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Symmetry Analysis and Conservation Laws of a Generalization of the Kelvin-Voigt Viscoelasticity Equation

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Abstract: In this paper, we study a generalization of the well-known Kelvin-Voigt viscoelasticity equation describing the mechanical behaviour of viscoelasticity. We perform a Lie symmetry analysis. Hence, we obtain the Lie point symmetries of the equation, allowing us to transform the partial differential equation into an ordinary differential equation by using the symmetry reductions. Furthermore, we determine the conservation laws of this equation by applying the multiplier method.

Keywords: viscoelasticity; Kelvin-Voigt equation; Lie symmetries; optimal system; group-invariant solutions; conservation laws; multiplier method

1. Introduction

The continuous development of mechanics and its engineering applications have increased remarkably the interest in non-linear phenomena, such as viscoelasticity. Viscoelastic materials are of interest in a wide variety of applications, from passive damping to aircraft tire construction. A good modelling of the material's behaviour is essential for the accurate design incorporating this material.

Viscoelastic behaviour appears in materials showing some sort of liquid-like elastic behaviour. However, a simple Hooke's law linear elastic constitutive relationship is not an accurate representation of viscoelastic material's behaviour. Viscoelastic materials are commonly said to have "memory" because of their rheological properties.

The rheological models, such as the Kelvin-Voigt model or the Maxwell model, are usually used to describe the viscoelastic behaviour. The Kelvin-Voigt model consists of a lumped parameter model similar to a spring and dashpot in parallel, while the Maxwell model describes a serially connected spring and dashpot. In addition, many papers have been published studying these models [1–4].

Many physical phenomena, as viscoelasticity, are described by non-linear partial differential equations (PDEs). In particular, the Kelvin-Voigt viscoelasticity equation is given by

$$u_{tt} - (C(x) u_x)_x - (B(x) u_{tx})_x = 0.$$

Nevertheless, in this paper we focus on a generalization of the Kelvin-Voigt viscoelasticity equation described by

$$u_{tt} - (C(x) f(u)_x)_x - (B(x) u_{tx})_x = 0, \quad (t,x) \in \mathbb{R} \times \Omega, \tag{1}$$

where Ω is an open subset of $\mathbb{R}$ and u a scalar real-valued function. Also, $f(u)$ is a smooth enough non-linear function, and $C(x) \neq 0$, $B(x) \neq 0$ are smooth enough functions too, depending on the variable $x \in \Omega$. Throughout the paper the subscripts denote partial derivatives.

There is no general theory for solving non-linear PDEs. Therefore, in this work, we use Lie theory to analyse Equation (1). Lie group analysis is a powerful tool to find general solutions for PDEs. This theory, originally defined by Sophus Lie at the end of the nineteenth century, develops solutions for PDEs by the transformation groups of Lie [5–8]. The fundamental basis of the Lie group method is that if a differential equation is invariant under a Lie group of transformations, then a reduction transformation exists. For instance, for PDEs with two independent variables like Equation (1), a single group reduction can transform the PDE into an ordinary differential equation (ODE), easier to solve.

Furthermore, a very important concept in the analysis of PDEs is the notion of conservation law. Conservation laws determine conserved quantities and constants of motion. They also detect integrability and check accuracy of numerical solutions method. Recently, Anco and Bluman [9,10] developed a method that does not need the existence of Lagrangians because it is based on adjoint equations for non-linear equations and avoids the integrals of functions. This method called the multiplier method allows finding all local conservation laws admitted by any evolution equation. Many papers have been published in the last few years using this method [11–19].

The paper is organized as follows: In Section 2 we determine the Lie point symmetries of Equation (1). Then, in Section 3 we use the Lie point symmetries admitted by Equation (1) to obtain an optimal system of one-dimensional subalgebras. Afterwards, in Section 4 we find symmetry reductions for the one-dimensional subalgebras calculated previously. These reductions allow us to transform Equation (1) into an ODE. In Section 5 we derive the conservation laws of Equation (1) by applying the multiplier method. Finally, in Section 6 some conclusions are presented.

