory integral, $\int_{-1}^{1} f(x)e^{ikx}dx$ has been computed by many methods such as asymptotic expansion, Filon method, Levin collocation method and numerical steepest descent method [6–10]. For instant, Dominguez et al. [11] for function $f(x)$ with integrable singularities have proposed an error bound, calculated in Sobolev spaces H^m, for composite Filon-Clenshaw-Curtis quadrature. Error bound depends on the derivative of $f(x)$ and length of the interval M, for some $C_1(f)$ defined as $E_N \leq C_1(f)(\frac{1+|\log(k)|}{k^{1+\beta}})^r(\log M)^{1+\beta-r}(\frac{1}{M})^{N+1-r}$ for $\beta \in (-1,0), r \in [0,1+\beta]$.

On the other hand, one methodology for numerical evaluation of integral $⨍_{-1}^{1} \frac{f(x)e^{ikx}}{x-t}dx$ is replacing $f(x)$ by different kind of polynomials [12,13]. Another technique is based on analytic continuation of the integral if the integrand $f(x)$ is analytic in the complex region [14]. As far as for $k = 0$ solution methods and properties of the solution for relative non-homogenous integrals have been discussed by using Brestain polynomials and Chebyshev polynoimals of all four kinds in [3,15].

For integral $\int_{-1}^{1} f(x)\log((x-\alpha)^2)e^{ikx}dx$ Clenshaw-Curtise rule is applied for numerical calculation. Wherein the convergence rate is independent of k but depends on the number of nodes of quadrature rule and function $f(x)$ [16]. Furthermore, Piessense and Branders [17] established the Clenshaw-Curtis quadrature rule, relies on the recurrence relation for $\int_{-1}^{1} f(x)e^{ikx}(x+1)^{\alpha}\log(x+1)dx$. They replaced the nonoscillatory and nonsingular part of the integrand by Chebyshev series. Chen [18] presented the numerical approximation of the integral $I[f] = \fint_{-1}^{1} \frac{f(x)e^{ikx}}{(x+1)^{\alpha}(x-1)^{\beta}\prod_{m=1}^{n}(x-\tau_m)^{\gamma_m}}dx$, with $\alpha, \beta < 1$, $a < \gamma_m < b$ and $\gamma_m \leq 1$. For analytic function $f(x)$ the integral was rewritten in the form of sum of line integrals, wherein the integrands do not oscillate and decay exponentially. Moreover, Fang [19] established the Clenshaw-Curtis quadrature for $\fint_{-1}^{1} \frac{(x+1)^{\alpha}(x-1)^{\beta}f(x)\log(x+1)e^{ikx}}{x-t}dx$ for general function $f(x)$ where steepest descent method is illustrated for analytic function $f(x)$. Recently, John [20] introduced the algorithm for integral approximation of Cauchy-singular, logarithmic-singular, Hadamard type and nearly singular integrals having integrable endpoints singularities i.e., $(1-x)^{\alpha}(1+x)^{\beta}$, $(\alpha, \beta > -1)$. Composed Gauss-Jacobi quadrature consists of approximating the function $f(x)$ by Jacobi polynomials $\{P_n^{\alpha,\beta}\}_{n=0}^{N-1}$ of degree $N-1$.

However, all these proposed method are inadequate to apply directly on integral (1) in the presence of oscillation and other singularities. This work presents Clenshaw-Curtis quadrature to get recurrence relation to compute the modified moments, that takes just $O(N\log N)$ operations. The initial Cauchy singular values for recurrence relation are obtained by the steepest descent method, as it prominently renowned to evaluate highly oscillatory integrals when the integrands are analytic in sufficiently large region.

The rest of the paper is organized as follows. Section 2 delineates the quadrature algorithm for integral (1). Numerical calculation of the modified moments with recurrence relation by using some Chebyshev properties is defined. Also steepest descent method is established for Cauchy singularity where later the obtained line integrals are further approximated by generalized Gauss quadrature. Section 3 alludes some error bounds derived in terms of Clenshaw-Curtis points and the rate of oscillation k. In Section 4, numerical examples are provided to demonstrate the efficiency and accuracy of the presented method.

2. Numerical Methods

In the computation of integral $I^{\alpha}[f]$, the Clenshaw-Curtis quadrature approach is extensively adopted. The scheme is postulated on interpolating the function $f(x)$ at Clenshaw-Curtis points set $X_{N+1} = \{x_j = cos\frac{j\pi}{N}\}_{j=0}^{N}$. Writing the interpolation polynomial as basis of Chebyshev series

$$f(x) \approx P_N(x) = \sum_{n=0}^{N}{}'' a_n T_n(x), \tag{2}$$

where $T_n(x)$ is the Chebyshev polynomial of first kind of degree N and double prime denotes a sum whose first and last terms are halved, the coefficients

$$a_n = \frac{2}{N}\sum_{j=0}^{N}{}'' f(x_j)T_n(x_j) \tag{3}$$

can be computed efficiently by FFT in $O(N \log N)$ operations [8,9]. This paper appertains to Clenshaw-Curtis quadrature, which depends on Hermite interpolating polynomial that allow us to get higher order accuracy

$$\tilde{P}(x_j) = f(x_j), j = 0, \cdots N; \qquad \tilde{P}(t) = f(t). \tag{4}$$

