Article

Slant Curves and Contact Magnetic Curves in Sasakian Lorentzian 3-Manifolds

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Abstract: In this article, we define Lorentzian cross product in a three-dimensional almost contact Lorentzian manifold. Using a Lorentzian cross product, we prove that the ratio of $\kappa$ and $\tau - 1$ is constant along a Frenet slant curve in a Sasakian Lorentzian three-manifold. Moreover, we prove that $\gamma$ is a slant curve if and only if $M$ is Sasakian for a contact magnetic curve $\gamma$ in contact Lorentzian three-manifold $M$. As an example, we find contact magnetic curves in Lorentzian Heisenberg three-space.

Keywords: slant curves; Legendre curves; magnetic curves; Sasakian Lorentzian manifold

1. Introduction

As a generalization of Legendre curve, we defined the notion of slant curves in [1,2]. A curve in a contact three-manifold is said to be slant if its tangent vector field has constant angle with the Reeb vector field. For a contact Riemannian manifold, we proved that a slant curve in a Sasakian three-manifold is that its ratio of $\kappa$ and $\tau - 1$ is constant. Baikoussis and Blair proved that, on a three-dimensional Sasakian manifold, the torsion of the Legendre curve is $+1$ ([3]).

A magnetic curve represents a trajectory of a charged particle moving on the manifold under the action of a magnetic field in [4]. A magnetic field on a semi-Riemannian manifold $(M, g)$ is a closed two-form $F$. The Lorentz force of the magnetic field $F$ is a $(1,1)$-type tensor field $\Phi$ given by

$$g(\Phi(X), Y) = F(X, Y), \quad \forall X, Y \in \Gamma(TM).$$

(1)

The magnetic trajectories of $F$ are curves $\gamma$ on $M$ that satisfy the Lorentz equation

$$\nabla_{\gamma'}\gamma' = \Phi(\gamma'),$$

(2)

where $\nabla$ is the Levi–Civita connection of $g$. The Lorentz equation generalizes the equation satisfied by the geodesics of $M$, namely $\nabla_{\gamma'}\gamma' = 0$. Since the Lorentz force $\Phi$ is skew-symmetric, we have

$$\frac{d}{dt} g(\gamma', \gamma') = 2g(\Phi(\gamma'), \gamma') = 0,$$

that is, magnetic curve have constant speed $|\gamma'| = v_0$. When the magnetic curve $\gamma(t)$ is arc-length parameterized, it is called a normal magnetic curve. Cabreizo et al. studied a contact magnetic field in three-dimensional Sasakian manifold ([5]).

In this article, we define the magnetic curve $\gamma$ with contact magnetic field $F_{\xi, q}$ of the length $q$ in three-dimensional Sasakian Lorentzian manifold $M^3$. We call it the contact magnetic curve or trajectories of $F_{\xi, q}$. 

In Section 3, we define a Lorentzian cross product in a three-dimensional almost contact Lorentzian manifold. Using the Lorentzian cross product, we prove that the ratio of \( \kappa \) and \( \tau - 1 \) is constant along a Frenet slant curve in a Sasakian Lorentzian three-manifold.

In Section 4, we prove that \( \gamma \) is a slant curve if and only if \( M \) is Sasakian for a contact magnetic curve \( \gamma \) in contact Lorentzian three-manifolds \( M \). For example, we find contact magnetic curves in Lorentzian Heisenberg three-space.

2. Preliminaries

Contact Lorentzian Manifold

Let \( M \) be a \((2n + 1)\)-dimensional differentiable manifold. \( M \) has an almost contact structure \((\varphi, \xi, \eta)\) if it admits a tensor field \( \varphi \) of \((1, 1)\), a vector field \( \xi \) and a 1-form \( \eta \) satisfying

\[
\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1. \tag{3}
\]

Suppose \( M \) has an almost contact structure \((\varphi, \xi, \eta)\). Then, \( \varphi\xi = 0 \) and \( \eta \circ \varphi = 0 \). Moreover, the endomorphism \( \varphi \) has rank \( 2n \).

If a \((2n + 1)\)-dimensional smooth manifold \( M \) with almost contact structure \((\varphi, \xi, \eta)\) admits a compatible Lorentzian metric such that

\[
g(\varphi X, \varphi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{4}
\]

then we say \( M \) has an almost contact Lorentzian structure \((\eta, \xi, \varphi, g)\). Setting \( Y = \xi \), we have

\[
\eta(X) = -g(X, \xi). \tag{5}
\]

Next, if the compatible Lorentzian metric \( g \) satisfies

\[
d\eta(X, Y) = g(X, \varphi Y), \tag{6}
\]

then \( \eta \) is a contact form on \( M \), \( \xi \) is the associated Reeb vector field, \( g \) is an associated metric and \((M, \varphi, \xi, \eta, g)\) is called a contact Lorentzian manifold.