2. Lie Point Symmetries

A one-parameter group of infinitesimal transformations in (x, t, u) is given by

$$
\begin{aligned}
x^* &= x + \varepsilon\,\xi(x, t, u) + \mathcal{O}(\varepsilon^2), \\
t^* &= t + \varepsilon\,\tau(x, t, u) + \mathcal{O}(\varepsilon^2), \\
u^* &= u + \varepsilon\,\phi(x, t, u) + \mathcal{O}(\varepsilon^2),
\end{aligned}
$$

where ε is the group parameter and $\xi(x, t, u)$, $\tau(x, t, u)$, and $\phi(x, t, u)$ are the infinitesimals.

Definition 1. *A vector field*

$$
X = \xi(x, t, u)\frac{\partial}{\partial x} + \tau(x, t, u)\frac{\partial}{\partial t} + \phi(x, t, u)\frac{\partial}{\partial u}, \tag{2}
$$

where $\xi(x, t, u)$, $\tau(x, t, u)$, and $\phi(x, t, u)$ are the infinitesimals, is a generator of a Lie point symmetry of Equation (1) if

$$
X^{(3)}\left(u_{tt} - (C(x)\,f(u)_x)_x - (B(x)\,u_{tx})_x\right) = 0, \tag{3}
$$

where $X^{(3)}$ is the third prolongation of the vector field (2) defined by

$$
X^{(3)} = X + \zeta_x\frac{\partial}{\partial u_x} + \zeta_t\frac{\partial}{\partial u_t} + \zeta_{xx}\frac{\partial}{\partial u_{xx}} + \zeta_{xt}\frac{\partial}{\partial u_{xt}} + \zeta_{tt}\frac{\partial}{\partial u_{tt}} + \zeta_{xxt}\frac{\partial}{\partial u_{xxt}},
$$

where the coefficients $\zeta_x, \zeta_t, \zeta_{xx}, \zeta_{xt}, \zeta_{tt}, \zeta_{xxt}$ are given by

$$
\begin{aligned}
\zeta_x &= D_x\phi - u_t D_x\tau - u_x D_x\xi, \\
\zeta_t &= D_t\phi - u_t D_t\tau - u_x D_t\xi, \\
\zeta_{xx} &= D_x(\zeta_x) - u_{xt} D_x\tau - u_{xx} D_x\xi, \\
\zeta_{xt} &= D_t(\zeta_x) - u_{xt} D_x\tau - u_{xx} D_t\xi, \\
\zeta_{tt} &= D_t(\zeta_t) - u_{tt} D_t\tau - u_{xt} D_t\xi, \\
\zeta_{xxt} &= D_x(\zeta_{xt}) - u_{ttx} D_x\tau - u_{xxt} D_x\xi.
\end{aligned}
$$

Here D_i stands for the total derivative operator.

Theorem 1. *The Lie point symmetries of the generalization of the Kelvin-Voigt Equation (1), with $f(u)$ non-linear function, and $C(x) \neq 0$, $B(x) \neq 0$ arbitrary functions, are generated by the operator*

$$
X_1 = \partial_t.
$$

For some particular functions of $f(u)$, $C(x)$, $B(x)$, there are additional generators given below.