For any fixed t, we can elect felicitous N such that $t \notin \{x_j\}_{j=0}^{N}$ and rewrite Hermite interpolating polynomial of degree $N+1$ in terms of Chebyshev series

$$\tilde{P}_{N+1}(x) = \sum_{n=0}^{N+1} c_n T_n(x) \tag{5}$$

c_n can be calculated in $O(N)$ operations once if a_n are known [13,21]. Finally Clenshaw-Curtis quadrature for integral $I^{\alpha}[f]$ is defined as

$$\begin{aligned} I_{N+1}^{\alpha}[f] &= \sum_{n=0}^{N+1} c_n \fint_{-1}^{1} \frac{T_n(x)\log(x-\alpha)e^{ikx}}{x-t} dx \\ &= \sum_{n=0}^{N+1} c_n D_n^{\alpha}(k,t) \end{aligned} \tag{6}$$

where

$$D_n^{\alpha}(k,t) = \fint_{-1}^{1} \frac{T_n(x)\log(x-\alpha)e^{ikx}}{x-t} dx \tag{7}$$

more specifically $D_n^{\alpha}(k,t)$ are called the modified moments. Efficiency of the Clenshaw-Curtis quadrature depends on the fast computation of the moments. In ensuing sub-section, we deduce the recurrence relation for $D_n^{\alpha}(k,t)$.

Computation of the $D_n^{\alpha}(k,t)$ Moments

A reputed property of Chebyshev polynomial [22]

$$T_n(x) = \frac{1}{2}(U_n(x) - U_{n-2}(x)), \tag{8}$$

leads the modified moments $D_n^{\alpha}(k,t) = \fint_{-1}^{1} \frac{T_n(x)\log(x-\alpha)e^{ikx}}{x-t} dx$ to

$$\begin{aligned} \fint_{-1}^{1} \frac{T_n(x)\log(x-\alpha)e^{ikx}}{x-t} dx &= \frac{1}{2}\left(\fint_{-1}^{1} \frac{U_n(x)\log(x-\alpha)e^{ikx}}{x-t} dx - \fint_{-1}^{1} \frac{U_{n-2}(x)\log(x-\alpha)e^{ikx}}{x-t} dx \right) \\ D_n^{\alpha}(k,t) &= \frac{1}{2}(Q_n^{\alpha}(k,t) - Q_{n-2}^{\alpha}(k,t)). \end{aligned} \tag{9}$$

Forthcoming theorem defines the procedure to calculate the moments $Q_n^{\alpha}(k,t) = \fint_{-1}^{1} \frac{U_n(x)\log(x-\alpha)e^{ikx}}{x-t} dx$.

Proposition 1. *The sequence $Q_n^{\alpha}(k,t) = \fint_{-1}^{1} \frac{U_n(x)\log(x-\alpha)e^{ikx}}{x-t} dx$ satisfies the recurrence relation*

$$\begin{aligned} Q_{n+1}^{\alpha}(k,t) &= 2Q_n^{\alpha}(k) + 2tQ_n^{\alpha}(k,t) - Q_{n-1}^{\alpha}(k,t), \qquad n \geq 1 \\ Q_1^{\alpha}(k,t) &= 2Q_0^{\alpha}(k) + 2tQ_0^{\alpha}(k,t). \end{aligned} \tag{10}$$

where

$$Q_n^\alpha(k) = \int_{-1}^{1} U_n(x)\log(x-\alpha)e^{ikx}dx, Q_0^\alpha(k) = \int_{-1}^{1}\log(x-\alpha)e^{ikx}dx. \tag{11}$$

Proof. Using Chebyshev recurrence relation

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x),$$

$$\begin{aligned}
Q_{n+1}^\alpha(k,t) &= ⨍_{-1}^{1} \frac{2(x-t+t)U_n(x)\log(x-\alpha)e^{ikx}}{x-t}dx - ⨍_{-1}^{1}\frac{U_{n-1}(x)\log(x-\alpha)e^{ikx}}{x-t}dx,\\
&= ⨍_{-1}^{1} 2U_n(x)\log(x-\alpha)e^{ikx}dx + ⨍_{-1}^{1}\frac{2tU_n(x)\log(x-\alpha)e^{ikx}}{x-t}dx\\
&\quad - ⨍_{-1}^{1}\frac{U_{n-1}(x)\log(x-\alpha)e^{ikx}}{x-t}dx,\\
Q_{n+1}^\alpha(k,t) &= 2Q_n^\alpha(k) + 2tQ_n^\alpha(k,t) - Q_{n-1}^\alpha(k,t).
\end{aligned}$$

□

The proof completes with the initial values $U_0(x) = 1, U_1(x) = 2x$. The starting values $Q_0^\alpha(k,t)$ and $Q_0^\alpha(k)$ of recurrence relation can be calculated by steepest descent method.

Proposition 2. *Suppose that $f(x)$ is an analytic function in the half-strip of the complex plan, $a \le \Re(x) \le b$ and $\Im(x) \ge 0$, and satisfies the condition for constant M and $0 \le k_0 < k$*

$$\int_{-1}^{1} |f(x+iR)|dx \le Me^{k_0 R},$$

then the integral (1) for $\alpha \in [-1,1]$ can be transformed into

$$\begin{aligned}
I^{\pm 1}[f] &= M_1^{\pm 1} + M_2^{\pm 1} + i\pi e^{ikt} f(t)\log(t-(\pm 1)),\\
I^{\alpha}[f] &= N_1 + N_2 + i\pi e^{ikt} f(t)\log(t-\alpha),
\end{aligned} \tag{12}$$

