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Relations among the Riemann Zeta and Hurwitz Zeta Functions, as Well as Their Products

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Abstract: In this paper, several relations are obtained among the Riemann zeta and Hurwitz zeta functions, as well as their products. A particular case of these relations give rise to a simple re-derivation of the important results of Katsurada and Matsumoto on the mean square of the Hurwitz zeta function. Also, a relation derived here provides the starting point of a novel approach which, in a series of companion papers, yields a formal proof of the Lindelöf hypothesis. Some of the above relations motivate the need for analysing the large \( \alpha \) behaviour of the modified Hurwitz zeta function \( \zeta_1(s, \alpha) \), \( s \in \mathbb{C}, \alpha \in (0, \infty) \), which is also presented here.

Keywords: Hurwitz zeta function; Riemann zeta function; asymptotics

MSC: 11M35; 11L07

1. Introduction

Let

\[ s = \sigma + it, \quad \sigma \in \mathbb{R}, \quad t \in \mathbb{R}. \quad (1) \]

The Riemann zeta function \( \zeta(s) \) and the modified Hurwitz zeta function \( \zeta_1(s, \alpha) \) are defined, respectively, by

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \sigma > 1, \quad (2) \]

\[ \zeta_1(s, \alpha) = \sum_{n=1}^{\infty} \frac{1}{(n + \alpha)^s}, \quad \sigma > 1, \quad \alpha \geq 1, \quad (3) \]

and by analytic continuation for \( s \in \mathbb{C}, s \neq 1 \).

The modified Hurwitz zeta function is simply related to the Hurwitz zeta function, \( \zeta(s, \alpha) \):

\[ \zeta(s, \alpha) = \frac{1}{\alpha^s} + \zeta_1(s, \alpha), \quad s \in \mathbb{C}, \alpha > 0. \quad (4) \]

In this paper, we present certain relations between \( \zeta(s) \) and \( \zeta_1(s, \alpha) \), as well as between products of these functions. The following results are presented in Sections 2–5 in more detail.

In Section 2, it is shown that the modified Hurwitz zeta function satisfies the identity

\[ |\zeta_1(s, \alpha)|^2 = \zeta_1(2\sigma, \alpha) + \frac{1}{2\pi i} \int_{(c)} \left( \frac{\Gamma(s+z)}{\Gamma(s)} \frac{\Gamma(s+z)}{\Gamma(s)} \right) \Gamma(-z)\zeta(-z)\zeta_1(2\sigma+z, \alpha) \, dz \]

\[ \sigma > 1, \quad t > 0, \quad \max(-\sigma, 1-2\sigma) < c < -1, \quad (5) \]
where $\Gamma(s)$, $s \in \mathbb{C}$, denotes the Gamma function and ($c$) denotes the vertical line in the complex $z$-plane on which $\text{Re}(z) = c$.

It is shown in [1] that (5) yields a singular integral equation for $|\zeta(s)|^2$, $0 < \sigma < 1$, $t > 0$, and this equation provides the starting point for the proof of the analogue of Lindelöf's hypothesis for a certain Riemann zeta-type double exponential sum describing the leading asymptotics of $|\zeta(s)|^2$; namely, it includes $m_1^{-\sigma-it}m_2^{-\sigma-it}$, but the limits of the summation are different from the double exponential sum relevant to $|\zeta(s)|^2$ — see [1]. We recall that Lindelöf’s hypothesis concerns the growth of $\zeta(s)$ as $t \to \infty$ along the critical line $\sigma = 1/2$, and states that $\zeta(\frac{1}{2} + it) = O(t^\varepsilon)$ for every positive $\varepsilon$. The Riemann hypothesis implies Lindelöf’s hypothesis, and conversely, Lindelöf’s hypothesis implies that very few zeros could disobey Riemann’s hypothesis [2]. Significant progress has been made by developing ingenuous ways of estimating exponential sums $\sum e^{2\pi i f(n)}$ using generalisations of the Vinogradov method [3]. Until recently, the best result in this direction was obtained by Huxley [4], where it is proved that $\zeta(\frac{1}{2} + it) = O((32/205)^{\varepsilon})$. Just recently, Bourgain announced a further improvement [5] where the exponent $32/205$ was reduced to $53/342$.

In Section 3 it is shown that there exists the following asymptotic relation between the Riemann zeta function and the modified Hurwitz zeta function:

$$\zeta(s) = \int_0^1 B_N(\alpha) \zeta_1(s, \alpha) \, d\alpha + \chi(s) \int_0^1 B_N(-\alpha) \zeta_1(1-s, \alpha) \, d\alpha + O(t^{-\sigma/2} \log t), \quad 0 < \sigma < 1, \quad t \to \infty, \quad (6)$$

where $B_N(\alpha)$ is defined by

$$B_N(\alpha) = \sum_{1 \leq n \leq N} e^{2\pi i n\alpha}, \quad \alpha > 0 \quad (7)$$

and $N = \sqrt{t/2\pi}$. A direct consequence of (6) is the following theorem (c.f. [6–8] and references therein):

**Theorem 1.** Let $I_k(t)$ denote the $2k$-th power mean of the modified Hurwitz zeta function, namely

$$I_k(t) = \int_0^1 |\zeta_1(\frac{1}{2} + it, \alpha)|^{2k} \, d\alpha, \quad k \in \mathbb{N}. \quad (8)$$

Then,

$$|\zeta(\frac{1}{2} + it)| \ll t^{1/4k} I_k(t)^{1/2k}, \quad k \in \mathbb{N} \quad (9)$$

in which the implicit constant is independent of $k$.

This result immediately implies that Lindelöf’s hypothesis is true, provided that for each $\varepsilon > 0$ $I_k(t) \ll_{k, \varepsilon} t^\varepsilon$ for each $k \in \mathbb{N}$.

In connection with (9), we recall that an equivalent formulation of the Lindelöf hypothesis involves estimating the $2k$-th power mean of the Riemann-zeta function

$$J_k(T) = \frac{1}{T} \int_0^T |\zeta(\frac{1}{2} + it)|^{2k} \, dt. \quad (10)$$

It can be shown ([9] Th. 13.2) that the Lindelöf hypothesis holds true if, and only if $J_k(T) = O(T^\varepsilon)$ for each $\varepsilon > 0$ and for each $k \in \mathbb{N}$.

In Section 4, the following identities are presented:

$$\int_0^1 \zeta_1(u_1, \alpha) \zeta_1(u_2, \alpha) \, d\alpha = \frac{1}{u_1 + u_2 - 1} + \int_1^\infty a^{-u_1} \zeta_1(u_2, \alpha) \, d\alpha + \int_1^\infty a^{-u_2} \zeta_1(u_1, \alpha) \, d\alpha, \quad (11)$$
\[
\int_0^1 \xi_1(u_1, a)\xi_1(u_2, a)\xi_1(u_3, a) \, da = \frac{1}{u_1 + u_2 + u_3 - 1} + \int_1^\infty a^{-u_1-1} \xi_1(u_2, a)\xi_1(u_3, a) \, da + 
\]
\[
\int_1^\infty a^{-u_1} \xi_1(u_1, a)\xi_1(u_3, a) \, da + \int_1^\infty a^{-u_2-1} \xi_1(u_1, a)\xi_1(u_2, a) \, da + 
\]
\[
\int_1^\infty a^{-u_1-u_2} \xi_1(u_3, a) \, da + \int_1^\infty a^{-u_2} \xi_1(u_1, a) \, da + \int_1^\infty a^{-u_3} \xi_1(u_2, a) \, da,
\] (12)

and
\[
\int_0^1 \prod_{i=1}^4 \xi_1(u_i, a) \, da = \frac{1}{u_1 + u_2 + u_3 + u_4 - 1} + \sum_{\text{perms}} \int_1^\infty a^{-u_1+u_2+u_3+u_4} \xi_1(u_i, a) \, da + 
\]
\[
\sum_{\text{perms}} \int_1^\infty a^{-u_1+u_2} \xi_1(u_3, a)\xi_1(u_4, a) \, da + \sum_{\text{perms}} \int_1^\infty a^{-u_1-u_2} \xi_1(u_2, a)\xi_1(u_3, a)\xi_1(u_4, a) \, da,
\] (13)

where \( \text{Re} \,(u_1 + u_2) > 1, \text{Re} \,(u_1 + u_2 + u_3) > 1, \) and \( \text{Re} \,(u_1 + u_2 + u_3 + u_4) > 1, \) respectively. The above formulae can be generalised in a straightforward way.