For a contact Lorentzian manifold \( M \), one may define naturally an almost complex structure \( J \) on \( M \times \mathbb{R} \) by

\[
J(X, f \frac{d}{dt}) = (\varphi X - f \xi, \eta(X) \frac{d}{dt}),
\]

where \( X \) is a vector field tangent to \( M \), \( t \) is the coordinate of \( \mathbb{R} \) and \( f \) is a function on \( M \times \mathbb{R} \). When the almost complex structure \( J \) is integrable, the contact Lorentzian manifold \( M \) is said to be normal or Sasakian. A contact Lorentzian manifold \( M \) is normal if and only if \( M \) satisfies

\[
[\varphi, \varphi] + 2d\eta \otimes \xi = 0,
\]

where \([\varphi, \varphi]\) is the Nijenhuis torsion of \( \varphi \).

Proposition 1 ([6,7]). An almost contact Lorentzian manifold \((M^{2n+1}, \eta, \xi, \varphi, g)\) is Sasakian if and only if

\[
(\nabla_X \varphi)Y = g(X, Y)\xi + \eta(Y)X. \tag{7}
\]

Using the similar arguments and computations in [8], we obtain
Proposition 2 ([6,7]). Let \((M^{2n+1}, \eta, \xi, \varphi, g)\) be a contact Lorentzian manifold. Then,

\[
\nabla_X \xi = \varphi X - \varphi hX. \tag{8}
\]

If \(\xi\) is a killing vector field with respect to the Lorentzian metric \(g\). Then, we have

\[
\nabla_X \xi = \varphi X. \tag{9}
\]

3. Slant Curves in Contact Lorentzian Three-Manifolds

Let \(\gamma: I \to M^3\) be a unit speed curve in Lorentzian three-manifolds \(M^3\) such that \(\gamma'\) satisfies \(g(\gamma', \gamma') = \varepsilon_1 = \pm 1\). The constant \(\varepsilon_1\) is called the causal character of \(\gamma\). A unit speed curve \(\gamma\) is said to be a spacelike or timelike if its causal character is 1 or \(-1\), respectively.

A unit speed curve \(\gamma\) is said to be a Frenet curve if \(g(\gamma'', \gamma'') \neq 0\). A Frenet curve \(\gamma\) admits an orthonormal frame field \(\{E_1 = \dot{\gamma}, E_2, E_3\}\) along \(\gamma\). The constants \(\varepsilon_2\) and \(\varepsilon_3\) are defined by

\[
g(E_i, E_i) = \varepsilon_i, \quad i = 2, 3
\]

and called second causal character and third causal character of \(\gamma\), respectively. Thus, \(\varepsilon_1\varepsilon_2 = -\varepsilon_3\) is satisfied.

Then, the Frenet–Serret equations are the following ([9,10]):

\[
\begin{aligned}
\nabla_\dot{\gamma} E_1 &= \varepsilon_2 \kappa E_2, \\
\nabla_\dot{\gamma} E_2 &= -\varepsilon_1 \kappa E_1 - \varepsilon_3 \tau E_3, \\
\nabla_\dot{\gamma} E_3 &= \varepsilon_2 \tau E_2,
\end{aligned} \tag{10}
\]

where \(\kappa = |\nabla_\dot{\gamma} \dot{\gamma}|\) is the geodesic curvature of \(\gamma\) and \(\tau\) its geodesic torsion. The vector fields \(E_1, E_2\) and \(E_3\) are called tangent vector field, principal normal vector field, and binormal vector field of \(\gamma\), respectively.

A Frenet curve \(\gamma\) is a geodesic if and only if \(\kappa = 0\). A Frenet curve \(\gamma\) with constant geodesic curvature and zero geodesic torsion is called a pseudo-circle. A pseudo-helix is a Frenet curve \(\gamma\) whose geodesic curvature and torsion are constant.

3.1. Lorentzian Cross Product

C. Camci ([11]) defined a cross product in three-dimensional almost contact Riemannian manifolds \((\tilde{M}, \eta, \xi, \varphi, \tilde{g})\) as following:

\[
X \wedge Y = -\tilde{g}(X, \varphi Y)\xi - \eta(Y)\varphi X + \eta(X)\varphi Y. \tag{11}
\]

If we define the cross product \(\wedge\) as Equation (11) in three-dimensional almost contact Lorentzian manifold \((M, \eta, \xi, \varphi, g)\), then

\[
g(X \wedge Y, X) = 2\eta(X)g(X, \varphi Y) \neq 0.
\]

In fact, we see already the cross product for a Lorentzian three-manifold as following:

Proposition 3. Let \(\{E_1, E_2, E_3\}\) be an orthonomal frame field in a Lorentzian three-manifold. Then,

\[
E_1 \wedge_L E_2 = \varepsilon_3 E_3, \quad E_2 \wedge_L E_3 = \varepsilon_1 E_1, \quad E_3 \wedge_L E_1 = \varepsilon_2 E_2. \tag{12}
\]

Now, in three-dimensional almost contact Lorentzian manifold \(M^3\), we define Lorentzian cross product as the following:
Definition 1. Let \((M^3, \varphi, \xi, \eta, g)\) be a three-dimensional almost contact Lorentzian manifold. We define a Lorentzian cross product \(\wedge_L\) by
\[
X \wedge_L Y = g(X, \varphi Y)\xi - \eta(Y)\varphi X + \eta(X)\varphi Y,
\]
(13)
where \(X, Y \in T M\).