1. *If $f(u)$ is an arbitrary function, $C(x) = c_1$ and $B(x) = b_1$, with c_1, b_1 arbitrary constants,*

$$
X_2^1 = \partial_x.
$$

2. *If $f(u)$ is an arbitrary function, $C(x) = c_1$ and $B(x) = b_1 x + b_2$, with $c_1, b_1 \neq 0$, b_2 arbitrary constants,*

$$
X_2^2 = (b_1 x + b_2)\partial_x + b_1 t\,\partial_t.
$$

3. *If $f(u)$ is an arbitrary function, $C(x) = \frac{4}{(nx+c_1)^2}$ and $B(x) = b_1$, with c_1, b_1 arbitrary constants, and n a positive integer,*

$$
X_2^3 = (nx + c_1)\partial_x + 2nt\,\partial_t.
$$

4. *If $f(u)$ is an arbitrary function, $C(x) = c_2(c_1 - x)^n$ and $B(x) = \frac{(-c_1+x)c_1\sqrt{(c_1-x)^n c_2}}{n}$, with c_1, c_2 arbitrary constants, and n a positive integer,*

$$
X_2^4 = (c_1 - x)\partial_x + \frac{1}{2}(n-2)t\,\partial_t.
$$

5. *If $f(u) = \dfrac{f_0^2 e^{\frac{-ke^k}{e^k-1}}\left(\frac{(e^k-1)u}{f_0} + \frac{f_1 e^k - f_1}{f_0}\right)^{\frac{e^k}{e^k-1}+1} + 2f_2 e^k - f_2}{2e^k - 1}$, $C(x) = c_1$ and $B(x) = b_1$, with k a positive integer, f_0, f_1, f_2 positive constants, and c_1, b_1 arbitrary constants, besides $X_2^5 = X_2^1$,*

$$
X_3^5 = x\partial_x + 2t\,\partial_t + 2(-1 + e^{-k})(u + f_1)\partial_u.
$$

6. *If $f(u) = \dfrac{f_1^2 e^{\frac{-f_0 e^{f_0}}{e^{f_0}-1}}\left(\frac{(e^{f_0}-1)u}{f_1} + \frac{f_2 e^{f_0} - f_2}{f_1}\right)^{\frac{e^{f_0}}{e^{f_0}-1}+1}}{(e^{f_0}-1)\left(\frac{e^{f_0}}{e^{f_0}-1}+1\right)} + f_3$, $C(x) = c_2(c_1 - x)^n$ and $B(x) = b_1$, with f_0, f_1, f_2, f_3 positive constants, c_1, c_2, b_1 arbitrary constants, and n a positive integer,*

$$
X_2^6 = (c_1 - x)\partial_x - 2t\,\partial_t - (1 + e^{-f_0})(u + f_2)(n+2)\partial_u.
$$

7. If $f(u) = \dfrac{f_0 e^{\frac{-ke^k}{e^k-1}}\left(\frac{(e^k-1)(u+f_1)}{f_0}\right)^{\frac{2e^k-1}{e^k-1}}+2f_2e^k-f_2}{2e^k-1}$, $C(x) = c_2(b_2-x)^{-m}$ and $B(x) = b_3(b_2-x)^n$, with k a positive integer, f_0, f_1, f_2 positive constants, b_2, b_3 arbitrary constants, and n, m positive integers,

$$X_2^7 = (b_2-x)\partial_x + t(n-2)\partial_t + (-1+e^{-k})(u+f_1)(m+2n-2)\partial_u.$$

Proof of Theorem 1. Expanding (3), we obtain an overdetermined system satisfying the determining equations for the symmetry group. From $B^2\,(\tau_u) = 0$ and $B^2\,(\tau_x) = 0$, we find that τ depends only on t. The equations $B^2\,(\xi_t) = 0$ and $B^2\,(\xi_u) = 0$ reveal that ξ is a function of x alone. The remaining determining equations are