As a direct application of (11), we present in Section 4 a new derivation of the following exact identity in [10]:
\[
\int_0^1 \xi_1(u, a)\xi_1(v, a) \, da = \frac{1}{u + v - 1} + \Gamma(u + v - 1) \xi_1(u + v - 1) \left[ \frac{\Gamma(1-u)}{\Gamma(u)} + \frac{\Gamma(1-v)}{\Gamma(v)} \right] + 
\]
\[
\frac{\xi(u) - 1}{v - 1} + \frac{\xi(v) - 1}{u - 1} + \frac{u}{v - 1} \int_0^1 a^1-v \xi_1(u+1, a) \, da + \frac{v}{u - 1} \int_0^1 a^1-u \xi_1(v+1, a) \, da
\] (14)

valid for \( \text{Re} \, u < 2 \) and \( \text{Re} \, v < 2. \) From this, one can obtain the estimate
\[
I_1(t) = \log \left( \frac{t}{2\pi} \right) + \gamma + O \left( \frac{1}{t^2} \right), \quad t \to \infty.
\] (15)

It is shown in [11] that (12) plays a crucial role for the derivation of an interesting identity between certain double exponential sums. Indeed, it is well-known that if \( 0 \leq \alpha < 1, \) then the large \( t \)-asymptotics of \( \xi_1(s, \alpha) \) is dominated by the sum \( S_1 \), defined by
\[
S_1(\sigma, t, \alpha) = \sum_{1 \leq n < t/2\pi} e^{-2\pi i n a} m^{s-1}, \quad 0 < \sigma < 1, t > 0.
\] (16)

However, if \( 1 \leq \alpha < \infty, \) the large \( t \)-asymptotics of \( \xi_1(s, \alpha) \) is dominated by the sum \( S_1 \) defined in (16), as well as by a different sum, \( S_2(\sigma, t, \alpha) \) [12] (c.f. [13]). Thus, the large \( t \)-asymptotics of Equation (12) provides a relation between two double sums generated from \( S_1 \) and \( S_2 \), and the explicit formulae obtained from the large \( t \)-asymptotics of the linear and quadratic terms.

Similarly, Equation (13) yields novel relations between cubic exponential sums.

Before using Equations (12) and (13) in the cases of \( \text{Re} \,(u_1 + u_2 + u_3) < 1 \) and \( \text{Re} \,(u_1 + u_2 + u_3 + u_4) < 1, \) respectively, it is necessary to regularize the terms involving \( a \to \infty; \) this regularisation is discussed in Section 4. Finally, in Section 5, by considering the Fourier series of the product \( \xi_1(u, a)\xi_1(v, a) \) with complex numbers \( u, v \) satisfying \( \text{Re} \, u > 0, \text{Re} \, v > 0, \) by using certain elementary estimates for the resulting coefficients, and by employing Theorem 1 together with Parseval’s identity, we obtain the following asymptotic result.

**Theorem 2.** For each \( \eta > 0, \) we have
\[
|\xi(\frac{1}{2} + it)|^4 \ll_{\eta} t^{1/2} \sum_{|n| \leq t/\pi} \left| \int_1^{t/2\pi + \eta} a^{-1/2+it} \xi_1(\frac{1}{2} + it, a) e^{-2\pi i n a} \, da \right|^2.
\]
In particular, if the sum is $O(\epsilon^t)$ for each $\epsilon > 0$, then $|\zeta(\frac{1}{2} + it)| = O(\epsilon^{1/8+\epsilon}).$

2. An Identity Involving the Hurwitz Zeta Functions

In order to derive (5), we let

$$\alpha \geq 0, \quad u \in \mathbb{C}, \quad v \in \mathbb{C}, \quad \text{Re} \, u > 1, \quad \text{Re} \, v > 1.$$  \hspace{1cm} (17)

The definition of the modified Hurwitz zeta function, namely Equation (3), implies

$$\zeta_1(u, \alpha) \zeta_1(v, \alpha) = \zeta_1(u + v, \alpha) + f(u, v, \alpha) + f(v, u, \alpha)$$ \hspace{1cm} (18)

where

$$f(u, v, \alpha) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (n + \alpha)^{-v}(n + m + \alpha)^{-u}.$$ \hspace{1cm} (19)

Assuming that

$$|\arg(-w)| < \pi, \quad -\text{Re} \, a < b < 0,$$

we observe the Mellin-Barnes type integral identity

$$\Gamma(a)(1-w)^{-a} = \frac{1}{2\pi i} \int_{(b)} \Gamma(z+a) \Gamma(-z)(-w)^z \, dz.$$ \hspace{1cm} (21)

Letting in Equation (21)

$$w = -\frac{m}{n + \alpha}, \quad a = u,$$

we find

$$\Gamma(u)(n + \alpha)^u(m + n + \alpha)^{-u} = \frac{1}{2\pi i} \int_{(c)} \Gamma(u+z) \Gamma(-z)m^z(n + \alpha)^{-z} \, dz.$$  

Thus,

$$(n + \alpha)^{-v}(n + m + \alpha)^{-u} = \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(u+z)}{\Gamma(u)} \Gamma(-z)m^z(n + \alpha)^{-u-v-z} \, dz.$$  

Summing over $m$ and $n$, we obtain

$$f(u, v, \alpha) = \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(u+z)}{\Gamma(u)} \Gamma(-z)\zeta(-z)\zeta_1(u + v + z, \alpha) \, dz.$$ \hspace{1cm} (22)

Substituting (22) into (18) and then letting $u = \delta, v = \delta$ in the resulting expression, we find (5).

3. An Asymptotic Relation between the Riemann and Hurwitz Functions

The approximate functional equation for the Riemann zeta function provides the starting point for the estimation of the $\zeta(s)$ along the critical line. In this section, we derive a weak analogue of this equation. Throughout, we will set $N = \lfloor \sqrt[4]{t/2\pi} \rfloor$ and assume $\sqrt[4]{t/2\pi} \notin \mathbb{N}$ so that $N^2 < t/2\pi$. We refer to the sum

$$B_N(\alpha) = \sum_{1 \leq n \leq N} e^{2\pi i n a} = \frac{e^{2\pi (N+1) a}}{\sin(\pi a)} \sin(N\pi a).$$

This function is similar to the classical Dirichlet kernel that arises in the Fourier analysis. As such, we have the following well-known estimates.

**Lemma 1.** $\|B_N\|_p = O(\log N)$ if $p = 1$ and $\|B_N\|_p = O(N^{1-1/p})$ for $p > 1$. 

Our first result expresses the approximate functional equation for \( \zeta(s) \) as an integral equation involving the Hurwitz zeta function \( \zeta_1(s, \alpha) \). The proof of Theorem 1 will follow directly from this result.

**Lemma 2.** Let \( s = \sigma + it \) and \( B_N \), as previously defined. Then, we have

\[
\zeta(s) = \int_0^1 B_N(\alpha) \zeta_1(s, \alpha) \, d\alpha + \chi(s) \int_0^1 B_N(-\alpha) \zeta_1(1-s, \alpha) \, d\alpha \\
- \int_0^1 B_N(\alpha) \sum_{1 \leq n \leq N} (n+\alpha)^{-s} \, d\alpha - \chi(s) \int_0^1 B_N(-\alpha) \sum_{1 \leq n \leq N} (n+\alpha)^{s-1} \, d\alpha \\
+ O \left( t^{-\sigma/2} \log t \right),
\]

(23)

when \( 0 < \sigma < 1 \). Furthermore,

\[
\zeta(s) = \int_0^1 B_N(\alpha) \zeta_1(s, \alpha) \, d\alpha + \chi(s) \int_0^1 B_N(-\alpha) \zeta_1(1-s, \alpha) \, d\alpha + O \left( t^{-\sigma/2} \log t \right), \quad t \to \infty,
\]

(24)

where \( \chi(s) = 2^{s-1} \pi s \sin(\pi s/2) \Gamma(1-s) \) with \( \Gamma(s) \) denoting the Gamma function, and \( \zeta_1(s, \alpha) \) denoting the modified Hurwitz zeta function.