The Lorentzian cross product \(\wedge_L\) has the following properties:

Proposition 4. Let \((M^3, \varphi, \xi, \eta, g)\) be a three-dimensional almost contact Lorentzian manifold. Then, for all \(X, Y, Z \in T M\) the Lorentzian cross product has the following properties:

(1) The Lorentzian cross product is bilinear and anti-symmetric.
(2) \(X \wedge_L Y\) is perpendicular both of \(X\) and \(Y\).
(3) \(X \wedge_L \varphi Y = -g(X, Y)\xi - \eta(X)Y\).
(4) \(\varphi X = \xi \wedge_L X\).
(5) Define a mixed product by \(\det(X, Y, Z) = g(X \wedge_L Y, Z)\) Then,
\[
\det(X, Y, Z) = -g(X, \varphi Y)\eta(Z) - g(Y, \varphi Z)\eta(X) - g(Z, \varphi X)\eta(Y)
\]
and \(\det(X, Y, Z) = \det(Y, Z, X) = \det(Z, X, Y)\).
(6) \(g(X, \varphi Y)Z + g(Y, \varphi Z)X + g(Z, \varphi X)Y = -(X, Y, Z)\xi\).

Proof. (We can prove by a similar way as in [11])

(1) and (2) are trivial.

(3) using Equations (3), (5) and (13),
\[
X \wedge_L \varphi Y = g(X, -Y + \eta(Y)\xi)\xi + \eta(X)(-Y + \eta(Y)\xi)
= -g(X, Y)\xi - \eta(X)Y.
\]

(4) by Equation (13),
\[
\xi \wedge_L X = g(\xi, \varphi X)\xi - \eta(X)\varphi \xi + \eta(\xi)\varphi X = \varphi X.
\]

(5) from Equations (5) and (13),
\[
g(X \wedge_L Y, Z) = g(g(X, \varphi Y)\xi - \eta(Y)\varphi X + \eta(X)\varphi Y, Z)
= -g(X, \varphi Y)\eta(Z) - g(Y, \varphi Z)\eta(X) - g(Z, \varphi X)\eta(Y).
\]

(6) is easily obtained by (5).  

From Equations (7) and (9), we have:

Proposition 5. Let \((M^3, \varphi, \xi, \eta, g)\) be a three-dimensional Sasakian Lorentzian manifold. Then, we have
\[
\nabla_Z(X \wedge_L Y) = (\nabla_Z X) \wedge_L Y + X \wedge_L (\nabla_Z Y),
\]
(14)
for all \(X, Y, Z \in T M\).
Proof. From Equation (13), we get

\[
\nabla_Z (X \wedge_L Y) = \nabla_Z (-g(X, \varphi Y)\xi + \eta(Y)\varphi X - \eta(X)\varphi Y)
\]

\[
= g(\nabla_Z X, \varphi Y)\xi + g(X, (\nabla_Z \varphi)Y)\xi + g(X, \varphi \nabla_Z Y)\xi + g(X, \varphi Y)\nabla_Z \xi
\]

\[
- \eta(\nabla_Z Y)\varphi X + g(Y, \nabla_Z \xi)\varphi X + \eta(Y)(\nabla_Z \varphi)X + \eta(Y)\varphi \nabla_Z X
\]

\[
+ \eta(\nabla_Z X)\varphi Y - g(X, \nabla_Z \xi)\varphi Y - \eta(X)(\nabla_Z \varphi)Y - \eta(X)\varphi \nabla_Z Y
\]

\[
= (\nabla_Z X) \wedge_L Y + X \wedge_L (\nabla_Z Y) + P(X, Y, Z),
\]

where

\[
P(X, Y, Z) = g(X, (\nabla_Z \varphi)Y)\xi + g(X, \varphi Y)\nabla_Z \xi + g(Y, \nabla_Z \xi)\varphi X - \eta(Y)(\nabla_Z \varphi)X
\]

\[
- g(X, \nabla_Z \xi)\varphi Y + \eta(X)(\nabla_Z \varphi)Y.
\]

Since \( M \) is a three-dimensional Sasakian Lorentzian manifold, it satisfies Equations (7) and (9). Hence, we have

\[
P(X, Y, Z) = g(X, \varphi Y)\varphi Z + g(Y, \varphi Z)\varphi X + g(Z, \varphi X)\varphi Y.
\]

Using Equation (6) of Proposition 4, we obtain \( P(X, Y, Z) = 0 \) and Equation (14). \( \square \)

3.2. Frenet Slant Curves

In this subsection, we study a Frenet slant curve in contact Lorentzian three-manifolds.