$$
\begin{aligned}
B^2\,(\xi_{uu}) = 0,\ B^2\,(\xi_{tu}) = 0,\ B^2\,(\tau_{uu}) = 0,\ B\,(\tau_{uu}) = 0, B^2\,(\tau_{uuu}) = 0,\ B^2\,(\xi_{uuu}) = 0, &= 0, \\
B\,(2\,(\tau_{ux})\,B - \xi_u) &= 0, \\
B\,(2\,(\tau_{uux})\,B - \xi_{uu}) &= 0, \\
B\,((\tau_{ux})\,B + \xi_u) &= 0, \\
B\,((\tau_{uxx})\,B + \phi_{uu} - 2\,(\tau_{tu})) &= 0, \\
B\,((\xi_u)\,(f_{uu})C - (\xi_{uu})\,(f_u)C - (\xi_{tuu})\,B) &= 0, \\
B\,(f_u\,(\phi_{xx})\,C + (\phi_{txx})\,B - \phi_{tt}) &= 0, \\
B\,((\tau_u)\,(f_u)C + (\phi_{uu})\,B - (\tau_{tu})\,B - 2\,(\xi_{ux})\,B) &= 0, \\
B\,((\tau_u)\,(f_u)C - (\phi_{uu})\,B + (\tau_{tu})\,B + 2\,(\xi_{ux})\,B) &= 0, \\
B\,((\tau_{uu})\,(f_u)C - (\phi_{uuu})\,B + (\tau_{tuu})\,B + 2\,(\xi_{uux})\,B) &= 0, \\
\xi\,(B') - (\tau_{xx})\,B^2 + (\tau_t)\,B - 2\,(\xi_x B) &= 0, \\
B\,(2\,(\tau_x)\,(f_u)C - 2\,(\phi_{ux})\,B + 2\,(\tau_{tx})\,B + (\xi_{xx})\,B - 2\,(\xi_t)) &= 0, \\
B\,((\tau_{xx})\,(f_u)C - (\phi_{uxx})\,B + (\tau_{txx})\,B + 2\,(\phi_{tu}) - \tau_{tt}) &= 0, \\
\xi(f_u)B\,(C') - \xi f_u\,(B')\,C + ((f_{uu}))\,\phi BC + (\tau_t)\,(f_u)BC + (\phi_{tu})\,B^2 - 2\,(\xi_{tx})\,B^2 &= 0, \\
B\,(2\,(\tau_x)\,(f_{uu})C + 2\,(\tau_{ux})\,(f_u)C - 2\,(\phi_{uux})\,B + 2\,(\tau_{tux})\,B + (\xi_{uxx})\,B - 2\,(\xi_{tu})) &= 0, \\
B\,(2(f_{uu})\,(\phi_x)\,C + 2f_u\,(\phi_{ux})\,C - (\xi_{xx})\,(f_u)C + 2\,(\phi_{tux})\,B - (\xi_{txx})\,B + \xi_{tt}) &= 0, \\
\xi(f_{uu})B\,(C') - \xi\,(f_{uu})\,(B')\,C + f_u\,(\phi_{uu})\,BC + (f_{uu})\,(\phi_u)\,BC + ((f_{uuu}))\,\phi BC + (\tau_t)\,(f_{uu})BC \\
-2\,(\xi_{ux})\,(f_u)BC + (\phi_{tuu})\,B^2 - 2\,(\xi_{tux})\,B^2 &= 0.
\end{aligned}
$$

Solving this system of equations we find the infinitesimals ξ, τ and ϕ of (2). $\quad\square$

3. Optimal Systems

It is important to classify invariant solutions according to the classification of the associated symmetry generators. Then, one generator from each class is used to determine the desired set of invariant solutions. An optimal system of generators is defined as a set consisting of exactly one generator from each class [20].

The problem of obtaining an optimal system of subgroups is equivalent to that of obtaining an optimal system of subalgebras, and so we concentrate on the latter. For one-dimensional subalgebras, this classification problem is essentially equal to classifying the orbits of the adjoint representation [5].

The most important operator on vector fields is their Lie bracket or commutator. If X_i and X_j are vector fields, then their Lie bracket $[X_i,\ X_j]$ is the unique vector field satisfying

$$[X_i,\ X_j] = X_i(X_j) - X_j(X_i).$$

The commutator table for the Lie algebra of Case 5 of Theorem (1) is shown in Table 1. The (i, j)-th entry of the table expresses the Lie bracket $[X_i,\ X_j]$, for $i, j = 1, 2, 3$.