Let us first establish (23). The identity in (24) will be a consequence of this.

**Proof.** First, we recall the approximate functional equations for \( \zeta(s) \) and \( \zeta_1(s, \alpha) \) (see [14] and references therein)

\[
\zeta(\sigma + it) = \sum_{1 \leq n \leq N} n^{-s} + \chi(s) \sum_{1 \leq n \leq N} n^{s-1} + O \left( t^{-\sigma/2} \right),
\]

(25)

and

\[
\zeta_1(s, \alpha) = \sum_{1 \leq n \leq N} (n+\alpha)^{-s} + \chi(s) \sum_{1 \leq n \leq N} e^{-2\pi in\alpha} n^{s-1} + O \left( t^{-\sigma/2} \right),
\]

(26)

uniformly in \( 0 < \alpha < 1 \). The following identity is valid

\[
\sum_{1 \leq n \leq N} n^z = \int_0^1 B_N(\alpha) \left( \sum_{1 \leq m \leq N} e^{-2\pi im\alpha} m^z \right) \, d\alpha, \quad z \in \mathbb{C}.
\]

(27)

Indeed, the left-hand side of (27) can be rewritten in the form

\[
\sum_{1 \leq m,n \leq N} \delta_{mn} m^z = \sum_{1 \leq m,n \leq N} m^z \int_0^1 e^{2\pi i (n-m)\alpha} \, d\alpha \\
= \int_0^1 \left( \sum_{1 \leq m \leq N} e^{2\pi im\alpha} \right) \left( \sum_{1 \leq n \leq N} e^{-2\pi im\alpha} m^z \right) \, d\alpha,
\]

which is the right-hand side of (27). Using \( z = s - 1 \) and employing (26), we find

\[
\chi(s) \sum_{1 \leq n \leq N} n^{s-1} = \int_0^1 B_N(\alpha) \left[ \zeta_1(s, \alpha) - \sum_{1 \leq n \leq N} (n+\alpha)^{-s} \right] \, d\alpha + \int_0^1 \left[ O \left( t^{-\sigma/2} \right) B_N(\alpha) \right] \, d\alpha.
\]

(28)

We note that

\[
\left| \int_0^1 \left[ O \left( t^{-\sigma/2} \right) B_N(\alpha) \right] \, d\alpha \right| \ll t^{-\sigma/2} \int_0^1 |B_N(\alpha)| \, d\alpha = O(t^{-\sigma/2} \log t).
\]
Equation (27) implies

$$\sum_{1 \leq n \leq N} n^s = \int_0^1 B_N(-\alpha) \left( \sum_{1 \leq n \leq N} e^{2\pi i n u} n^s \right) \, d\alpha, \quad u \in \mathbb{C}. \tag{29}$$

Replacing \( s \) by \( 1 - s \) in (26) and taking the complex conjugate of the resulting equation, we find

$$\xi_1(1 - s, \alpha) = \sum_{1 \leq n \leq N} (n + \alpha)^{s-1} + \chi(1 - s) \sum_{1 \leq n \leq N} e^{2\pi i n} n^{-s} + \mathcal{O}\left(t^{-1-\epsilon}/2\right). \tag{30}$$

Using (29) with \( z = -s \) and employing (30), we find

$$\sum_{1 \leq n \leq N} n^{-s} = \int_0^1 B_N(-\alpha) \left[ \chi(s) \xi_1(1 - s, \alpha) - \chi(s) \sum_{1 \leq n \leq N} (n + \alpha)^{s-1} \right] \, d\alpha + \mathcal{O}\left(t^{-\sigma/2} \log t\right). \tag{31}$$

Here, we have used \( \chi(s) = \mathcal{O}\left(t^{1/2-\epsilon}\right) \) and a similar estimate as before:

$$\left| \chi(s) \int_0^1 \mathcal{O}\left(t^{-1-\epsilon}/2\right) B_N(-\alpha) \, d\alpha \right| \ll t^{-\sigma/2} \int_0^1 |B_N(-\alpha)| \, d\alpha = \mathcal{O}\left(t^{-\sigma/2} \log t\right).$$

Using Equations (28) and (31) in (25), we arrive at the result in the lemma. \( \Box \)

**Lemma 3.** With \( B_N(\alpha) \) defined as before, we have

$$\int_0^1 B_N(\alpha) \sum_{1 \leq n \leq N} (n + \alpha)^{-s} \, d\alpha = \frac{i}{2\pi N} \sum_{1 \leq n \leq N} \frac{1}{2\pi n} - n + \frac{1}{2\pi i} \sum_{1 \leq n \leq N} \frac{1}{2\pi n} - n + \mathcal{O}\left(t^{-\sigma/2} \log t\right).$$

**Proof.** Using the periodicity of \( B_N(\alpha) \) on \( \mathbb{R}/\mathbb{Z} \), we find

$$\int_0^1 B_N(\alpha) \sum_{1 \leq n \leq N} (n + \alpha)^{-s} \, d\alpha = \sum_{1 \leq n \leq N} \int_0^{n+1} B_N(\alpha) \alpha^{-s} \, d\alpha = \int_1^N B_N(\alpha) \alpha^{-s} \, d\alpha + \int_N^{N+1} B_N(\alpha) \alpha^{-s} \, d\alpha. \tag{32}$$

We next estimate the first integral on the right-hand side of (32), which we denote by \( I_N(s) \):

$$I_N(s) = \sum_{1 \leq n \leq N} \int_1^N e^{2\pi i n \alpha - i \log \alpha} \alpha^{-s} \, d\alpha, \quad 0 < \sigma < 1. \tag{33}$$

The integral in the above sum does not have any stationary points. Indeed, candidates for stationary points are the points \( \alpha^* \), where

$$\alpha^* = \frac{t}{2\pi n} > \frac{N^2}{n}.$$

Thus, since \( 1 \leq n \leq N \), \( \alpha^* \) is outside the range of integration. Hence, the above integral can be estimated using integration by parts:

$$\int_1^N e^{2\pi i n \alpha - i \log \alpha} \alpha^{-s} \, d\alpha = i \int_1^N \frac{d}{d\alpha} \left(e^{2\pi i n \alpha - i \log \alpha}\right) \alpha^{1-\sigma} \, d\alpha = -\frac{1}{2\pi i} \left(\frac{N^{-s}}{2\pi n} - \frac{1}{2\pi} - n\right) + E_n(s), \tag{34}$$
where
\[ E_n(s) = \int_1^N \frac{d}{da} \left( e^{2\pi i(na-\log a)} \left[ \frac{(1-\sigma)\alpha^{1-\sigma}}{(t-2\pi i\alpha)^2} + \frac{2\pi i\alpha^{1-\sigma}}{(t-2\pi i\alpha)^3} \right] da. \]

The error term \( E_n(s) \) can be evaluated using the second mean value theorem for integrals. For instance, for some \( \xi \in (1, N) \), we have
\[
|\text{Re} \ E_n(s)| = \left| (1-\sigma)N^{1-\sigma} \left( \frac{2\pi iN^2}{(t-2\pi iN)^2} \right) + \frac{2\pi iN}{(t-2\pi iN)^3} \right| \text{Re} \int_{\xi}^{N} \frac{d}{da} \left( e^{2\pi i(na-\log a)} \right) da
\]

\[ = \mathcal{O} \left( \frac{N^{1-\sigma}}{(t-2\pi iN)^2} \right) + \mathcal{O} \left( \frac{nN^{2-\sigma}}{(t-2\pi iN)^3} \right), \]

and similarly for \( \text{Im} \ E_n(s) \). It is now straightforward to show
\[
\sum_{1 \leq n \leq N} E_n(s) = \mathcal{O} \left( t^{-(1+\sigma)/2} \right). \tag{35}
\]