A curve in a contact Lorentzian three-manifold is said to be slant if its tangent vector field has constant angle with the Reeb vector field (i.e., \( \eta(\gamma') = -g(\gamma', \xi) \) is a constant).

Since the Reeb vector field \( \xi \) is denoted by

\[
\xi = \sum_{i=1}^{3} \varepsilon_i g(\xi, E_i) E_i = -\sum_{i=1}^{3} \varepsilon_i \eta(E_i) E_i,
\]

using Equation (4) of Proposition 4 and Proposition 3, we have:

Proposition 6. Let \((M^3, \varphi, \xi, \eta, g)\) be a three-dimensional almost contact Lorentzian manifold. Then, for a Frenet curve \( \gamma \) in \( M^3 \), we have

\[
\varphi E_1 = \varepsilon_2 \varepsilon_3 (\eta(E_2)E_3 - \eta(E_3)E_2),
\]

\[
\varphi E_2 = \varepsilon_3 \varepsilon_1 (\eta(E_3)E_1 - \eta(E_1)E_3),
\]

\[
\varphi E_3 = \varepsilon_1 \varepsilon_2 (\eta(E_1)E_2 - \eta(E_2)E_1).
\]

By using Proposition 6, we find that differentiating \( \eta(E_i) \) (for \( i = 1, 2, 3 \)) along a Frenet curve \( \gamma \)

\[
\eta(E_1)' = \varepsilon_2 \kappa \eta(E_2) + g(E_1, \varphi h E_1),
\]

\[
\eta(E_2)' = -\varepsilon_1 \kappa \eta(E_1) - \varepsilon_3 (\tau - 1) \eta(E_3) + g(E_2, \varphi h E_1),
\]

\[
\eta(E_3)' = \varepsilon_2 (\tau - 1) \eta(E_2) + g(E_3, \varphi h E_1).
\]
Now, we assume that $M^3$ is a Sasakian Lorentzian manifold; then,

$$\eta(E_1)' = \varepsilon_2 \kappa \eta(E_2), \quad (15)$$
$$\eta(E_2)' = -\varepsilon_1 \kappa \eta(E_1) - \varepsilon_3 (\tau - 1) \eta(E_3), \quad (16)$$
$$\eta(E_3)' = \varepsilon_2 (\tau - 1) \eta(E_2). \quad (17)$$

From Equation (15), if $\gamma$ is a geodesic curve, that is $\kappa = 0$, in a Sasakian Lorentzian three-manifold $M^3$, then $\gamma$ is naturally a slant curve. Now, let us consider a non-geodesic curve $\gamma$; then, we have:

**Proposition 7.** A non-geodesic Frenet curve $\gamma$ in a Sasakian Lorentzian three-manifold $M^3$ is slant curve if and only if $\eta(E_2) = 0$.

From Equations (15) and (17) and Proposition 7, we get that $\eta(E_1)$ and $\eta(E_3)$ are constants. Hence, using Equation (16), we obtain:

**Theorem 1.** The ratio of $\kappa$ and $\tau - 1$ is a constant along a non-geodesic Frenet slant curve in a Sasakian Lorentzian three-manifold $M^3$.

Next, let us consider a Legendre curve $\gamma$ as a spacelike curve with spacelike normal vector. For a Legendre curve $\gamma$, $\eta(\gamma') = \eta(E_1) = 0, \eta(E_2) = 0$ and $\eta(E_3)$ is a constant. Hence, using Equation (16), we have:

**Corollary 1.** Let $M$ be a three-dimensional Sasakian Lorentzian manifold $(M^3, \eta, \xi, \varphi, g)$. Then, the torsion of a Legendre curve is 1.

From this, we see that the ratio of $\kappa$ and $\tau - 1$ is a constant along non-geodesic Frenet slant curve containing Legendre curve.

### 3.3. Null Slant Curves

In this section, let us consider a null curve $\gamma$ that has a null tangent vector field $g(\gamma', \gamma') = 0$ and $\gamma$ is not a geodesic (i.e., $g(\nabla_{\gamma'} \gamma', \nabla_{\gamma'} \gamma') \neq 0$). We take a parameterization of $\gamma$ such that $g(\nabla_{\gamma'} \gamma', \nabla_{\gamma'} \gamma') = 1$. Then, Duggal, K.L. and Jin, D.H ([12]) proved that there exists only one Cartan frame $\{T, N, W\}$ and the function $\tau$ along $\gamma$ whose Cartan equations are

$$\nabla_{\tau} T = N, \quad \nabla_{\tau} W = \tau N, \quad \nabla_{\tau} N = -\tau T - W,$$

where

$$T = \gamma', \quad N = \nabla_{\tau} T, \quad \tau = \frac{1}{2} g(\nabla_{\tau} N, \nabla_{\tau} N), \quad W = -\nabla_{\tau} N - \tau T. \quad (18)$$

