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Brief Report

Symmetric Criticality and Magnetic Monopoles in General Relativity

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Abstract: The Weyl method for finding solutions in general relativity using symmetry by varying an action with respect to a reduced set of field variables is known to fail in some cases. We add to the list of failures by considering an application of the Weyl method to a magnetically charged spherically symmetric source, obtaining an incorrect geometry. This is surprising, because the same method, applied to electrically charged central bodies correctly produces the Reissner-Nordström spacetime.

Keywords: Weyl method; Palais principle of symmetric criticality; solutions to Einstein's equations; magnetic monopole

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1. Introduction

We often use symmetry to simplify the field equations of general relativity (GR) and help solve them. There is a particular approach, the Weyl method [1], that benefits from an early application of assumed symmetry and can lead to striking simplification. The method has been used to successfully generate the spherically symmetric vacuum spacetime of general relativity, its first application. It has also been applied to modified theories like GR with cosmological constant, Einstein-Gauss-Bonnet gravity [2] and conformal gravity, all of which are developed and/or reviewed in Reference [3]. For axial symmetry, the $2+1$ dimensional "Kerr" solution for gravity with (negative) cosmological constant (BTZ) is obtained in Reference [3], with $3+1$ dimensional Kerr obtained using a targeted form of the technique in Reference [4].

But we must be careful, the method does not always work, as was detailed in Reference [5]. In this note, we review the method, providing some of its successful examples and discuss its failure in specific cases. We show that while the method is successful in finding the spacetime associated with an electrically charged spherical mass, it fails when the electric charge is replaced by magnetic charge (i.e., a magnetic monopole).

2. The Weyl Method

The Weyl method refers to the approach, invented and advertised by Weyl in Reference [1], of using information, in particular symmetry information, prior to varying an action in order to reduce the number, and simplify the form, of the field equations. Spherical symmetry in the Einstein-Hilbert action provides a good first example. Starting from the spherically symmetric line element, with two unknown function of r, the radial coordinate,

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{1}$$

we can form the Lagrangian for the action (primes indicate r-derivatives),

$$
\mathcal{L} = \sqrt{-g}R = \frac{\sin\theta}{2\left(A(r)B(r)\right)^{3/2}}\Big[r^2 B(r) A'(r)^2 + 4A(r)^2\left(-B(r) + B(r)^2 + rB'(r)\right) + rA(r)\left(rA'(r)B'(r) - 2B(r)\left(2A'(r) + rA''(r)\right)\right)\Big], \tag{2}
$$

and then we can use the Euler-Lagrange equations for $A(r)$ and $B(r)$,

$$
\begin{aligned}
0 &= \frac{\partial \mathcal{L}}{\partial A(r)} - \frac{d}{dr}\left(\frac{\partial \mathcal{L}}{\partial A'(r)}\right) + \frac{d^2}{dr^2}\left(\frac{\partial \mathcal{L}}{\partial A''(r)}\right) = \frac{\sin\theta}{\sqrt{A(r)B(r)^3}}\left[B(r)\left(-1 + B(r)\right) + rB'(r)\right] \\
0 &= \frac{\partial \mathcal{L}}{\partial B(r)} - \frac{d}{dr}\left(\frac{\partial \mathcal{L}}{\partial B'(r)}\right) = \frac{\sin\theta}{\sqrt{A(r)B(r)^3}}\left[A(r)\left(-1 + B(r)\right) - rA'(r)\right]
\end{aligned} \tag{3}
$$

to find $A(r)$ and $B(r)$. Solving the top equation for $B(r)$, we get

$$
B(r) = \frac{1}{1 - \frac{\alpha}{r}} \tag{4}
$$

for constant α. Then using this in the second equation, we can solve for $A(r)$,

$$
A(r) = 1 - \frac{\alpha}{r}. \tag{5}
$$

We have recovered the Schwarzschild solution, with constant α awaiting its usual physical interpretation, $\alpha = 2M$, with $G \to 1$, $c \to 1$.

The beauty of the Weyl approach is that the assumed form of the line element can simplify (or complexify) the field equations for the unknown functions. For example, if we started with the two-function ($a(r)$, $b(r)$ now) line element as in Reference [3],

$$
ds^2 = -a(r)b(r)^2 dt^2 + 1/a(r)dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{6}
$$

motivated by, for example, the single-function form of the determinant $\sqrt{-g} = b(r)r^2\sin\theta$ piece of the action, then the Lagrangian is

$$
\mathcal{L} = -\sin\theta\left[b(r)\left(-2r + \left(r^2 a(r)\right)'\right)' + \left(2r^2 a(r)b'(r)\right)' + r^2 a'(r)b'(r)\right] \tag{7}
$$

with field equations (obtainable even by dropping the total r-derivative in $\mathcal{L}$),

$$
2r\sin\theta b'(r) = 0 \qquad 2\sin\theta\left(1 - (ra(r))'\right) = 0 \tag{8}
$$

a decoupled set that's even easier to solve than those in (3) and leads, of course, to the same Schwarzschild spacetime.