where

$$\begin{aligned}
N_1 &= \frac{i}{k}e^{-ik}\int_{kr}^{kR}\frac{f(-1+\frac{i}{k}x)\log(-1+\frac{i}{k}x-\alpha)e^{-x}}{-1+\frac{i}{k}x-t}dx,\\
N_2 &= -\frac{i}{k}e^{ik}\int_{kr}^{kR}\frac{f(1+\frac{i}{k}x)\log(1+\frac{i}{k}x-\alpha)e^{-x}}{1+\frac{i}{k}x-t}dx,\\
M_1^{\pm 1} &= \pm\frac{i}{k}e^{\mp ik}\int_{kr}^{kR}\frac{f(\mp 1+\frac{i}{k}x)\log(\mp 2+\frac{i}{k}x)e^{-x}}{\mp 1+\frac{i}{k}x-t}dx,\\
M_2^{\pm 1} &= \mp\frac{i}{k}e^{\pm ik}\int_{kr}^{kR}\frac{f(\pm 1+\frac{i}{k}x)\log(\frac{i}{k}x)e^{-x}}{\pm 1+\frac{i}{k}x-t}dx.
\end{aligned} \tag{13}$$

Proof. Following proof asserts the results for case $\alpha = 1$, and for $\alpha \in [-1, 1)$ the same technique can be used as well. Since the integrand $\frac{f(x)\log(x-1)e^{ikx}}{x-t}$ is analytic in the half strip of the complex plane, by Cauchy's Theorem, we have

$$\int_{\Gamma_1+\Gamma_2+\Gamma_3+\Gamma_4-\Gamma_5-\Gamma_6-\Gamma_7} \frac{f(x)\log(x-1)e^{ikx}}{x-t}dx = 0, \tag{14}$$

with all the contours taken in clockwise direction (Figure 1).

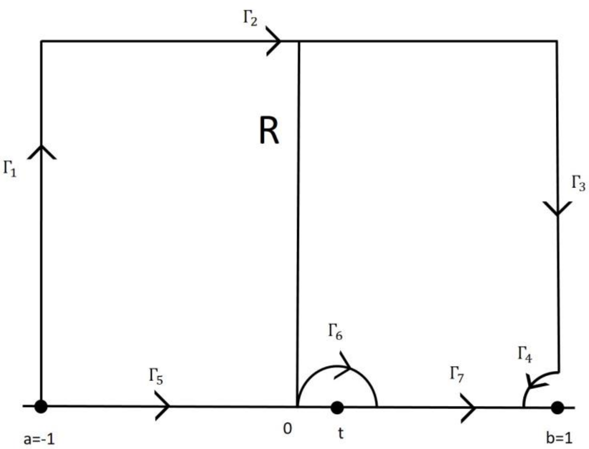

Figure 1. Illustration of integration path of $I^{+1}[f]$.

Setting $\hat{I}_i = \int_{\Gamma_i} \frac{f(x)\log(x-1)e^{ikx}}{x-t}dx, i = 1, 2, \cdots 7$, we obtain that

$$\hat{I}_1 + \hat{I}_2 + \hat{I}_3 + \hat{I}_4 = \hat{I}_5 + \hat{I}_6 + \hat{I}_7. \tag{15}$$

$$\begin{aligned}\hat{I}_1 &= \int_r^R \frac{f(-1+ip)\log(-1+ip-1)e^{ik(-1+ip)}}{-1+ip-t}idp \\ &= \frac{i}{k}e^{-ik}\int_{kr}^{kR} \frac{f(-1+\frac{i}{k}x)\log(-2+\frac{i}{k}x)e^{-x}}{-1+\frac{i}{k}x-t}dx.\end{aligned}$$

Similarly for $\hat{I}_3$, we get

$$\begin{aligned}\hat{I}_3 &= -\int_r^R \frac{f(1+ip)\log(1+ip-1)e^{ik(1+ip)}}{1+ip-t}idp \\ &= -\frac{i}{k}e^{ik}\int_{kr}^{kR} \frac{f(1+\frac{i}{k}x)\log(\frac{i}{k}x)e^{-x}}{1+\frac{i}{k}x-t}dx.\end{aligned}$$

From the statement of the theorem, $\int_{-1}^{1} |f(x+iR)| \le Me^{w_0 R}$,

$$\begin{aligned}\hat{I}_2 &= \int_{-1}^{1} \frac{f(x+iR)\log(x+iR-1)e^{ik(x+iR)}}{x+iR-t}dx \\ &= \frac{1}{R}\int_{-1}^{1} f(x+iR)\log(x+iR-1)e^{ik(x+iR)}dx \\ &\to 0 \qquad asR \to \infty.\end{aligned}$$

Let $x-1=re^{i\theta}$, then

$$\begin{aligned}\hat{I}_4 &= \int_0^{\frac{\pi}{2}} \frac{f(re^{i\theta}+1)\log(re^{i\theta})e^{ik(re^{i\theta}+1)}}{1+re^{i\theta}-t} ire^{i\theta}d\theta \\ &= ir\int_0^{\frac{\pi}{2}} \frac{f(re^{i\theta}+1)\log(re^{i\theta})e^{ik(re^{i\theta}+1)}}{1+re^{i\theta}-t} e^{i\theta}d\theta \\ &\to 0 \qquad as r \to 0.\end{aligned}$$

In addition

$$\begin{aligned}\hat{I}_6 &= \int_0^{\pi} \frac{f(re^{i\theta}+t)\log(t+re^{i\theta}-1)e^{ik(re^{i\theta}+t)}}{re^{i\theta}} ire^{i\theta}d\theta \\ r &\to 0 \\ &= i\pi e^{ikt} f(t)\log(t-1).\end{aligned}$$