Hence,

$$g(T, W) = g(N, N) = 1, \quad g(T, T) = g(T, N) = g(W, W) = g(W, N) = 0.$$

For a null Legendre curve $\gamma$, we easily prove that $\gamma$ is geodesic. Hence, we suppose that $\gamma$ is non-geodesic; then, we have:
**Theorem 2.** Let $\gamma$ be a non-geodesic null slant curve in a Sasakian Lorentzian three-manifold. We assume that $\kappa = 1$, then we have

$$
N = \pm \frac{1}{a} \varphi \gamma', \quad \tau = \frac{1}{2a^2} \mp 1, \quad W = \frac{1}{2a^2} \gamma' - \frac{1}{a} \xi,
$$

(19)

where $a = \eta(\gamma')$ is non-zero constant.

**Proof.** Let $\varphi T = lT + mN + nW$ for some $l, m, n$. We find $l = g(\varphi T, T) = 0$, then $\varphi T = mN + nW$. From this, we get

$$
g(\varphi T, \varphi T) = m^2 = a^2 \quad \text{and} \quad 0 = g(\varphi T, \xi) = n(\alpha \tau + m).
$$

Hence, $m = \pm a$ and $n = 0$ or $m = -a \tau$.

If $n = 0$, then $N = \frac{1}{a} \varphi T = \pm \frac{1}{2} \varphi T$. Using the Cartan equation, we find that $\tau = \frac{1}{2a^2} \mp 1$ and $W = \frac{1}{2a^2} \gamma' - \frac{1}{a} \xi$.

Next, if $n \neq 0$ and $m = -a \tau$ then since $\gamma$ is a slant curve, differentiating $g(\varphi T, N) = m = \pm a$, we have $n = g(\varphi T, W) = 0$, which gives a contradiction. □

From the second equation of Equation (19), we have:

**Remark 1.** Let $\gamma$ be a non-geodesic null slant curve in a Sasakian Lorentzian three-manifold. We assume that $\kappa = 1$ then $\tau$ is constant such that $\tau = \frac{1}{2a^2} \mp 1$.

### 4. Contact Magnetic Curves

In a three-dimensional Sasakian Lorentzian manifold $M^3$, the Reeb vector field $\xi$ is Killing. By Equation (6), the 2-form $\Phi$ is $d\eta$, that is $d\eta(X, Y) = g(X, \varphi Y)$, for all $X, Y \in \Gamma(TM)$.

Let $\gamma : I \to M$ be a smooth curve on a contact Lorentzian manifold $(M, \varphi, \xi, \eta, g)$. Then, we define a magnetic field on $M$ by

$$
F_{\xi, \varphi}(X, Y) = -q d\eta(X, Y),
$$

where $X, Y \in \mathfrak{X}(M)$ and $q$ is a non-zero constant. We call $F_{\xi, \varphi}$ the contact magnetic field with strength $q$.

Using Equations (1), (4) and (6) we get $\Phi(X) = q \varphi X$. Hence, from Equation (2) the Lorentz equation is

$$
\nabla_{\gamma'} \gamma' = q \varphi \gamma'.
$$

(20)

This is the generalized equation of geodesics under arc length parameterization, that is $\nabla_{\gamma'} \gamma' = 0$.

For $q = 0$, we find that the contact magnetic field vanishes identically and the magnetic curves are geodesics of $M$. The solutions of Equation (20) are called contact magnetic curve or trajectories of $F_{\xi, \varphi}$.

By using Equations (8) and (20), differentiating $g(\xi, \gamma')$ along a contact magnetic curve $\gamma$ in contact Lorentzian three-manifold

$$
\frac{d}{dt} g(\xi, \gamma') = g(\nabla_{\gamma'} \xi, \gamma') + g(\xi, \nabla_{\gamma'} \gamma')
$$

$$
= g(\varphi \gamma' - \varphi h \gamma', \gamma') + g(\xi, q \varphi \gamma')
$$

$$
= -g(\varphi h \gamma', \gamma').
$$

Hence, we have:

**Theorem 3.** Let $\gamma$ be a contact magnetic curve in a contact Lorentzian three-manifold $M$. $\gamma$ is a slant curve if and only if $M$ is Sasakian.
Next, we find the curvature $\kappa$ and torsion $\tau$ along non-geodesic Frenet contact magnetic curves $\gamma$. We suppose that $\eta(E_1) = a$, for a constant $a$. Then, using Equations (4), (10) and (20), we get

$$\varepsilon_2 \kappa^2 = q^2 g(\varphi \gamma', \varphi \gamma') = q^2 (\varepsilon_1 + a^2).$$

Hence, we find that $\gamma$ has a constant curvature

$$\kappa = |q| \sqrt{\varepsilon_2 (\varepsilon_1 + a^2)},$$

and, from Equations (10), (20) and (21), the binormal vector field

$$E_2 = \frac{q}{\varepsilon_2 \kappa} \varphi_\gamma' = -\frac{\delta \varepsilon_2}{\sqrt{\varepsilon_2 (\varepsilon_1 + a^2)}} \varphi_\gamma',$$

where $\delta = q/|q|$.