Table 1. The commutator table for Case 5 of Theorem (1).

$[X_i,\ X_j]$	X_1	X_2^5	X_3^5
X_1	0	0	$2X_1$
X_2^5	0	0	X_2^5
X_3^5	$-2X_1$	$-X_2^5$	0

The adjoint representation can be constructed by summing the Lie series

$$
\begin{aligned}
Ad(\exp(\epsilon\ X_i))X_j &= \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!}(ad\ X_i)^n(X_j), \\
&= X_j - \epsilon[X_i,\ X_j] + \frac{\epsilon^2}{2}[X_i,\ [X_i,\ X_j]] - \cdots .
\end{aligned}
\tag{4}
$$

To compute the adjoint representation, we use the Lie series (4) in conjunction with the commutator table in Table 1. The adjoint table of this Lie algebra is shown in Table 2, with the (i, j)-th entry indicating $Ad(\exp(\epsilon\ X_i))X_j$.

Table 2. The adjoint table for Case 5 of Theorem (1).

$Ad(\exp(\epsilon\ X_i))X_j$	X_1	X_2^5	X_3^5
X_1	X_1	X_2^5	$X_3^5 - 2\epsilon X_1$
X_2^5	X_1	X_2^5	$X_3^5 - \epsilon X_2^5$
X_3^5	$e^{2\epsilon} X_1$	$e^{\epsilon} X_2^5$	X_3^5

Theorem 2. *A one-dimensional optimal system for the generalization of the Kelvin-Voigt Equation (1) is given by*

$$
\lambda X_1 + \mu X_2^1, \qquad X_2^2, \qquad X_2^3, \qquad X_2^4, \qquad X_3^5, \qquad X_2^6, \qquad X_2^7,
$$

where λ and μ are arbitrary constants.

Proof of Theorem 2. Let $\mathcal{G}$ be the symmetry algebra of Equation (1), with the adjoint representation for Case 5 in Theorem (1) determined in Table 2. Let $v = a_1 v_1 + a_2 v_2 + a_3 v_3$ be a non-zero vector field of $\mathcal{G}$. For each case, we simplify the coefficients a_i, $i = 1, 2, 3$, as much as possible through proper adjoints applications on v. $\square$

4. Symmetry Reductions

In this section, we use the optimal system of one-dimensional subalgebras of Theorem (2) to determine the symmetry reductions of Equation (1).

The symmetry variables are found by solving the invariant surface condition

$$
\Phi \equiv \xi \partial_x + \tau \partial_t - \phi = 0.
$$

Reduction 1. From $\lambda X_1 + \mu X_2^1$, we obtain the travelling wave reduction

$$
z = \mu x - \lambda t, \qquad u = h(z),
$$

where $h(z)$ satisfies

$$
\lambda\,\mu^2\,b_1\,h''' - \mu^2\,c_1\,f'\,h'' + \lambda^2\,h'' - \mu^2\,c_1\,f''\,(h')^2 = 0.
$$

Reduction 2. From X_2^2, we obtain the invariant solution

$$
z = \frac{b_1 x + b_2}{t}, \qquad u = h(z),
$$

where $h(z)$ satisfies
$$b_1^2 h''' z^2 + h'' z^2 + 3 b_1^2 h'' z + 2 h' z + b_1^2 h' = 0.$$

Reduction 3. From X_2^3, we obtain the invariant solution
$$z = \frac{nx + c_1}{\sqrt{t}}, \qquad u = h(z),$$

where $h(z)$ satisfies
$$2 b_1 h''' n^2 z + h'' z^2 + 4 b_1 h'' n^2 + 3 h' z = 0.$$