Thus, we complete the proof with

$$\begin{aligned}I^{+1}[f] &= \lim_{r\to 0, R\to\infty} (\hat{I}_1+\hat{I}_2+\hat{I}_3+\hat{I}_4-\hat{I}_6) \\ &= M_1^{+1}+M_2^{+1}+i\pi e^{ikt}f(t)\log(t-1).\end{aligned} \tag{16}$$

□

From Proposition 2.2 numerical scheme for the line integrals $M_1^{\pm1}, M_2^{\pm1}$ can be evaluated by generalized Gauss-Laguerre quadrature rule, using command lagpts in Chebfun [23]. Let $\{x_j^{(\beta)}, w_j^{(\beta)}\}_{j=1}^N$ be the nodes and weights of the weight function $x^\beta e^{-x}$ and let $\{x_j^{(\beta,l)}, w_j^{(\beta,l)}\}_{j=1}^N$ be the nodes and weights of the weight function $x^\beta(x-1-\ln(x))e^{-x}$. The line integrals $M_1^{\pm1}$ and $M_2^{\pm1}$ can be approximated by

$$\begin{aligned}M_1^{\pm1} &\approx R_{\{1,N\}}^{\pm1} = \pm\frac{i}{k}e^{\mp ik}\sum_{j=1}^N \frac{w_j^{(\beta)} f(\mp1+\frac{i}{k}x_j^{(\beta)})\log(\mp2+\frac{i}{k}x_j^{(\beta)})}{\mp1+\frac{i}{k}x_j^{(\beta)}-t}dx, \\ M_2^{\pm1} &\approx R_{\{2,N\}}^{\pm1} = \mp\frac{i}{k}e^{ik}\Bigg[\log(\frac{i}{k})-1)\sum_{j=1}^N w_j^{(\beta)}\frac{f(\pm1+\frac{i}{k}x_j^{(\beta)})}{\pm1+\frac{i}{k}x_j^{\beta}-t}dx \\ &+\sum_{j=1}^N w_j^{(\beta+1)}\frac{f(\pm1+\frac{i}{k}x_j^{(\beta+1)})}{\pm1+\frac{i}{k}x_j^{(\beta+1)}-t}dx - \sum_{j=1}^N w_j^{(\beta,l)}\frac{f(\pm1+\frac{i}{k}x_j^{(\beta,l)})}{\pm1+\frac{i}{k}x_j^{(\beta,l)}-t}dx\Bigg].\end{aligned} \tag{17}$$

For simplicity

$$I^{\pm1}[f] = R_{\{1,N\}}^{\pm1} + R_{\{2,N\}}^{\pm1} + i\pi f(t)\log(t-(\pm1)). \tag{18}$$

By the same argument N_1 and N_2 can also be approximated with generalized Gauss-Laguerre quadrature rule. Aforementioned theorem enlightens the another interesting fact that $I^\alpha[f]$ can also be computed by it if $f(x)$ is an analytic function.

Computation of the moments $Q_n^\alpha(k)$ is derived as, by using Chebyshev property (8)

$$\begin{aligned}\frac{1}{2}(Q_n^\alpha(k) - Q_{n-2}^\alpha(k)) &= D_n^\alpha(k) \\ &= \int_{-1}^{1}(T_n(x) - T_n(\alpha))\log(x-\alpha)e^{ikx}dx + T_n(\alpha)\int_{-1}^{1}\log(x-\alpha)e^{ikx}dx.\end{aligned} \tag{19}$$

For $\alpha \neq \pm 1$, integrating by parts, we derive

$$\int_{-1}^{1}(T_n(x) - T_n(\alpha))\log(x-\alpha)e^{ikx}dx = \frac{1}{ik}\Bigg[(T_n(x) - T_n(\alpha))\log(x-\alpha)e^{ikx}|_{-1}^{1}$$
$$-\int_{-1}^{1} T_n'(x)\log(x-\alpha)e^{ikx}dx - ⨍_{-1}^{1}\frac{(T_n(x)-T_n(\alpha))}{(x-\alpha)}e^{ikx}dx\Bigg]$$

$$\begin{aligned}&= \frac{1}{ik}\Bigg[(1 - T_n(\alpha))\log(1-\alpha)e^{ikx} + ((-1)^{n+1} + T_n(\alpha))\log(-1-\alpha)e^{-ik} \\ &- n\int_{-1}^{1}U_{n-1}(x)\log(x-\alpha)e^{ikx}dx - 2\int_{-1}^{1}U_{n-1}(x)e^{ikx}dx - 2\sum_{j=0}^{n-2}T_{n-1-j}(\alpha)\int_{-1}^{1}U_j(x)e^{ikx}dx\Bigg].\end{aligned} \tag{20}$$

We deduce the following recurrence relation by inserting (20) in (19)

$$Q_n^\alpha(k) - \frac{2n}{ik}Q_{n-1}^\alpha(k) + Q_{n-2}^\alpha(k) = \delta_n^\alpha(k) \tag{21}$$

where

$$\begin{aligned}\delta_n^\alpha(k) = &\frac{2}{ik}\Big[(1-T_n(\alpha))\log(1-\alpha)e^{ikx} + ((-1)^{n+1} + T_n(\alpha))\log(-1-\alpha)e^{-ik}\Big] \\ &- \frac{2}{ik}\left[2\sum_{j=0}^{n-2}T_{n-1-j}(\alpha)B_j(k) + B_{n-1}(k)\right] + 2T_n(\alpha)Q_0^\alpha(k),\end{aligned} \tag{22}$$

and

$$B_j(k) = \int_{-1}^{1}U_j(x)e^{ikx}dx, \qquad j = 0,\cdots,n-1. \tag{23}$$

It is worth to mention that $(B_j(k))_{j=0}^N$ can be computed in $O(N)$ operations [12]. For $\alpha = \pm 1$ we obtain the $\delta_n^{\pm 1}(k)$ as

$$\delta_n^{\pm 1}(k) = \begin{cases} 2\log(\mp 2)e^{\mp ik} & n = odd \\ 0 & n = even. \end{cases} \tag{24}$$

Unfortunately, practical experiments demonstrate that the recurrence relation for $Q_n^\alpha(k)$ is numerically unstable in the forward direction for $n > k$, in this sense so-called Oliver's algorithm is stable and used to rewrite the recurrence relation in the tridiagonal form [24].