Using Proposition 3 and Equation (22), the binormal $E_3$ is computed as

$$\varepsilon_3 E_3 = E_1 \wedge L E_2 = \gamma' \wedge L \left( -\frac{\delta \varepsilon_2}{\sqrt{\varepsilon_2 (\varepsilon_1 + a^2)}} \varphi_\gamma' \right) = -\frac{\delta \varepsilon_2}{\sqrt{\varepsilon_2 (\varepsilon_1 + a^2)}} (\varepsilon_1 \xi + a \gamma').$$

Differentiating binormal vector field $E_3$, we have

$$\nabla_{\gamma'} E_3 = -\frac{\delta \varepsilon_2 \varepsilon_3}{\sqrt{\varepsilon_2 (\varepsilon_1 + a^2)}} \nabla_{\gamma'} (\varepsilon_1 \xi + a \gamma') = -\frac{\delta \varepsilon_2 \varepsilon_3}{\sqrt{\varepsilon_2 (\varepsilon_1 + a^2)}} (\varepsilon_1 + qa) \varphi \gamma'.$$

On the other hand, by Equation (10), we have

$$\nabla_{\gamma'} E_3 = \varepsilon_2 \tau E_2 = \tau \frac{\delta \varphi \gamma'}{\sqrt{\varepsilon_2 (\varepsilon_1 + a^2)}}.$$

From Equations (23) and (24), since $\varepsilon_1 \varepsilon_2 \varepsilon_3 = -1$, we obtain

$$\tau = 1 + \varepsilon_1 qa.$$ (25)

Moreover, if $\gamma$ is a non-geodesic curve, then

$$\frac{\tau - 1}{\kappa} = \frac{\delta \varepsilon_1 a}{\sqrt{\varepsilon_2 (\varepsilon_1 + a^2)}}.$$

Therefore, we obtain:

**Theorem 4.** Let $\gamma$ be a non-geodesic Frenet curve in a Sasakian Lorentzian three-manifold $M$. If $\gamma$ is a contact magnetic curve, then it is slant pseudo-helix with curvature $\kappa = |q| \sqrt{\varepsilon_2 (\varepsilon_1 + a^2)}$ and torsion $\tau = 1 + \varepsilon_1 qa$. Moreover, the ratio of $\kappa$ and $\tau - 1$ is a constant.
Since a Legendre curve is a spacelike curve with spacelike normal vector field and $\eta(\gamma') = a = 0$, we assume that $\gamma$ is a Legendre curve and we have:

**Corollary 2.** Let $\gamma$ be a non-geodesic Legendre curve in a Sasakian Lorentzian three-manifold $M$. If $\gamma$ is a contact magnetic curve, then it is Legendre pseudo-helix with curvature $\kappa = |q|$ and torsion $\tau = 1$.

Now, from the geodesic curvature in Equation (21), if $\varepsilon_1 = 1$, then $\eta(\gamma') = a$ and $1 \leq 1 + a^2$, and we have $\varepsilon_2 = 1$. Moreover, using $\varepsilon_3 = -\varepsilon_1 \cdot \varepsilon_2$, we obtain $\varepsilon_3 = -1$. Next, if $\varepsilon_1 = -1$, then $\eta(\gamma') = a = \cosh \alpha_0$. Since $\gamma$ is a geodesic for $a = \cosh \alpha_0 = 1$, we assume that $\gamma$ is non-geodesic, and we get $a^2 > 1$. Hence, $-1 + a^2 > 0$ and we get $\varepsilon_2 = \varepsilon_3 = 1$. Therefore, we obtain:

**Theorem 5.** Let $\gamma$ be a non-geodesic Frenet curve in a Sasakian Lorentzian three-manifold $M$. If $\gamma$ is a contact magnetic curve, then $\gamma$ is one of the following:

(i) a spacelike curve with spacelike normal vector field; or
(ii) a timelike curve.

Moreover, we have:

**Corollary 3.** Let $\gamma$ be a non-geodesic Frenet curve in a Sasakian Lorentzian three-manifold $M$. If $\gamma$ is a contact magnetic curve, then there does not exist a spacelike curve with timelike normal vector field.

In a similar with a Frenet curve, we study null contact magnetic curves in a Sasakian Lorentzian three-manifold $M$. Hence, we find that there exist a null contact magnetic curve with $q = \pm a$ and same the result with Theorem 2.

**Example**

The Heisenberg group $\mathbb{H}_3$ is a Lie group which is diffeomorphic to $\mathbb{R}^3$ and the group operation is defined by

\[(x, y, z) \ast (x', y', z') = (x + x', y + y', z + z' + \frac{xy}{2} - \frac{x'y}{2}).\]

The mapping

\[
\mathbb{H}_3 \rightarrow \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\} : (x, y, z) \mapsto \begin{pmatrix} 1 & x & z + \frac{xy}{2} \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}
\]

is an isomorphism between $\mathbb{H}_3$ and a subgroup of $GL(3, \mathbb{R})$.