3. Symmetric Criticality

There is a problem with the Weyl approach, one that goes back to the idea of action variation itself. Symmetries can be applied at the level of a field equation and lead to correct simplifications. Indeed, the symmetry of a solution is implied by the form of the field equation and (more importantly), the boundary conditions we impose on its solutions. Simplifications of this sort belong to the PDE problem that the field equations and boundaries define. But any information that derives from the field equations must be treated carefully when used prior to varying an action, that is, prior to developing the field equations, precisely what Weyl invites us to do.

As a reductio ad absurdum example from classical mechanics, suppose we take the free particle action in one dimension,

$$S[x(t)] = \int_{t_0}^{t_f} \frac{1}{2} m\dot{x}(t)^2 dt \tag{9}$$

and vary the action to get the equation of motion, $m\ddot{x}(t) = 0$ from which we learn that $x(t) = ft + g$ for constants f and g. If we insert this solution back into the action, we get

$$S = \int_{t_0}^{t_f} \frac{1}{2} m f^2 dt \tag{10}$$

which cannot itself be varied to recover a valid equation of motion governing $x(t)$. This is the logic that shows the potential flaw in the Weyl procedure. We have fixed all the degrees of freedom by solving the equation of motion, leaving us with nothing to vary in the action when that solution has been introduced.

The previous example is contrived and extreme but consider the slightly more disguised error in the following: We note that for the Schwarzschild solution (4) and (5), $B(r) = 1/A(r)$ (this is what suggests the two-function form of the line element in (6)). Suppose we use that information in developing the Lagrangian, that is, start with the line element

$$ds^2 = -a(r)dt^2 + 1/a(r)dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{11}$$

Then the Lagrangian becomes

$$\mathcal{L} = -\sin\theta\left(-2 + 2a(r) + 4ra'(r) + r^2a''(r)\right) = -\frac{d}{dr}\left[\sin\theta\left(-2r + \left(r^2a(r)\right)'\right)\right] \tag{12}$$

which, since it is a total derivative, leads to a trivial field equation ($0 = 0$) leaving $a(r)$ unconstrained. One might naïvely conclude that any function $a(r)$ solves the field equation in the spherically symmetric case. This is, of course, incorrect. The actual field equation, Einstein's in vacuum, $R_{\mu\nu} = 0$, has non-zero entries:

$$R_{00} = \frac{a(r)}{2}\left(\frac{2a'(r)}{r} + a''(r)\right) \qquad R_{rr} = -R_{00}/a(r)^2 \qquad R_{\theta\theta} = 1 - (ra(r))' \qquad R_{\phi\phi} = \sin^2\theta R_{\theta\theta}, \tag{13}$$

and these are solved by the usual $a(r) = 1 - \alpha/r$. While we can start with (11) and get the correct result from the field equations themselves, we have used *too much* simplifying information to recover that result from the Weyl method [6]. It is easy to go back and check that a solution obtained via the Weyl method is valid by running it through Einstein's equation. What is more difficult is to determine, a priori, whether a particular simplifying assumption will lead to problems. The equivalence of "varying an action, then imposing symmetry assumptions" and "imposing symmetry assumptions and then varying an action" is an example of Palais' "principle of symmetric criticality" [5]. He cautions that the principle is not universal and the current case provides an example of its failure.

Another case in which the principle fails is in establishing Birkhoff's theorem in general relativity. Birkhoff's theorem says that the spherically symmetric vacuum solution to Einstein's equation (Schwarzschild) is static, with no time dependence. If you started with an ansatz like (1) but allowed the functions A and B to depend on time, you would find no constraint on their temporal dependence using the Weyl approach, while Einstein's equations explicitly require $\dot{A}(r,t) = 0 = \dot{B}(r,t)$ (dots denoting t-derivatives), a statement of Birkhoff's theorem. The Weyl method can be redeemed in this case using an auxiliary field as detailed in Reference [7] (with the same fix applied to Lovelock gravity establishing Birkhoff's theorem there in Reference [8]) but using just the spherical symmetry by itself is not enough to establish Birkhoff's theorem. A similar auxiliary field is used in (6), where $b(r) = 1$ is an uninteresting solution to a trivial field equation, yet the function $b(r)$ is necessary to constrain $a(r)$

to its correct value by preventing the collapse of the Lagrangian to a total derivative as in (12) which lacked the $b(r)$ starting field.