We also have the elementary estimate
\[
\left| \int_0^1 B_N(a) |a^{-s} da \right| \ll t^{-\sigma/2} \int_0^1 |B_N(a)| da = \mathcal{O} \left( t^{-\sigma/2} \log t \right).
\]

This combined with (35) and (34) gives the desired result. \( \square \)

The leading order terms in the above expansion are \( \mathcal{O} \left( t^{-\sigma/2} \log t \right) \), thus they can be absorbed into the error term. Indeed, using Lemma 3, it is now straightforward to see that
\[
\int_0^1 B_N(a) \sum_{1 \leq n \leq N} (n + a)^{-s} da = \mathcal{O} \left( t^{-\sigma/2} \log t \right).
\]

Using similar arguments, we also find
\[
\chi(s) \int_0^1 B_N(-a) \sum_{1 \leq n \leq N} (n + a)^{s-1} da = \mathcal{O} \left( t^{-\sigma/2} \log t \right).
\]

Combining this observation with the result of Lemma 2, we conclude that
\[
\zeta(s) = \int_0^1 B_N(a) \chi_1(s,a) da + \chi(s) \int_0^1 B_N(-a) \chi_1(1-s,a) da + \mathcal{O} \left( t^{-\sigma/2} \log t \right), \quad t \to \infty
\]

for \( 0 < \sigma < 1 \). The proof to Theorem 1 now follows from Lemmas 2 and 3 with \( s = 1/2 \) and the application of Hölder’s inequality with exponents
\[
p = 2k, \quad q = \frac{2k}{2k-1}.
\]

In particular, using the estimates in Lemma 1, we have
\[
\left| \int_0^1 B_N(a) \chi_1(s,a) da \right| \leq \left( \int_0^1 |B_N(a)|^{2k/(2k-1)} da \right)^{1-1/2k} \left( \int_0^1 |\chi_1(s,a)|^{2k} da \right)^{1/2k}
\]

\[ \ll N^{1/2k-1} I_k(t)^{1/2k} \ll n^{1/4k-1} I_k(t)^{1/2k}. \]

This gives rise to the result in Theorem 1.
4. Relations among Products of the Hurwitz Zeta Functions

4.1. Quadratic Formula

**Lemma 4.** Let \( \zeta_1(u, \alpha) \), \( u \in \mathbb{C}, \alpha > 0 \), denote the modified Hurwitz function, i.e.,

\[
\zeta_1(u, \alpha) = \sum_{m=1}^{\infty} \frac{1}{(m+\alpha)^u}, \quad \alpha > 0 \quad \text{Re } u > 1.
\]  

Then, for \( \text{Re } (u + v) > 2 \),

\[
\int_0^1 \zeta_1(v, \alpha) \zeta_1(u, \alpha) \, d\alpha = \frac{1}{u + v - 1} + \int_1^\infty \alpha^{-v} \zeta_1(u, \alpha) \, d\alpha + \int_1^\infty \alpha^{-u} \zeta_1(v, \alpha) \, d\alpha.
\]  

**Proof.** Let \( q_0(v, u) \) denote the LHS of Equation (37).

Using the integral representation of the modified Hurwitz function, namely

\[
\zeta_1(s, \alpha) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{e^{-\rho \alpha} \rho^{s-1}}{\rho - 1} \, d\rho, \quad \alpha > 0, \quad \text{Re } s > 1,
\]  

we find

\[
q_0(v, u) = \frac{1}{\Gamma(u)\Gamma(v)} \int_0^\infty \rho_1 \int_0^\infty \rho_2 \frac{\rho_1^{v-1} \rho_2^{u-1} - e^{-(\rho_1 + \rho_2)}}{\rho_1 + \rho_2} \frac{1 - e^{-(\rho_1 + \rho_2)}}{(\rho_1 - 1)(\rho_2 - 1)}.
\]

Inserting in this equation the identity

\[
\frac{1 - e^{-(\rho_1 + \rho_2)}}{(\rho_1 - 1)(\rho_2 - 1)} = e^{-(\rho_1 + \rho_2)} \left[ 1 + \frac{1}{\rho_1 - 1} + \frac{1}{\rho_2 - 1} \right],
\]

we find

\[
q_0(v, u) = f_0(v, u) + I_0(v, u) + I_0(u, v),
\]  

where

\[
f_0(v, u) = \frac{1}{\Gamma(u)\Gamma(v)} \int_0^\infty \rho_1 \int_0^\infty \rho_2 \frac{\rho_1^{v-1} \rho_2^{u-1} e^{-(\rho_1 + \rho_2)}}{\rho_1 + \rho_2},
\]

and

\[
I_0(v, u) = \frac{1}{\Gamma(u)\Gamma(v)} \int_0^\infty \rho_1 \int_0^\infty \rho_2 \frac{\rho_1^{v-1} \rho_2^{u-1} e^{-(\rho_1 + \rho_2)}}{(\rho_1 + \rho_2)(\rho_2 - 1)}.
\]

Next, we will show that

\[
I_0(v, u) = \frac{1}{u + v - 1}.
\]

Indeed, using the integral representations of \( \Gamma(u) \) and \( \Gamma(v) \), we find

\[
\Gamma(u)\Gamma(v) = \int_0^\infty \rho_1 \int_0^\infty \rho_2 \rho_1^{u-1} \rho_2^{v-1} e^{-(\rho_1 + \rho_2)}.
\]  

Replacing in the RHS of (43) \( \rho_1 \) and \( \rho_2 \) by \( ax_1 \) and \( ax_2 \), multiplying the resulting expression by \( a^{-(u+v)} \), and integrating with respect to \( a \) from \( a = 1 \) to \( a = \infty \), we find

\[
\frac{\Gamma(u)\Gamma(v)}{u + v - 1} = \int_0^\infty dx_1 \int_0^\infty dx_2 \frac{x_1^{u-1} x_2^{v-1} e^{-(x_1 + x_2)}}{x_1 + x_2},
\]

which gives (42).

Finally, we will show that

\[
I_0(v, u) = \int_1^\infty \alpha^{-v} \zeta_1(u, \alpha) \, d\alpha.
\]
Indeed, using the integral representations of $\Gamma(v)$ and of $\zeta_1(u, \alpha)$, we find

$$
\Gamma(v)\zeta_1(u, \alpha) = \int_0^\infty e^{-\rho} \rho^{v-1} \, d\rho \times \frac{1}{\Gamma(u)} \int_0^\infty \frac{e^{-\alpha \rho}}{\rho^2 - 1} \, d\rho.
$$

(45)

Replacing in the RHS of the above equation $\rho$ by $\alpha \rho_1$, multiplying the resulting equation by $\alpha^{-v} / \Gamma(v)$, and then integrating with respect to $\alpha$ from $\alpha = 1$ to $\alpha = \infty$, we find (44).

Inserting in Equation (39) the expressions for $I_0(v, u)$, for $I_0(v, u)$ and for $I_0(u, v)$ from Equations (42) and (44), we find (37).

Equation (12) can be derived following the approach used in Lemmas 4 and 5, and thus it is omitted.

4.2. Quadruple Formula

*Lemma 5.* Let $\zeta_1(s, \alpha)$ be defined as in (38). Then for Re $u_i > 1$, $i = 1, 2, 3, 4$, the following identity is valid:

$$
\int_0^1 \prod_{i=1}^4 \zeta_1(u_i, \alpha) \, d\alpha = \frac{1}{u_1 + u_2 + u_3 + u_4 - 1} + \sum_{\text{perms}} \int_1^\infty \alpha^{-(u_1+u_2+u_3)} \zeta_1(u_4, \alpha) \, d\alpha
$$

$$
+ \sum_{\text{perms}} \int_1^\infty \alpha^{-(u_1+u_2)} \zeta_1(u_3, \alpha) \zeta_1(u_4, \alpha) \, d\alpha + \sum_{\text{perms}} \int_1^\infty \alpha^{-u_1} \zeta_1(u_2, \alpha) \zeta_1(u_3, \alpha) \zeta_1(u_4, \alpha) \, d\alpha,
$$

(46)

where the sums run over permutations of $(1, 2, 3, 4)$ so that the first and third sums contain 4 terms, whilst the second sum contains 6 terms.