3. Error Analysis

Lemma 1. *([9,13,14]) Suppose $f \in C^{m+1}[-1,1]$, for a non-negative integer m with $f(t) = 0$, then*

$$\left| \left(\frac{f(x)}{x-t} \right)^{(m)} \right| \leq \frac{2^{m+1}-1}{m+1} \| f^{(m+1)} \|_\infty. \tag{25}$$

Lemma 2. *([9,14]) Let $f(x)$ be a Lipschitz continous function on [−1,1] and let $P_N[f]$ be the interpolation polynomial of $f(t)$ at $N+1$ Clenshaw-Curtis points. Then it follows that*

$$\lim_{N \to +\infty} \| f - P_N[f] \|_\infty = 0. \tag{26}$$

In particular, if $f(x)$ is analytic with $|f(t)| \leq M$ in an Bernstein ellipse ε_ρ with foci ± 1 and major and minor semiaxis lengths summing to $\rho > 1$, then

$$\| f - P_N[f] \|_\infty \leq \frac{4M}{\rho^N(\rho-1)}. \tag{27}$$

if $f(x)$ has an absolutely continuous $(\kappa_0 - 1)$st derivative and $f^{(\kappa_0)}$ of bounded variation V_{κ_0} on [-1,1] for some $\kappa_0 \geq 1$, then for $N \geq \kappa_0 + 1$

$$\| f - P_N[f] \|_\infty \leq \frac{4V_{\kappa_0}}{\kappa_0 \pi N(N-1)\cdots(N-\kappa_0+1)}. \tag{28}$$

Lemma 3. *(van der Corput Lemma [25]) Suppose that $f \in C^1[0,b]$, then for each $\beta > -1$, it follows*

$$\left| \int_0^b x^\beta e^{ikx} dx \right| \leq W_1(k) \left(|f(b)| + \int_0^b |f'(x)| dx \right), \left| \int_0^b x^\beta \log(x) e^{ikx} dx \right| \leq W_2(k) \left(|f(b)| + \int_0^b |f'(x)| dx \right), \tag{29}$$

where

$$W_1(k) = \begin{cases} O\left(|k|^{-1-\beta}\right), & -1 < \beta \leq 0 \\ O(|k|^{-1}), & \beta > 0 \end{cases}, W_2(k) = \begin{cases} O\left(|k|^{-1-\beta}(1 + |\log(k)|)\right), & -1 < \beta \leq 0 \\ O(|k|^{-1}), & \beta > 0 \end{cases}.$$

Moreover, for some special cases we have

Lemma 4. *Suppose that $f \in C^1[0,1]$, then it follows for all k that*

$$\int_0^1 x(1-x)\log(x) e^{ikx} dx = O(|k|^{-2}(1 + \log|k|)), \int_0^1 x(1-x)\log(x-1) e^{ikx} dx = O(|k|^{-2}(1 + \log|k|)). \tag{30}$$

Proof. For simplicity, here we prove the first identity in (3.29). Similar proof can be directly applied to the second identity in (3.29).

Since

$$\int_0^1 x(1-x)\log(x) e^{ikx} dx = \frac{1}{ik} \int_0^1 x(1-x)\log(x) de^{ikx} = -\frac{1}{ik} \int_0^1 e^{ikx} [(1-x)\log(x) - x\log(x) + (1-x)] dx,$$

it leads to the desired result by Lemma 3.3. □

Suppose that $t \notin X_{N+1}$, $f \in C^2[-1,1]$ and define

$$\phi(x) = \begin{cases} \frac{f(x)-f(t)}{(x-t)}, & x \neq t \\ f'(t), & x = t. \end{cases}$$

From Lemma 2.1, we see that $\phi \in C^1[-1,1]$ and $\|\phi'\|_\infty \leq \frac{3}{2}\|f''\|_\infty$, in addition, $g(x) = \frac{\tilde{P}_{N+1}(x)-f(t)}{x-t}$ is a polynomial of degree at most N with $g(x_j) = \phi(x_j)$ for $j = 0,1,\ldots,N$ [21]. Then the error on the Clenshaw-Curtis quadrature (6) can be estimated by

$$\begin{aligned} \left|E_{N+1}\right| = \left|I^\alpha[f] - I^\alpha_{N+1}[f]\right| &= \left|\int_{-1}^1 (\phi(x) - g(x))\log(x-\alpha)e^{ikx}dx\right| \\ &\leq \|\phi(x) - g(x)\|_\infty \left|\int_{-1}^1 \log(x-\alpha)dx\right| \\ &= O(\|\phi(x) - g(x)\|_\infty). \end{aligned}$$