One way of viewing the problem with proving Birkhoff's theorem is the focus on the two-dimensional $r - t$ subspace of spherically symmetric spacetimes that are, at least potentially, time dependent. The diagonal metric ansatz does not probe enough of that space to capture the time-independent constraint. A similar problem occurs if we attempt to carry out the procedure on a static, axially symmetric spacetime like the Weyl class of metrics. These typically start with line element

$$ds^2 = -e^{2a(s,z)}dt^2 + e^{-2a(s,z)+2b(s,z)}\left(ds^2 + dz^2\right) + s^2 e^{-2a(s,z)}d\phi^2 \tag{14}$$

for unknown functions $a(s,z)$ and $b(s,z)$ exhibiting cylindrical symmetry (no ϕ dependence). The Weyl method again fails to return a complete set of field equations, in this case because we have started off with the $s - z$ subspace in its (guaranteed) conformally flat form. Here, again, a Lagrange multiplier procedure can be used to restore the 3 independent field equations from Einstein's equation in vacuum but this must be done explicitly.

4. Reissner-Nordström and Magnetic Monopoles

Weyl's method works for extended sources as well as the simpler vacuum solutions provided the sources can themselves be fit into a field-theoretic action in combination with the Einstein-Hilbert action. The gravitational field variables show up in the auxiliary action in the usual way, both through the density $\sqrt{-g}$ and any explicit metric dependence, for example, $g^{\mu\nu}$ in the Lagrangian for a scalar field ϕ: $\phi_{,\mu}g^{\mu\nu}\phi_{,\nu}$ (the method is not available for non-Hilbert stress tensors, making it difficult to use in a cosmology context with fluid stress tensor sources). We can obtain the spherically symmetric spacetime for a charged massive spherical central body by starting with the combined Einstein-Hilbert and E&M action:

$$S = \int \sqrt{-g}\left(R + \sigma F^{\mu\nu}F_{\mu\nu}\right) d^4x \tag{15}$$

where σ is just a constant to set the coupling between gravity and E&M.

Now let's use the Weyl method to find the static, spherically symmetric solutions away from the massive source as in Reference [3]. Start with the ansatz from (6) for the gravitational piece, then the electromagnetic portion reads

$$F_{\mu\nu}F^{\mu\nu} \equiv F^{\mu\nu}F^{\alpha\beta}g_{\mu\alpha}g_{\nu\beta} = 2\left(B^2 - E^2 b(r)^2\right) \tag{16}$$

which depends on the metric used to contract the field strength tensor indices. The starting action is

$$S = \int \sqrt{-g}\left(R + 2\sigma\left(B^2 - E^2 b(r)^2\right)\right) d^4x \tag{17}$$

For a spherically symmetric electric charge source, $\mathbf{B} = 0$ and $\mathbf{E} = E(r)\hat{\mathbf{r}}$. The electric field comes from the $A_0(r)$ term in the vector potential A_μ where the lower form is the relevant one (since the field strength tensor is naturally covariant, $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$). In terms of this single non-zero term in the four-potential, $E(r) = A_0'(r)/b(r)^2$ and the Lagrangian is

$$\mathcal{L} = -\sin\theta\left[b(r)\left(-2r + \left(r^2 a(r)\right)'\right)' + \left(2r^2 a(r) b'(r)\right)' + r^2 a'(r) b'(r) + \frac{2\sigma r^2 A_0'(r)^2}{b(r)}\right] \tag{18}$$

Using the Euler-Lagrange equations that come from varying the associated action with respect to $a(r)$, $b(r)$ and $A_0(r)$ independently, we get

$$\begin{aligned}
0 &= 2r\sin\theta b'(r) \\
0 &= -2\sin\theta\left(-1+(ra(r))' - \frac{r^2\sigma A_0'(r)^2}{b(r)^2}\right) \\
0 &= \frac{4r\sigma\sin\theta}{b(r)^2}\left(-rb'(r)A_0'(r) + b(r)\left(2A_0'(r) + rA_0''(r)\right)\right)
\end{aligned} \tag{19}$$