*Proof.* Employing the representation (38) for each Hurwitz function and integrating over $(0, 1)$, we find that the left-hand side of (46), which we denote by $Q(u_1, u_2, u_3, u_4)$, is given by

$$
Q = \frac{1}{\prod_{i=1}^4 \Gamma(u_i)} \int_{(0,\infty)^4} \prod_{i=1}^4 \rho_i^{u_i-1} \, d\rho_i \frac{1 - e^{-(\rho_1+\rho_2+\rho_3+\rho_4)}}{\prod_{i=1}^4 R_i}.
$$

(47)

where the functions $\{R_i\}$ are defined by

$$
R_i = e^{\rho_i} - 1, \quad i = 1, 2, 3, 4.
$$

(48)

The following identity is valid:

$$
\frac{1 - e^{-(\rho_1+\rho_2+\rho_3+\rho_4)}}{\prod_{i=1}^4 R_i} = e^{-(\rho_1+\rho_2+\rho_3+\rho_4)} \left[ 1 + \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4} \right]
$$

$$
+ \frac{1}{R_1 R_2} + \frac{1}{R_1 R_3} + \frac{1}{R_1 R_4} + \frac{1}{R_2 R_3} + \frac{1}{R_2 R_4} + \frac{1}{R_3 R_4} + \frac{1}{R_1 R_2 R_3} + \frac{1}{R_1 R_2 R_4} + \frac{1}{R_1 R_3 R_4} + \frac{1}{R_2 R_3 R_4}.
$$

(49)

Using this in (47), we find

$$
Q = \frac{1}{\prod_{i=1}^4 \Gamma(u_i)} \int_{(0,\infty)^4} \prod_{i=1}^4 \rho_i^{u_i-1} \, d\rho_i \frac{1 - e^{-(\rho_1+\rho_2+\rho_3+\rho_4)}}{\prod_{i=1}^4 R_i} e^{-(\rho_1+\rho_2+\rho_3+\rho_4)} \left[ 1 + \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4} \right]
$$

$$
+ \frac{1}{R_1 R_2} + \frac{1}{R_1 R_3} + \frac{1}{R_1 R_4} + \frac{1}{R_2 R_3} + \frac{1}{R_2 R_4} + \frac{1}{R_3 R_4} + \frac{1}{R_1 R_2 R_3} + \frac{1}{R_1 R_2 R_4} + \frac{1}{R_1 R_3 R_4} + \frac{1}{R_2 R_3 R_4}.
$$

(50)

In order to simplify the right-hand side of (50), we first note the definition of the Gamma function—namely, the equation

$$
\Gamma(u) = \int_0^\infty r^{u-1} e^{-r} \, dr, \quad \text{Re } u > 0,
$$

(51)
implies that
\[ \prod_{i=1}^{4} \Gamma(u_i) = \int_{(0,\infty)^4} \prod_{i=1}^{4} (x_i^{u_i-1} \, dx_i) \, e^{-x_1-x_2-x_3-x_4}. \]

Using in the right-hand side of this equation the transformations
\[ x_i = a \rho_i, \quad i = 1, 2, 3, 4, \tag{52} \]
dividing by the product of the four Gamma functions, and multiplying the resulting expression by \( a^{-u_1-u_2-u_3-u_4} \), we find the identity
\[ a^{-u_1-u_2-u_3-u_4} = \frac{1}{\prod_{i=1}^{4} \Gamma(u_i)} \int_{(0,\infty)^4} \prod_{i=1}^{4} \left( \rho_i^{u_i-1} \, d\rho_i \right) \, e^{-\rho_1-\rho_2-\rho_3-\rho_4}. \tag{53} \]

Employing for \( \zeta_1(u, a) \) and \( \Gamma(u) \) Equations (38) and (51), respectively, we find
\[ \zeta_1(u_1, a) \Gamma(u_2) \Gamma(u_3) \Gamma(u_4) = \frac{1}{\Gamma(u_1)} \int_{(0,\infty)^4} d\rho_1 \, d\rho_2 \, d\rho_3 \, d\rho_4 \, \rho_1^{u_1-1} \frac{\rho_2^{u_2-1}}{R_1} \frac{\rho_3^{u_3-1}}{R_2} \frac{\rho_4^{u_4-1}}{R_3} \, e^{-\rho_1-\rho_2-\rho_3-\rho_4}. \]

Using in the right-hand side of this equation the transformations (52) but restricted only to \( i = 2, 3, 4 \), dividing by \( \Gamma(u_2) \Gamma(u_3) \Gamma(u_4) \), multiplying by \( a^{-(u_2+u_3+u_4)} \) and integrating with respect to \( a \) over \( (1, \infty) \), we obtain
\[ \int_{1}^{\infty} a^{-(u_2+u_3+u_4)} \zeta_1(u_1, a) \, da = \frac{1}{\prod_{i=1}^{4} \Gamma(u_i)} \int_{(0,\infty)^4} \prod_{i=1}^{4} \left( \rho_i^{u_i-1} \, d\rho_i \right) \, R_1 (\rho_1 + \rho_2 + \rho_3 + \rho_4). \tag{54} \]

Similarly, Equations (38) and (51) imply
\[ \zeta_1(u_1, a) \zeta_1(u_2, a) \Gamma(u_3) \Gamma(u_4) = \frac{1}{\Gamma(u_1) \Gamma(u_2)} \int_{(0,\infty)^4} d\rho_1 \, d\rho_2 \, d\rho_3 \, d\rho_4 \, \rho_1^{u_1-1} \rho_2^{u_2-1} \rho_3^{u_3-1} \rho_4^{u_4-1} \, e^{-\rho_1-\rho_2-\rho_3-\rho_4} \frac{R_1}{R_2}. \]

Using in the right-hand side of this equation the transformations (52) but only for \( i = 3, 4 \), dividing by \( \Gamma(u_3) \Gamma(u_4) \), multiplying by \( a^{-u_3-u_4} \) and integrating the resulting expression with respect to \( a \) over \( (1, \infty) \), we obtain
\[ \int_{1}^{\infty} a^{-u_3-u_4} \zeta_1(u_1, a) \zeta_1(u_2, a) \, da = \frac{1}{\prod_{i=1}^{4} \Gamma(u_i)} \int_{(0,\infty)^4} \prod_{i=1}^{4} \left( \rho_i^{u_i-1} \, d\rho_i \right) \, R_1 R_2 (\rho_1 + \rho_2 + \rho_3 + \rho_4). \tag{55} \]

A similar procedure yields the identity
\[ \int_{1}^{\infty} a^{-u_4} \zeta_1(u_1, a) \zeta_1(u_2, a) \zeta_1(u_3, a) \, da = \frac{1}{\prod_{i=1}^{4} \Gamma(u_i)} \int_{(0,\infty)^4} \prod_{i=1}^{4} \left( \rho_i^{u_i-1} \, d\rho_i \right) \, R_1 R_2 R_3 (\rho_1 + \rho_2 + \rho_3 + \rho_4). \tag{56} \]

Employing in Equation (50), Equations (53)–(56), and appropriate permutations of \( (1, 2, 3, 4) \), we find Equation (46).
4.3. A New Derivation of Power Mean Estimates for the Hurwitz Zeta Function

Here, we re-derive some of the results from [10]. The identity in (14) is a consequence of Equation (37) and of the following exact formula.