Corollary 1. *Suppose that $t \notin X_{N+1}$ and f'' is bounded on $[-1,1]$, then the Clenshaw-Curtis quadrature (6) is convergent*

$$\lim_{N\to+\infty} |E_{N+1}| = \lim_{N\to+\infty} |I^\alpha[f] - I^\alpha_{N+1}[f]| = 0. \tag{31}$$

In particular, if $f(x)$ is analytic and $|f'(x)| \leq M$ in a Bernstein ellipse ε_ρ, $\rho > 1$, then the error term satisfies

$$E_{N+1} = O\left(\frac{1}{\rho^N}\right). \tag{32}$$

If $f(x)$ has an absolutely continuous $(\kappa_0 - 1)$st derivative and $f^{(\kappa_0)}$ of bounded variation V_{κ_0} on [−1,1] for some $\kappa_0 \geq 1$, then for $N \geq \kappa_0 + 1$ $(\kappa_0 \geq 2)$

$$E_{N+1} = O\left(\frac{1}{N^{-\kappa_0+1}}\right). \tag{33}$$

Theorem 1. *The error bound for $I^\alpha_{N+1}[f]$ for integral $I^\alpha[f]$ can be estimated as*

$$E_{N+1} = \begin{cases} O\left(k^{-1}(1+|\log(k)|)\rho^{-N}\right), & f(x) \text{ analytic in the Bernstein ellipse } \varepsilon_\rho \\ O\left(k^{-1}(1+|\log(k)|)N^{-\kappa_0+2}\right), & f^{(\kappa_0+1)} \text{ of bounded variation} \end{cases}. \tag{34}$$

In addition, for $\alpha = \pm 1$, it follows

$$E_{N+1} = O\left(k^{-2}(1+|\log(k)|)\right) f \in C^2[-1,1]. \tag{35}$$

Proof. Since

$$\begin{aligned} E_{N+1} = \int_{-1}^1 (\phi(x) - P_N(x))\log(x-\alpha)e^{ikx}dx &= \int_{-1}^\alpha (\phi(x) - P_N(x))\log(x-\alpha)e^{ikx}dx \\ &+ \int_\alpha^1 (\phi(x) - P_N(x))\log(x-\alpha)e^{ikx}dx, \end{aligned}$$

by Lemma 3.3, it implies

$$E_{N+1} = O\left(k^{-1}(1+|\log(k)|)(\|\phi - P_N\|_\infty + \|\phi' - P'_N\|_\infty)\right),$$

which yields (3.33) together with the estimate on $\|\phi' - P'_N\|_\infty$ in [14].

The identity (3.34) follows from Lemma 3.4 due to that $\|\phi(x) - P_N(x)\| = (1+x)(1-x)h(x)$ for some $h \in C^1[-1,1]$. □

Remark 1. *From the convergence rates Corollary 3.1 and Theorem 3.1, compared with that in [19], the new scheme is of much fast convergence rate. It is also illustrated by the numerical results (see Section 4).*

4. Numerical Results

In this section, we will present several examples to illustrate the efficiency and accuracy of the proposed method. The exact values of an integral (36) are computed through Mathematica 11. Unless otherwise specifically stated, all the tested numerical examples are executed by using Matlab R2016a on a 4 GHz personal laptop with 8 GB of RAM.

Example 1. *Let us consider the integral*

$$I[f] = \fint_{-1}^{1} \frac{\sin(x)\log(x-\alpha)e^{ikx}}{x-t}dx, \tag{36}$$

for $\alpha = -1$, $t = 0.3$, *Table 1 shows the results for relative error compared with results of integral* (30) *[19] in Table 2.*

Table 1. The relative error of Clenshaw-Curtis quadrature rule for integral (36).

k	N = 4	N = 7	N = 11	N = 16
20	5.642×10^{-6}	1.432×10^{-8}	1.299×10^{-13}	1.795×10^{-14}
100	1.819×10^{-7}	8.954×10^{-10}	4.693×10^{-15}	1.051×10^{-15}
500	1.223×10^{-8}	5.462×10^{-11}	5.586×10^{-15}	5.276×10^{-15}
10,000	4.469×10^{-11}	1.054×10^{-13}	1.114×10^{-13}	1.115×10^{-13}

Table 2. The relative error of Clenshaw-Curtis quadrature rule for integral (30) [19].

k	N = 4	N = 7	N = 11	N = 16
20	3.710×10^{-3}	2.126×10^{-8}	2.846×10^{-13}	1.781×10^{-14}
100	3.016×10^{-3}	2.221×10^{-8}	1.473×10^{-13}	8.427×10^{-16}
500	2.924×10^{-3}	2.094×10^{-8}	1.408×10^{-13}	5.351×10^{-15}
10,000	3.047×10^{-3}	2.181×10^{-8}	1.836×10^{-13}	1.115×10^{-13}

Example 2. *Let integral*

$$\fint_{-1}^{1} \frac{e^{x}\log(x-\alpha)e^{ikx}}{x-t}dx \tag{37}$$

Tables 3–5 represent results for relative error computed by Clenshaw-Curtis quadrature. As exact value we just have used that returned by the rule when a huge number of points is used.