Reduction 4. From X_2^4, we obtain the invariant solution
$$z = -t^{\frac{2}{n-2}} (c_1 - x), \qquad u = h(z),$$

where $h(z)$ satisfies

$$
\begin{aligned}
& 2 c_1 \sqrt{c_2}\, h''' n\, (-z)^{\frac{n}{2}} z^2 - 4 c_1 \sqrt{c_2}\, h''' (-z)^{\frac{n}{2}} z^2 - 4 h'' n z^2 + c_1 \sqrt{c_2}\, h'' n^2 (-z)^{\frac{n}{2}} z \\
+ \ & 4 c_1 \sqrt{c_2}\, h'' n\, (-z)^{\frac{n}{2}} z - 12 c_1 \sqrt{c_2}\, h'' (-z)^{\frac{n}{2}} z + 2 h' n^2 z - 8 h' n z + c_1 \sqrt{c_2}\, h' n^2 (-z)^{\frac{n}{2}} \\
- \ & 4 c_1 \sqrt{c_2}\, h' (-z)^{\frac{n}{2}} = 0.
\end{aligned}
$$

Reduction 5. From X_3^5, we obtain the invariant solution
$$z = \frac{x}{\sqrt{t}}, \qquad u = \frac{t^{-1+e^{-k}}}{h(z)} - f_1,$$

where $h(z)$ must satisfy a non-autonomous equation.

Reduction 6. From X_2^6, we obtain the invariant solution
$$z = -(c_1 - x)\sqrt{t}, \qquad u = \frac{t^{\frac{(-1+e^{-f_0})(n+2)}{2}}}{h(z)} - f_2,$$

where $h(z)$ must satisfy a non-autonomous equation.

Reduction 7. From X_2^7, we obtain the invariant solution
$$z = -\frac{(b_2 - x)}{t^{\frac{1}{n-2}}}, \qquad u = \frac{t^{\frac{-(-1+e^{-k})(m+2n-2)}{n-2}}}{h(z)} - f_1,$$

where $h(z)$ must satisfy a non-autonomous equation.

The expressions of the reduced equations for X_3^5, X_2^6, and X_2^7 are omitted here to save space.

5. Conservation Laws

A local conservation law for the generalization of the Kelvin-Voigt viscoelasticity Equation (1) is a continuity equation
$$D_t T + D_x X = 0,$$

holding for all solutions of Equation (1), where the conserved density T and the spatial flux X are functions of x, t, u, and derivatives of u. Here D_t and D_x denote total derivatives with respect to t and x, respectively. The pair (T, X) is called a conserved current.

Two local conservation laws are considered to be locally equivalent [5,21] if they differ by a locally trivial conservation law $T = D_x\Theta$, $X = -D_t\Theta$, where T and X are evaluated on the set of solutions of Equation (1) and Θ is a function of x, t, u, and derivatives of u.

A non-trivial conservation law can be written in a general form as

$$\frac{d}{dt}\int_\Omega T\,dx = -X\Big|_{\partial\Omega},$$

where $\Omega \subseteq \mathbb{R}$ is any fixed spatial domain.

Any local conservation law can be stated by using the characteristic form arising from a divergence identity

$$D_t\tilde{T} + D_x\tilde{X} = (u_{tt} - (C(x)\,f(u)_x)_x - (B(x)\,u_{tx})_x)Q, \tag{5}$$

where $\tilde{T} = T + D_x\Theta$ and $\tilde{X} = X - D_t\Theta$ are locally equivalent to T and X. The function Q is called a multiplier. It satisfies

$$Q = E_u(\tilde{T}),$$

where E_u represents the Euler operator with respect to u [5], that is

$$E_u = \partial_u - D_x\partial_{u_x} - D_t\partial_{u_t} + D_xD_t\partial_{u_{xt}} + D_x^2\partial_{u_{xx}} + \cdots .$$