Now, we take the contact form $\eta = dz + (ydx - xdy)$.

Then, the characteristic vector field of $\eta$ is $\xi = \frac{\partial}{\partial z}$.

Now, we equip the Lorentzian metric as following:

\[g = dx^2 + dy^2 - (dz + (ydx - xdy))^2.\]
We take a left-invariant Lorentzian orthonormal frame field \((e_1, e_2, e_3)\) on \((\mathbb{H}_3, g)\):

\[
e_1 = \frac{\partial}{\partial x} - y \frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y} + x \frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial z},
\]

and the commutative relations are derived as follows:

\[
[e_1, e_2] = 2e_3, \quad [e_2, e_3] = [e_3, e_1] = 0.
\]

Then, the endomorphism field \(\varphi\) is defined by

\[
\varphi e_1 = e_2, \quad \varphi e_2 = -e_1, \quad \varphi e_3 = 0.
\]

The Levi–Civita connection \(\nabla\) of \((\mathbb{H}_3, g)\) is described as

\[
\nabla_{e_1} e_1 = \nabla_{e_2} e_2 = \nabla_{e_3} e_3 = 0, \quad \nabla_{e_1} e_2 = e_3 = -\nabla_{e_3} e_1, \quad \nabla_{e_2} e_3 = -e_1 = \nabla_{e_1} e_2, \quad \nabla_{e_3} e_1 = e_2 = \nabla_{e_2} e_3.
\] (26)

The contact form \(\eta\) satisfies \(d\eta(X, Y) = g(X, \varphi Y)\). Moreover, the structure \((\eta, \xi, \varphi, g)\) is Sasakian. The Riemannian curvature tensor \(R\) of \((\mathbb{H}_3, g)\) is given by

\[
R(e_1, e_2)e_1 = 3e_2, \quad R(e_1, e_2)e_2 = -3e_1, \\
R(e_2, e_3)e_2 = -e_3, \quad R(e_2, e_3)e_3 = -e_2, \\
R(e_3, e_1)e_3 = e_1, \quad R(e_3, e_1)e_1 = e_3,
\]

and the other components are zero.

The sectional curvature is given by [6]

\[
K(\xi, e_i) = -R(\xi, e_i, \xi, e_i) = -1, \text{ for } i = 1, 2,
\]

and

\[
K(e_1, e_2) = R(e_1, e_2, e_1, e_2) = 3.
\]

Thus, we see that the Lorentzian Heisenberg space \((\mathbb{H}_3, g)\) is the Lorentzian Sasakian space forms with constant holomorphic sectional curvature \(\mu = 3\).

Let \(\gamma\) be a Frenet slant curve in Lorentzian Heisenberg space \((\mathbb{H}_3, g)\) parameterized by arc-length. Then, the tangent vector field has the form

\[
T = \gamma' = \sqrt{\varepsilon_1 + a^2} \cos \beta e_1 + \sqrt{\varepsilon_1 + a^2} \sin \beta e_2 + a e_3,
\] (27)

where \(a = constant, \ \beta = \beta(s)\). Using Equation (26), we get

\[
\nabla_{\gamma'} \gamma' = \sqrt{\varepsilon_1 + a^2} (\beta' + 2a) (-\sin \beta e_1 + \cos \beta e_2).
\] (28)

Since \(\gamma\) is a non-geodesic, we may assume that \(\kappa = \sqrt{\varepsilon_1 + a^2} (\beta' + 2a) > 0\) without loss of generality. Then, the normal vector field

\[
N = -\sin \beta e_1 + \cos \beta e_2.
\]
The binormal vector field $\varepsilon \beta B = T \wedge L N = -a \cos \beta e_1 - a \sin \beta e_2 - \sqrt{\varepsilon_1 + a^2 \varepsilon_3}$. From Theorem 5, we see that $\varepsilon_2 = 1$, thus we have $\varepsilon_3 = -\varepsilon_1$. Hence,

$$B = \varepsilon_1 (a \cos \beta e_1 + a \sin \beta e_2 + \sqrt{\varepsilon_1 + a^2 \varepsilon_3}).$$

Using the Frenet–Serret Equation (10), we have

**Lemma 1.** Let $\gamma$ be a Frenet slant curve in Lorentzian Heisenberg space $(\mathbb{H}_3, g)$ parameterized by arc-length. Then, $\gamma$ admits an orthonormal frame field $\{T, N, B\}$ along $\gamma$ and

$$\kappa = \sqrt{\varepsilon_1 + a^2 (\beta' + 2a)},$$
$$\tau = 1 + \varepsilon_1 a (\beta' + 2a).$$