Table 3. The relative error of Clenshaw-Curtis quadrature rule for integral (37) for $\alpha = -1$, $f(x) = e^{x}$, $t = 0.5$.

k	Exact Value	N = 4	N = 8	N = 10	N = 20
20	1.360346130460585 − 1.837213701909973i	3.505×10^{-6}	1.356×10^{-10}	1.744×10^{-13}	1.744×10^{-13}
100	0.528568077016834 + 2.007019282199925i	3.418×10^{-7}	8.530×10^{-12}	1.983×10^{-14}	0.00
500	2.032501926854849 + 0.510184343854610i	1.619×10^{-8}	3.974×10^{-13}	7.640×10^{-16}	5.297×10^{-17}
10,000	2.074653919328735 + 0.324969073545833i	6.131×10^{-11}	1.418×10^{-15}	2.114×10^{-16}	4.237×10^{-16}

Table 4. The relative error of Clenshaw-Curtis quadrature rule for integral (37) for $\alpha = 1$, $f(x) = e^x$, $t = 0.5$.

k	Exact Value	N = 4	N = 8	N = 10	N = 20
20	11.034821521905808 + 12.628898944623328i	1.144×10^{-6}	2.364×10^{-11}	1.232×10^{-13}	2.118×10^{-16}
100	−16.418938229588949 + 1.005287081095468i	3.020×10^{-8}	1.064×10^{-12}	3.356×10^{-15}	2.620×10^{-16}
500	−7.387722497395380 + 14.855177327546180i	3.485×10^{-9}	8.256×10^{-14}	2.394×10^{-16}	1.197×10^{-16}
10,000	−6.063084167285699 +15.515830521473685i	1.132×10^{-11}	2.871×10^{-16}	1.192×10^{-16}	1.192×10^{-16}

Table 5. The relative error of Clenshaw-Curtis quadrature rule for integral (37) for $\alpha = 0$, $f(x) = e^x$, $t = 0.5$.

k	Exact Value	N = 4	N = 8	N = 10	N = 20
20	−1.928049736402945 + 2.990487262985703i	3.066×10^{-6}	8.567×10^{-11}	2.780×10^{-13}	1.248×10^{-16}
100	−0.934970743093483 − 3.460743549362822i	1.163×10^{-7}	2.942×10^{-12}	6.977×10^{-15}	2.770×10^{-16}
500	−3.485804022702049 − 0.864498281620865i	4.687×10^{-9}	1.177×10^{-13}	2.764×10^{-16}	9.274×10^{-17}
10,000	−3.547102638652960 − 0.555272021948841i	1.174×10^{-11}	2.473×10^{-16}	1.274×10^{-16}	3.092×10^{-17}

Example 3. *Let the integral be*

$$\fint_{-1}^{1} \frac{\cos(x)\log(x-\alpha)e^{ikx}}{x-t}dx \tag{38}$$

Tables 6–8 represent results for relative error computed by Clenshaw-Curtis quadrature. As exact value is calculated by using the rule for large number of points.

Table 6. The relative error of Clenshaw-Curtis quadrature rule for integral (38) for $\alpha = -1$, $f(x) = \cos(x)$, $t = 0.8$.

k	Exact Value	N = 4	N = 8	N = 10	N = 20
20	0.498125821203593 − 1.281802863555419i	6.031×10^{-6}	2.150×10^{-10}	3.026×10^{-13}	9.449×10^{-16}
100	1.264215353181015 − 0.141780191524840i	5.295×10^{-7}	1.348×10^{-11}	3.197×10^{-14}	3.728×10^{-16}
500	1.090289998226562 − 0.675652244977728i	2.584×10^{-8}	6.432×10^{-13}	1.596×10^{-15}	1.935×10^{-16}
10,000	−1.283945795748914 + 0.084367340279936i	9.738×10^{-11}	2.749×10^{-15}	3.649×10^{-16}	1.186×10^{-16}

Table 7. The relative error of Clenshaw-Curtis quadrature rule for integral (38) for $\alpha = 1$, $f(x) = \cos(x)$, $t = 0.8$.

k	Exact Value	N = 4	N = 8	N = 10	N = 20
20	5.342145332192533 + 5.729353825896764i	2.057×10^{-6}	4.505×10^{-11}	2.427×10^{-13}	9.620×10^{-16}
100	−2.621138174403697 + 7.318981197518284i	4.131×10^{-8}	1.918×10^{-12}	6.183×10^{-15}	3.657×10^{-16}
500	0.622301278817091 + 7.666316541113909i	6.007×10^{-9}	1.558×10^{-13}	4.418×10^{-16}	1.291×10^{-16}
10,000	3.064017684660896 − 7.095233976390074i	1.886×10^{-11}	4.632×10^{-16}	1.284×10^{-16}	1.284×10^{-16}

Table 8. The relative error of Clenshaw-Curtis quadrature rule for integral (38) for $\alpha = 0$, $f(x) = \cos(x)$, $t = 0.8$.

k	Exact Value	N = 4	N = 8	N = 10	N = 20
20	−0.112551138814753 + 0.430514461423602i	2.401×10^{-5}	6.824×10^{-10}	2.176×10^{-12}	3.111×10^{-15}
100	−0.477210698149339 + 0.058677959322354i	8.618×10^{-7}	2.188×10^{-11}	5.266×10^{-14}	4.883×10^{-16}
500	−0.417276484590423 + 0.257429625049396i	3.407×10^{-8}	8.609×10^{-13}	7.249×10^{-16}	1.382×10^{-15}
10,000	0.487266314746835 − 0.032032920039315i	8.567×10^{-11}	2.205×10^{-15}	6.556×10^{-16}	4.626×10^{-16}

5. Conclusions

Clearly, Tables 1–8 illustrate the relative error of the Clenshaw-Curtis quadrature taken as $\frac{|I^{\alpha}_{N+1}[f]-I^{\alpha}[f]|}{|I^{\alpha}[f]|}$. We can see that for proposed Clenshaw-Curtis quadrature based on Hermite interpolation polynomial, with small value of points higher precision of the numerical results of integrals is obtained in $O(N \log N)$ operations. Furthermore these tables show that more accurate results can be obtained as k increases with fixed value of N. Conversely, more accurate approximation can be achieved as N increases but k is fixed. Moreover, Tables demonstrate that results successfully satisfy the analysis derived in Section 3.