For evolution equations, there is a one-to-one relationship between non-zero multipliers and non-trivial conserved current vectors up to local equivalence [5,9]. In general, a function $Q(x, t, u, u_t, u_x, \dots)$ is a multiplier if it verifies that $(u_{tt} - (C(x)\,f(u)_x)_x - (B(x)\,u_{tx})_x)\,Q$ is a divergence expression for all function $u(x, t)$. Given a multiplier Q, the conserved density can be determined by

$$T = \int_0^1 u\,Q(x, t, \lambda u, \lambda u_x, \lambda u_{xx}, \dots)d\lambda.$$

The divergence condition yields to the determining equation

$$E_u\Big((u_{tt} - (C(x)\,f(u)_x)_x - (B(x)\,u_{tx})_x)\,Q \Big) = 0. \tag{6}$$

In order to give a complete classification of multipliers, we write and split the determining Equation (6) with respect to the variables $u_{tt}, u_{ttt}, u_{ttx}, u_{txx}, u_{xxx}, u_{tttx}, u_{ttxx}, u_{txxx}, u_{xxxx}$. Thus, we get a linear determining system for $Q(x, t, u, u_t, u_x, \dots)$. The multipliers are found by solving the system with the same algorithmic method used for the determining equations for infinitesimal symmetries. Then, integrating the characteristic Equation (5) for each multiplier, we find the conserved current.

Theorem 3. *The multipliers admitted by the generalization of the Kelvin-Voigt Equation (1), with $f(u)$ a smooth enough non-linear function, and $C(x) \neq 0$, $B(x) \neq 0$ smooth enough arbitrary functions, are given by*

$$Q_1 = 1, \qquad Q_2 = t, \qquad Q_3 = \int \frac{1}{C(x)}dx.$$

Theorem 4. *All non-trivial local conservation laws admitted by the generalization of the Kelvin-Voigt Equation (1), with $f(u)$ a smooth enough non-linear function, and $C(x) \neq 0$, $B(x) \neq 0$ smooth enough arbitrary functions, are given by*

1. *For the multiplier $Q_1 = 1$, the conserved density and the spatial flux are*

$$\begin{aligned} T_1 &= u_t, \\ X_1 &= -B(x)\,u_{tx} - C(x)\,f(u)_x. \end{aligned}$$

2. *For the multiplier $Q_2 = t$, the conserved density and the spatial flux are*

$$\begin{aligned} T_2 &= t\,u_t - u, \\ X_2 &= -t\,B(x)\,u_{tx} - t\,C(x)\,f(u)_x. \end{aligned}$$

3. For the multiplier $Q_3 = \int \frac{1}{C(x)} dx$, the conserved density and the spatial flux are

$$
\begin{aligned}
T_3 &= \left(\int \frac{1}{C(x)} dx \right) u_t + \left(\frac{B(x)\, C'(x)}{C(x)^2} - \frac{B'(x)}{C(x)} \right) u, \\
X_3 &= -\left(\int \frac{1}{C(x)} dx \right) B(x)\, u_{tx} - \left(\int \frac{1}{C(x)} dx \right) C(x)\, f(u)_x + \frac{B(x)\, u_t}{C(x)} + f(u).
\end{aligned}
$$

6. Conclusions

In this paper, we studied a generalization of the Kelvin-Voigt viscoelasticity equation given by the partial differential Equation (1). Firstly, we determined a complete Lie group classification. Then, we constructed the optimal system of one-dimensional subalgebras. These one-dimensional subalgebras have been used to find the symmetry reductions, allowing us to transform the partial differential equation into an ordinary differential equation. Moreover, we analysed all conservation laws for this equation by applying the multiplier method.

Author Contributions: A.P.M. and M.S.B. worked together in the derivation of the mathematical results. Both authors provided critical feedback and helped shape the research, analysis and manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

Abbreviations

The following abbreviations are used in this manuscript:

PDE Partial Differential Equation
ODE Ordinary Differential Equation

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