Lemma 6. Let $\zeta_1(u, \alpha), u \in \mathbb{C},$ and $\alpha > 0$ denote the modified Hurwitz function, and let $\zeta(u), u \in \mathbb{C}$ denote Riemann’s zeta function. Then,

$$\int_0^\infty \alpha^{-v} \zeta_1(u, \alpha) \, d\alpha = \frac{\Gamma(1 - v)}{\Gamma(u)} \Gamma(u + v - 1) \zeta(u + v - 1), \quad \text{Re} \, (u + v) > 2, \quad \text{Re} \, v > 1,$$

and

$$\int_0^1 \alpha^{-v} \zeta_1(u, \alpha) \, d\alpha = \frac{\zeta(u) - 1}{1 - v} + \frac{u}{1 - v} \int_0^1 \alpha^{-v} \zeta_1(u + 1, \alpha) \, d\alpha, \quad \text{Re} \, v < 1, \quad u \in \mathbb{C}. \quad (58)$$

**Proof.** In order to derive (57), we first assume that $\text{Re} \, u > 1,$ so that we can use the sum representation of $\zeta_1(u, \alpha).$ Furthermore, we assume that $\text{Re} \, v < 1,$ so that the relevant integral converges at $\alpha = 0.$ Then,

$$\int_0^\infty \alpha^{-v} \zeta_1(u, \alpha) \, d\alpha = \sum_{m=1}^\infty \int_0^\infty \alpha^{-v} (m + \alpha)^{-u} \, d\alpha = \sum_{m=1}^\infty m^{1-(u+v)} \int_0^\infty \beta^{-v} (1 + \beta)^{-u} \, d\beta,$$

where we have used the change of variables $\alpha = \beta m$ in the second equation. Then, the definition of Riemann’s zeta function, together with the identity

$$\int_0^\infty \frac{\beta^{-v}}{(1 + \beta)^u} \, d\beta = \frac{\Gamma(1 - v) \Gamma(u + v - 1)}{\Gamma(u)},$$

imply Equation (57).

In order to derive Equation (58), we use integration by parts:

$$\int_0^1 \alpha^{-v} \zeta_1(u, \alpha) \, d\alpha = \left. \frac{\alpha^{1-v}}{1-v} \zeta_1(u, \alpha) \right|_0^1 - \frac{1}{1-v} \int_0^1 \alpha^{1-v} \partial_\alpha \zeta_1(u, \alpha) \, d\alpha.$$

Thus, (58) follows. □

**Proof of identity (14).** Splitting the first integral in the RHS of (37) and then using Equations (57) and (58), as well as using the analogous equations where $u$ and $v$ are interchanged, we find Equation (14). □

**Remark 1.** In order to derive Equation (15) we let $\sigma = \frac{1}{2} + \frac{\epsilon}{2}$ in the LHS of (15), and employ the identities

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma + O(\epsilon), \quad \epsilon \to 0,$$

$$\frac{\Gamma'(\frac{1}{2} + it)}{\Gamma(\frac{1}{2} + it)} = \ln t + \frac{i \pi}{2} + O \left( \frac{1}{t^2} \right), \quad t \to \infty,$$

as well as the identity

$$\zeta(\epsilon) = -\frac{1}{2} \left[ 1 + \epsilon \ln(2\pi) + O(\epsilon^2) \right], \quad \epsilon \to 0.$$

**Remark 2.** By proving a simple estimate for the integrals in the RHS of (14), it is shown in [10] that these integrals do not contribute to the leading asymptotics of the LHS of Equation (14). Actually, it is straightforward to show that if

$$v = \sigma_1 - it, \quad u = \sigma_2 + it, \quad \sigma_1 < 2, \quad \sigma_1 > 0,$$
then
\[
\int_0^1 a^{1-v} \zeta_1(u+1, \alpha) \, d\alpha = \frac{1}{it} \sum_{m=1}^{\infty} \frac{1}{m(m+1)^u} + O\left(\frac{1}{t^2}\right), \quad t \to \infty.
\]  
(63)

Indeed,
\[
\int_0^1 a^{1-v} \zeta_1(u+1, \alpha) = \sum_{m=1}^{\infty} \int_0^1 a^{1-\sigma_1} (m+\alpha)^{-1-\sigma_2} e^{it[\ln \alpha - \ln(\alpha+m)]} \, d\alpha.
\]  
(64)

Noting that
\[
\frac{d}{d\alpha} [\ln \alpha - \ln(\alpha+m)] \neq 0, \quad m \geq 1, \quad 0 < \alpha < 1,
\]
it follows that integrals in the RHS of (64) do not possess any stationary points. Then, straightforward integration by parts yields (63).

5. A Relation between Quadratic Products of Hurwitz Zeta Functions and Their Fourier Series

Theorem 2 will be proved by examining the Fourier series for the function
\[
\zeta_1(u, \alpha) \zeta_1(v, \alpha)
\]
for \( \Re u, \Re v > 0 \). Following Rane [15], we first construct the Fourier series for \( \zeta_1(s, \alpha) \).

**Lemma 7.** Let \( \sigma \in (0, 1) \). Then, the Fourier series
\[
\frac{1}{s-1} + \sum_{n \neq 0} a_n(s) e^{2\pi i n \alpha}, \quad a_n(s) = \int_1^s \alpha^{-s} e^{-2\pi i n \alpha} \, d\alpha
\]
converges pointwise to \( \zeta_1(s, \alpha) \) for each \( \alpha \in (0, 1) \).

**Proof.** Since \( \zeta_1(s, \alpha) \) is a smooth function of \( \alpha \) (for fixed \( s \)), its Fourier series converges pointwise for \( \alpha \in (0, 1) \). The Fourier coefficients are defined by
\[
a_n(s) = \int_0^1 e^{-2\pi i n \alpha} \zeta_1(s, \alpha) \, d\alpha.
\]

Note that the Fourier series for \( \zeta(s, \alpha) = \alpha^{-s} + \zeta_1(s, \alpha) \) is well-known, and has Fourier coefficients \( \Gamma(1-s)(2\pi i n)^{s-1} \). Hence,
\[
a_n(s) = \Gamma(1-s)(2\pi i n)^{s-1} - \int_0^1 \alpha^{-s} e^{-2\pi i n \alpha} \, d\alpha.
\]

Using Euler’s integral representation of the Gamma function, we arrive at the desired result. \( \square \)

**Remark 3.** Since \( a_n(s) \) is expressible in terms of the incomplete Gamma function, we conclude that it has an analytic extension to all complex \( s \neq 1 \).

Note that for \( \sigma > 1 \), we have
\[
\frac{1}{s-1} = \int_1^s \alpha^{-s} \, d\alpha,
\]
so we may write
\[
a_0(s) = \int_1^s \alpha^{-s} \, d\alpha,
\]
for \( \sigma > 1 \), and by analytic continuation elsewhere. Now, we write
\[
\zeta_1(s, \alpha) = \sum_{\sigma} a_n(s) e^{2\pi i n \alpha}, \quad \sigma > 0
\]
where the $a_n(s)$ are defined accordingly.

**Lemma 8.** Let $\Re u, \Re v > 1$ and define the functions

$$q_n(u, v) = a_n(u + v) + \int_1^{\infty} \zeta_1(u, a) a^{-v} e^{-2\pi i n a} \, da + \int_1^{\infty} \zeta_1(v, a) a^{-u} e^{-2\pi i n a} \, da \quad n \in \mathbb{Z}.$$  

Then, the Fourier series $\sum_n q_n(u, v) e^{2\pi i n a}$ converges pointwise to $\zeta_1(u, a)\zeta_1(v, a)$ for $\alpha \in (0, 1)$.