Author Contributions: The Authors have equally contributed to this paper.

Funding: This work was supported by National Science Foundation of China (No. 11771454) and the Mathematics and Interdisciplinary Sciences Project of Central South University.

Conflicts of Interest: The authors declare no conflict of interest.The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript, or in the decision to publish the results.

References

1. Karczmarek, P.; Pylak, D.; Sheshko, M.A. Application of Jacobi polynomials to approximate solution of a singular integral equation with Cauchy kernel. *Appl. Math. Comput.* **2006**, *181*, 694–707. [CrossRef]
2. Cuminato, N.A. On the uniform convergence of a perturbed collocation method for a class of Cauchy integral equations. *Appl. Numer. Math.* **1995**, *16*, 439–455. [CrossRef]
3. Eshkuvatov, Z.K.; Long, N.N.; Abdulkawi, M. Approximate solution of singular integral equations of the first kind with Cauchy kernel. *Appl. Math. Lett.* **2009**, *22*, 651–657. [CrossRef]
4. Ursell, F. Integral equations with a rapidly oscillating kernel. *J. Lond. Math. Soc.* **1969**, *1*, 449–459. [CrossRef]
5. Polyanin, A.D.; Manzhirov, A.V. *Handbook of Integral Equations*; CRC Press: Boca Raton, FL, USA, 1998. [CrossRef]
6. Olver, S. Numerical Approximation of Highly Oscillatory Integrals. Ph.D. Thesis, University of Cambridge, Cambridge, UK, 2008.
7. Iserles, A.; Norsett, S.P. On quadrature methods for highly oscillatory integrals and their implementation. *BIT Numer. Math.* **2004**, *44*, 755–772. [CrossRef]
8. Trefethen, L.N. Is Gauss quadrature better than Clenshaw Curtis? *SIAM Rev.* **2008**, *50*, 67–87. [CrossRef]
9. Trefethen, L.N. Chebyshev Polynomials and Series, Approximation theory and approximation practice. *Soc. Ind. Appl. Math.* **2013**, *128*, 17–19.
10. Dominguez, V.; Graham, I.G.; Smyshlyaev, V.P. Stability and error estimates for Filon-Clenshaw-Curtis rules for highly oscillatory integrals. *SIMA J. Numer. Anal.* **2011**, *31*, 1253–1280. [CrossRef]
11. Dominguez, V.; Graham, I.G.; Kim, T. Filon Clenshaw Curtis rules for highly oscillatory integrals with algebraic singularities and stationary points. *SIMA J. Numer. Anal.* **2013**, *51*, 1542–1566. [CrossRef]
12. Wang, H.; Xiang, S. Uniform approximations to Cauchy principal value integrals of oscillatory functions. *Appl. Math. Comput.* **2009**, *215*, 1886–1894. [CrossRef]
13. He, G.; Xiang, S. An improved algorithm for the evaluation of Cauchy principal value integrals of oscillatory functions and its application. *J. Comput. Appl. Math.* **2015**, *280*, 1–13. [CrossRef]
14. Xiang, S.; Chen, X.; Wang, H. Error bounds for approximation in Chebyshev points. *Numer. Math.* **2010**, *116*, 463–491. [CrossRef]
15. Setia, A. Numerical solution of various cases of Cauchy type singular integral equation. *Appl. Math. Comput.* **2014**, *230*, 200–207. [CrossRef]
16. Dominguez, V. Filon Clenshaw Curtis rules for a class of highly oscillatory integrals with logarithmic singularities. *J. Comput. Appl. Math.* **2014**, *261*, 299–319. [CrossRef]
17. Piessens, R.; Branders, M. On the computation of Fourier transforms of singular functions. *J. Comput. Appl. Math.* **1992**, *43*, 159–169. [CrossRef]

18. Chen, R. Fast computation of a class of highly oscillatory integrals. *Appl. Math. Comput.* **2014**, *227*, 494–501. [CrossRef]
19. Fang, C. Efficient methods for highly oscillatory integrals with weak and Cauchy singularities. *Int. J. Comput. Math.* **2016**, *93*, 1597–1610. [CrossRef]
20. Tsalamengas, J.L. Gauss Jacobi quadratures for weakly, strongly, hyper and nearly singular integrals in boundary integral equation methods for domains with sharp edges and corners. *J. Comput. Phys.* **2016**, *325*, 338–357. [CrossRef]
21. Liu, G.; Xiang, S. Clenshaw Curtis-type quadrature rule for hypersingular integrals with highly oscillatory kernels. *Appl. Math. Comput.* **2019**, *340*, 251–267. [CrossRef]
22. Boyd, J.P. *Chebyshev and Fourier Spectral Methods*; Courier Corporation: The North Chelmsford, MA, USA, 2001.
23. Trefethen, L.N. Chebfun. Available online: https://www.chebfun.org (accessed on 27 April 2019).
24. Oliver, J. The numerical solution of linear recurrence relations. *Numer. Math.* **1968**, *11*, 349–360. [CrossRef]
25. Xiang, S.; He, G.; Cho, Y.N. On error bounds of Filon Clenshaw Curtis quadrature for highly oscillatory integrals. *Adv. Comput. Math.* **2015**, *41*, 573–597. [CrossRef]