Next, if $\gamma$ is a null slant curve in the Lorentzian Heisenberg space $(\mathbb{H}_3, g)$, then the tangent vector field has the form

$$T = \gamma' = a \cos \beta e_1 + a \sin \beta e_2 + ae_3,$$

where $a = \text{constant}$, $\beta = \beta(s)$. Using Equation (26), we get

$$\nabla_{\gamma'} \beta' = a (\beta' + 2a) (-\sin \beta e_1 + \cos \beta e_2).$$

Since $\gamma$ is non-geodesic, using Equation (18) we have $|a(\beta' + 2a)| = 1$ and

$$N = -\sin \beta e_1 + \cos \beta e_2.$$

Differentiating $N$, we get

$$\nabla_{\gamma'} N = -(\beta' + a) \cos \beta e_1 - (\beta' + a) \sin \beta e_2 + ae_3.$$

From Equation (18), $\tau = \frac{1}{2a} (\nabla_{\gamma'} N, \nabla_{\gamma'} N) = \frac{1}{2} (\beta')^2 + a \beta'$. Since $W = -\nabla_{\gamma'} N - \tau T$, we have

$$W = \{-\frac{1}{2} (\beta')^2 + (\frac{1}{a} - a) \beta' + 1\} T - (\beta' + 2a) \xi = \frac{1}{2a} (\cos \beta e_1 + \sin \beta e_2 - e_3).$$

Therefore, we have

**Lemma 2.** Let $\gamma$ be a non-geodesic null slant curve in the Lorentzian Heisenberg space $(\mathbb{H}_3, g)$. We assume that $\kappa = |a(\beta' + 2a)| = 1$. Then, its torsion is constant such that $\tau = 1 - \frac{1}{2a} + 1$.

Let $\gamma(s) = (x(s), y(s), z(s))$ be a curve in Lorentzian Heisenberg space $(\mathbb{H}_3, g)$. Then, the tangent vector field $\gamma'$ of $\gamma$ is

$$\gamma' = \left( \frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds} \right) = \frac{dx}{ds} \frac{\partial}{\partial x} + \frac{dy}{ds} \frac{\partial}{\partial y} + \frac{dz}{ds} \frac{\partial}{\partial z}.$$

Using the relations:

$$\frac{\partial}{\partial x} = e_1 + ye_3, \quad \frac{\partial}{\partial y} = e_2 - xe_3, \quad \frac{\partial}{\partial z} = e_3.$$
If $\gamma$ is a slant curve in $(\mathbb{H}_3, g)$, then from Equation (27) the system of differential equations for $\gamma$ is given by
\[
\frac{dx}{ds}(s) = \sqrt{\epsilon_1 + a^2} \cos \beta(s), \\
\frac{dy}{ds}(s) = \sqrt{\epsilon_1 + a^2} \sin \beta(s), \\
\frac{dz}{ds}(s) = a + \sqrt{\epsilon_1 + a^2} (x(s) \sin \beta(s) - y(s) \cos \beta(s)).
\]

Now, we construct a magnetic curve $\gamma$ (containing Frenet and null curve) in the Lorentzian Heisenberg space $(\mathbb{H}_3, g)$. From Equations (20) and (28), we have:

**Proposition 8.** Let $\gamma : I \rightarrow (\mathbb{H}_3, g)$ be a magnetic curve parameterized by arc-length in the Lorentzian Heisenberg space $(\mathbb{H}_3, g)$. Then,
\[
\beta' = q - 2a, \quad \text{for } a = \eta(\gamma').
\]
Namely, $\beta'$ is a constant, e.g., $A$, hence $\beta(s) = As + b$, $b \in \mathbb{R}$. If $\gamma$ is a null curve, then $q = \pm \frac{1}{A}$. Finally, from Equations (32) and (33), we have the following result:

**Theorem 6.** Let $\gamma : I \rightarrow (\mathbb{H}_3, g)$ be a non-geodesic curve parameterized by arc-length $s$ in the Lorentzian Heisenberg group $(\mathbb{H}_3, g)$. If $\gamma$ is a contact magnetic curve, then the parametric equations of $\gamma$ are given by
\[
\begin{align*}
    x(s) &= \frac{1}{A} \sqrt{\epsilon_1 + a^2} \sin(As + b) + x_0, \\
    y(s) &= -\frac{1}{A} \sqrt{\epsilon_1 + a^2} \cos(As + b) + y_0, \\
    z(s) &= \left\{ a + \frac{\epsilon_1 + a^2}{A} \right\} s - \frac{\sqrt{\epsilon_1 + a^2}}{A} \left\{ x_0 \cos(As + b) + y_0 \sin(As + b) \right\} + z_0,
\end{align*}
\]
where $b, x_0, y_0, z_0$ are constants. If $\epsilon_1 = 0$ then $\gamma$ is a null curve.

In particular, for a Frenet Legendre curve $\gamma$, we get $\beta' = q = A$.

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**References**

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