**Proof.** Since the Fourier coefficients for $\zeta_1(u, a)$ are $a_n(u)$, the Fourier coefficients of the product $\zeta_1(u, a)\zeta_1(v, a)$ are given by the convolution

$$q_n(u, v) = \sum_m a_m(u) a_{n-m}(v).$$

For $\Re u, \Re v > 1$, we have

$$\sum_m a_m(u) a_{n-m}(v) = \sum_m \int_1^{\infty} \int_1^{\infty} \beta^{\alpha - u} \beta^{-v} e^{-2\pi i m(\alpha - \beta)} e^{-2\pi i n \beta} \, d\beta \, d\alpha,$$

the double integral being absolutely convergent. Now, we recall the distributional result

$$\sum_m e^{-2\pi i m(\alpha - \beta)} = \sum_m \delta(\beta - \alpha - m).$$

Using this in the above, we find

$$q_n(u, v) = \sum_{m \geq 1} \int_1^{\infty} \alpha^{\alpha - u} (m + \alpha)^{-v} e^{-2\pi i n a} \, da$$

$$+ \int_1^{\infty} \alpha^{-u - v} e^{-2\pi i n a} \, da + \sum_{m \geq 1} \int_1^{\infty} \alpha^{-v} (m + \alpha)^{-u} e^{-2\pi i n a} \, da. \quad (65)$$

Since $\Re u, \Re v > 1$, the integrands of the first and third terms can be dominated by the integrable functions $\alpha^{-\Re u}$ and $\alpha^{-\Re v}$, respectively, allowing us to pass the sum inside the integral,

$$q_n(u, v) = a_n(u + v) + \int_1^{\infty} \zeta_1(u, a) a^{-v} e^{-2\pi i n a} \, da + \int_1^{\infty} \zeta_1(v, a) a^{-u} e^{-2\pi i n a} \, da.$$

We note that both the integrals are absolutely convergent for $\Re u, \Re v > 1$. $\square$

To establish the main result in this section, we must first perform an analytic continuation of the functions $q_n(u, v)$ valid for $\Re u, \Re v > 0$. To this end, we recall the following result [15]:

$$\zeta_1(s, a) = \frac{\alpha^{1-s}}{s - 1} - \frac{\alpha^{-s}}{2} + \lim_{N \to \infty} \sum_{0 < |m| < N} \left( \int_\alpha^\infty x^{-s} e^{2\pi i m x} \, dx \right) e^{-2\pi i m a}, \quad (66)$$

where $s = \sigma + it$ and $\sigma > 0$. This result can be derived using the Euler-Maclaurin formula. We will need the following lemma to control the final term.

**Lemma 9.** Let $s = \sigma + it$ with $\sigma > 0$. Then if $\alpha \geq \eta > 1/2\pi$ we have

$$\lim_{N \to \infty} \sum_{0 < |m| < N} \left( \int_\alpha^\infty x^{-s} e^{2\pi i m x} \, dx \right) e^{-2\pi i m a} \ll \eta \, t^{\sigma - 1}.$$
Proof. For $\alpha$ in the stated range, we can integrate by parts using
\[
\int_a^\infty x^{-\alpha} e^{2\pi i m (x-\alpha)} \, dx = \int_a^\infty \frac{x^{-\sigma}}{i(2\pi m - t/x)} \frac{d}{dx} \left( x^{-\sigma} e^{2\pi i m (x-\alpha)} \right) \, dx
\]
\[
= -\frac{\alpha^{-\sigma}}{i(2\pi m - t/\alpha)} + i \int_a^\infty \frac{d}{dx} \left( \frac{x^{-\sigma}}{2\pi m - t/x} \right) x^{-\sigma} e^{2\pi i m (x-\alpha)} \, dx.
\]
We can estimate the sum arising from the first term
\[
\left| \sum_{0 < |m| < N} \frac{\alpha^{-\sigma}}{i(2\pi m - t/\alpha)} \right| = \alpha^{-\sigma} \sum_{0 < |m| < N} \frac{2/\alpha}{(4\pi^2 m^2 - t^2/\alpha^2)^{1/2}} \ll \eta \alpha^{-\sigma-1}.
\]
Computing the derivative, the second term becomes
\[
\int_a^\infty \frac{x^{-\sigma-1}(2\pi m + (1-\sigma)t/x)}{i(2\pi m - t/x)^3} \frac{d}{dx} \left( x^{-\sigma} e^{2\pi i m (x-\alpha)} \right) \, dx.
\]
An application of the second mean value theorem for integrals on the real and imaginary parts of this term show it to be $O_\eta(\alpha^{-\sigma-1} m^{-2}) + O_\eta(\alpha^{-\sigma-2} m^{-3})$. In particular
\[
\left| \sum_{0 < |m| < N} \int_a^\infty \frac{d}{dx} \left( \frac{x^{-\sigma}}{2\pi m - t/x} \right) x^{-\sigma} e^{2\pi i m (x-\alpha)} \, dx \right| \ll_\eta \alpha^{-\sigma-1} \text{ for } \alpha \geq \eta > t/2\pi
\]
so we have established our estimate. \(\square\)

Now, we return to the analytic continuation of $q_\alpha(u, v)$ for Re $u$, Re $v > 0$. The previous lemma establishes that
\[
\left| \alpha^{-\nu} \xi_1(u, \alpha) - \frac{\alpha^{1-u-v}}{u-1} + \frac{\alpha^{-u-v}}{2} \right| \leq \eta \alpha^{-\Re u - \Re v - 1}, \text{ for } \alpha \geq \eta > t/2\pi. \tag{67}
\]
In particular, the left-hand side is an absolutely integrable function of $\alpha$ on $(1, \infty)$, provided that Re $u$, Re $v > 0$. This suggests the splitting
\[
\int_1^\infty \xi_1(u, \alpha) \alpha^{-\nu} e^{-2\pi i n u} \, d\alpha = \int_1^\infty \left( \alpha^{-\nu} \xi_1(u, \alpha) - \frac{\alpha^{1-u-v}}{u-1} + \frac{\alpha^{-u-v}}{2} \right) e^{-2\pi i n \alpha} \, d\alpha
\]
\[
+ \frac{\alpha_n(u + v - 1)}{u - 1} - \frac{\alpha_n(u + v)}{2},
\]
which is valid for Re $u$, Re $v > 1$. This gives rise to the representation
\[
q_\alpha(u, v) = \left[ \frac{1}{u-1} + \frac{1}{v-1} \right] a_n(u + v - 1) + \int_1^\infty \left( \alpha^{-\nu} \xi_1(u, \alpha) - \frac{\alpha^{1-u-v}}{u-1} + \frac{\alpha^{-u-v}}{2} \right) e^{-2\pi i n \alpha} \, d\alpha
\]
\[
+ \int_1^\infty \left( \alpha^{-\nu} \xi_1(v, \alpha) - \frac{\alpha^{1-u-v}}{v-1} + \frac{\alpha^{-u-v}}{2} \right) e^{-2\pi i n \alpha} \, d\alpha,
\]
which provides an analytic continuation of $q_\alpha(u, v)$ for Re $u$, Re $v > 0$.

Remark 4. Using $\partial_\alpha \xi_1(s, \alpha) = -s \xi_1(s + 1, \alpha)$ in (66), we see that
\[
\frac{\partial \xi_1}{\partial \alpha}(u, \alpha) = -\alpha^{-s} + \frac{s \alpha^{-s-1}}{2} - \lim_{N \to \infty} \sum_{0 < |m| < N} \left( s \int_a^\infty x^{-s-1} e^{2\pi i n x} \, dx \right) e^{-2\pi i n}.
\]
Using the previous lemma, this then implies that for \( \alpha \geq \eta > t/2\pi \)
\[
\left| \frac{\partial}{\partial \alpha} \left( \zeta_1(s, \alpha) - \frac{\alpha^{1-s}}{s-1} + \frac{\alpha^{-s}}{2} \right) \right| \ll \eta^2 \alpha^{-\sigma - 2}. \tag{68}
\]

**Lemma 10.** If \( u = \sigma + it \) and \( v = \sigma - it \), then for each \( \eta > 0 \)
\[
\left| \int_{t/2\pi + \eta}^{\infty} \left( a^{-v} \zeta_1(u, \alpha) - \frac{\alpha^{1-u-v}}{u-1} + \frac{\alpha^{-u-v}}{2} \right) e^{-2\pi i n a} \right| \ll \frac{t^{-2\sigma}}{n} \]
where the implied constant is independent of \( t \).

**Proof.** Integrating by parts, we find the above integral can be rewritten as
\[
\frac{1}{2\pi i n} \left( a^{-v} \zeta_1(u, \alpha) - \frac{\alpha^{1-u-v}}{u-1} + \frac{\alpha^{-u-v}}{2} \right) e^{-2\pi i n a} \Bigg|_{\alpha=t/2\pi+\eta}^{\infty} + \frac{1}{2\pi i n} \int_{t/2\pi + \eta}^{\infty} \frac{\partial}{\partial \alpha} \left( a^{-v} \zeta_1(u, \alpha) - \frac{\alpha^{1-u-v}}{u-1} + \frac{\alpha^{-u-v}}{2} \right) e^{-2\pi i n a} \, d\alpha.
\]
The first term can be estimated using (67), giving
\[
\left| \frac{1}{2\pi i n} \left( a^{-v} \zeta_1(u, \alpha) - \frac{\alpha^{1-u-v}}{u-1} + \frac{\alpha^{-u-v}}{2} \right) e^{-2\pi i n a} \right|_{\alpha=t/2\pi+\eta}^{\infty} \ll \frac{t^{-2\sigma}}{n}.
\]
And the second term can be estimated using (68)
\[
\left| \frac{1}{2\pi i n} \int_{t/2\pi + \eta}^{\infty} \frac{\partial}{\partial \alpha} \left( a^{-v} \zeta_1(u, \alpha) - \frac{\alpha^{1-u-v}}{u-1} + \frac{\alpha^{-u-v}}{2} \right) e^{-2\pi i n a} \, d\alpha \right| \ll \frac{t^{2}}{n} \int_{t/2\pi + \eta}^{\infty} a^{-2\sigma - 2} \, d\alpha.
\]
Performing the final integration shows that this term is \( O_{\eta}(t^{1-2\sigma}/n) \). \[\square\]

The previous Lemma gives the following
\[
q_n(u, v) = \left[ \frac{1}{u-1} + \frac{1}{v-1} \right] a_n(u + v - 1) + \int_{1}^{t/2\pi + \eta} \left( a^{-v} \zeta_1(u, \alpha) - \frac{\alpha^{1-u-v}}{u-1} + \frac{\alpha^{-u-v}}{2} \right) e^{-2\pi i n a} \, d\alpha \\
+ \int_{1}^{t/2\pi + \eta} \left( a^{-u} \zeta_1(v, \alpha) - \frac{\alpha^{1-u-v}}{v-1} + \frac{\alpha^{-u-v}}{2} \right) e^{-2\pi i n a} \, d\alpha + O_{\eta} \left( \frac{1^{1-2\sigma}}{n} \right),
\]
valid for \( u = \sigma + it, v = \sigma - it \) and \( \sigma > 0 \). We note that for \( n \neq 0 \), a simple integration-by-parts argument provides an analytic continuation for \( a_n(s) \) in \( \text{Re } s > -1 \). That is,
\[
a_n(s) = \int_{1}^{\infty} \alpha^{-s} e^{-2\pi i n a} \, d\alpha \\
= \frac{1}{2\pi i n} - \frac{s}{2\pi i n} \int_{1}^{\infty} \alpha^{-s-1} e^{-2\pi i n a} \, d\alpha.
\]
In particular,
\[
a_n(2\sigma - 1) = \frac{1}{2\pi i n} - \frac{(2\sigma - 1)}{2\pi i n} \int_{1}^{\infty} \alpha^{-2\sigma} e^{-2\pi i n a} \, d\alpha = O \left( \frac{1}{n} \right).
\]
Using integration by parts, we also have for \( 0 < \sigma \leq 1/2 \),
\[
\frac{1}{u-1} \int_{1}^{t/2\pi + \eta} \alpha^{1-u-v} e^{-2\pi i n a} \, d\alpha = O \left( \frac{t^{1-2\sigma}}{n} \right), \quad \int_{1}^{t/2\pi + \eta} \alpha^{-u} e^{-2\pi i n a} = O \left( \frac{1}{n} \right).
\]
So for $0 < \sigma \leq 1/2$ and $(u, v) = (\sigma + it, \sigma - it)$, we have

$$q_n(u, v) = \int_1^{t/2\pi + \eta} a^{-v} \zeta_1(u, a) e^{-2\pi i n a} \, da + \int_1^{t/2\pi + \eta} a^{-v} \zeta_1(u, a) e^{-2\pi i n a} \, da + \mathcal{O}\left(\frac{t^{1-2\sigma}}{n}\right).$$

Finally, we show that terms with $|n| > t/2\pi$ are easily controllable. For this, we once again use the approximate functional equation for the Hurwitz zeta function in the form

$$\zeta_1(s, a) = \sum_{1 \leq m \leq M} (a + m)^{-s} + \frac{(M + a)^{1-s}}{s-1} - \frac{1}{2} (M + a)^{-s} - s \int_M^\infty (a + x)^{-s-1} (x - [x] - \frac{1}{2}) \, dx$$

which holds for $M > 1$. This follows directly from the Euler-Maclaurin formula when $\text{Re } s > 1$, and then by analytic continuation for $\text{Re } s > 0$. We will require the following lemma:

**Lemma 11.** For $y > 1, \sigma_1, \sigma_2 > 0$, and $|n| > t/2\pi$, we have

$$\left|\int_1^{t/2\pi + \eta} a^{-v_1 + i t} (\bar{a} + y)^{-v_2 - i t} e^{-2\pi i n a} \, da\right| \ll \frac{y^{-\sigma_2}}{|n - t/2\pi|},$$

$$\left|\int_1^{t/2\pi + \eta} a^{-v_1 - i t} (\bar{a} + y)^{-v_2 + i t} e^{-2\pi i n a} \, da\right| \ll \frac{y^{-\sigma_2}}{|n + t/2\pi|}.$$

**Proof.** The proof is essentially the same as that used for Lemma 9. The oscillatory term does not have stationary points if $|n| > t/2\pi$, so integrating by parts yields the desired estimate. \qed

Applying this lemma and using the approximate functional equation, we find that for $|n| > t/2\pi$, the following estimate is valid for each $M > 1$:

$$\int_1^{t/2\pi + \eta} a^{-v} \zeta_1(u, a) e^{-2\pi i n a} \, da = \mathcal{O}\left(\frac{M^{1-\sigma}}{|n - t/2\pi|}\right) + \mathcal{O}\left(\frac{M^{1-\sigma}}{|n + t/2\pi|}\right) + \mathcal{O}\left(\frac{M^{\sigma}}{|n - t/2\pi|}\right) + \mathcal{O}\left(\frac{tM^{\sigma}}{|n - t/2\pi|}\right).$$

By choosing $M = \mathcal{O}(t)$, we find

$$\left|\int_1^{t/2\pi + \eta} a^{-v} \zeta_1(u, a) e^{-2\pi i n a} \, da\right| \ll \frac{t^{1-\sigma}}{|n - t/2\pi|}. $$

Similarly,

$$\left|\int_1^{t/2\pi + \eta} a^{-u} \zeta_1(v, a) e^{-2\pi i n a} \, da\right| \ll \frac{t^{1-\sigma}}{|n - t/2\pi|}. $$

So for $u = \sigma + it$ and $v = \bar{a}$,

$$\sum_{n > t/\pi} |q_n(u, v)|^2 \ll \eta \sum_{|n| > t/\pi} \left[\frac{t^{2-2\sigma}}{|n - t/2\pi|^2} + \frac{t^{2-2\sigma}}{|n + t/2\pi|^2} + \frac{t^{2-4\sigma}}{n^2}\right] = \mathcal{O}\left(t^{1-2\sigma}\right).$$
Now, by Parseval’s theorem,
\[
\int_0^1 |\zeta_1(u, \alpha)|^4 \, d\alpha = \sum_n |q_n(u, \bar{u})|^2
\]
\[
\ll \eta \sum_{|n| \leq t/\pi} \left| \int_1^{t/2\pi+\eta} a^{-\bar{a}} \zeta_1(u, a) e^{-2\pi ina} \, da + O\left( \frac{t^{-2\sigma}}{n} \right) \right|^2 + O\left( t^{1-2\sigma} \right)
\]
\[
\ll \eta \sum_{|n| \leq t/\pi} \left| \int_1^{t/2\pi+\eta} a^{-\bar{a}} \zeta_1(u, a) e^{-2\pi ina} \, da \right|^2 + O\left( t^{-2\sigma} \right) + O\left( t^{1-2\sigma} \right)
\]

Taking $\sigma = \frac{1}{2}$ and using this in Theorem 1 gives rise to the estimate in Theorem 2.

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