The first equation is trivially solved by setting $b(r) = b_0$ a constant (that can be set to one by coordinate rescaling). The third equation, simplified using the first, is $2A_0'(r) + rA_0''(r) = (r^2A_0'(r))'/r = 0$. Its solution is $A_0(r) = V_0 - \beta/r$ for constant V_0, the value of the potential at spatial infinity and a constant β that is proportional to the electric charge. With these two in place and taking $V_0 \to 0$, the middle equation reads

$$-(ra(r))' + \left(1+\sigma\frac{\beta^2}{r^2}\right) = 0 \longrightarrow a(r) = 1 - \frac{\alpha}{r} - \sigma\frac{\beta^2}{r^2}, \tag{20}$$

where α is related to the mass of the central body as in the Schwarzschild case. The line element and potential are

$$\begin{aligned}
ds^2 &= -\left(1-\frac{\alpha}{r}-\sigma\frac{\beta^2}{r^2}\right)dt^2 + \frac{1}{\left(1-\frac{\alpha}{r}-\sigma\frac{\beta^2}{r^2}\right)}dr^2 + r^2\left(d\theta^2+\sin^2\theta d\phi^2\right) \\
A_0(r) &= -\frac{\beta}{r},
\end{aligned} \tag{21}$$

which is the correct Reissner-Nordström solution. Note that $A_0(r)$ is related to the electric field magnitude, for $b(r) = 1$, by $E(r) = A_0'(r) = \beta/r^2$, the usual Coulomb field associated with a spherically symmetric charge (the covariant zero-component of the four-potential, A_0, plays the role of $-V(r)$ for the usual electrostatic potential $V(r)$).

Let's now consider the spacetime associated with a massive spherical central body with magnetic monopole charge (but no electric charge). All that changes is that we take $\mathbf{E} = 0$ and $\mathbf{B} = B(r)\hat{\mathbf{r}}$, this time with $B(r) = W'(r)/b(r)^2$ for a magnetic monopole potential $W(r)$ replacing $A_0(r)$ from above. Looking at (17), it is clear that the sign associated with the magnetic field is opposite that of the electric case and that ends up introducing a minus sign in the line element that solves the field equations. The Lagrangian is now

$$\mathcal{L} = -\sin\theta\left[b(r)\left(-2r+\left(r^2a(r)\right)'\right)' + \left(2r^2a(r)b'(r)\right)' + r^2a'(r)b'(r) - \frac{2\sigma r^2W'(r)^2}{b(r)^3}\right] \tag{22}$$

In addition to the sign change, there is a factor of $1/b(r)^3$ attached to the potential, as opposed to the $1/b(r)$ in (18). This ends up introducing an essentially irrelevant factor of 3 in the metric's dependence on magnetic charge.

For constant $\bar{\beta}$ associated with the magnetic charge, the line element and potential that comes from the Weyl method applied to (22) is:

$$\begin{aligned}
ds^2 &= -\left(1-\frac{\alpha}{r}+3\sigma\frac{\bar{\beta}^2}{r^2}\right)dt^2 + \frac{1}{\left(1-\frac{\alpha}{r}+3\sigma\frac{\bar{\beta}^2}{r^2}\right)}dr^2 + r^2\left(d\theta^2+\sin^2\theta d\phi^2\right) \\
W(r) &= -\frac{\bar{\beta}}{r} \longrightarrow B(r) = \frac{\bar{\beta}}{r^2}
\end{aligned} \tag{23}$$

From this, we would conclude that the Reissner-Nordström solution for a magnetic monopole has a fundamentally different structure than the electric monopole case, with $\beta^2 \to -3\bar{\beta}^2$ taking us from one metric to the other. Again it is the minus sign that is important here, that's what changes the structure of the spacetime (in particular, the horizon structure is different between the two).

Einstein's field equations tell a different story—the correct one, of course [9,10]. For the electromagnetic sourcing, we consider the full field equations,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}, \tag{24}$$

and note that the elements of the electromagnetic stress tensor,

$$\begin{aligned} T^{00} &= \frac{1}{2}\left(E^2 + B^2\right) \\ T^{ij} &= \left(\frac{1}{2}\delta^{ij}E^2 - E^i E^j\right) + \left(\frac{1}{2}\delta^{ij}B^2 - B^i B^j\right) \end{aligned} \tag{25}$$

are symmetric in $\mathbf{E} \leftrightarrow \mathbf{B}$, while the Poynting vector contribution, $T^{0i} \sim (\mathbf{E} \times \mathbf{B})^i$ vanishes when considering either field in isolation (this is true even in the extended setting in which magnetic monopoles are incorporated in Maxwell's equation from the start). Then the role of an electric or magnetic monopole in gravity is the same, we have $\beta^2 \to \bar{\beta}^2$ in (21). The field equations give a different result than the Weyl method, so this is another example where the Palais principle is violated.

The problem in this case comes from the move from the electromagnetic action, $\mathcal{L}_{\text{EM}} \sim \sqrt{-g} F_{\mu\nu} F^{\mu\nu}$, to the electromagnetic stress tensor (obtained by Hilbert's procedure),

$$T_{\mu\nu} \equiv \frac{2}{\sqrt{-g}} \frac{\partial \mathcal{L}_{\text{EM}}}{\partial g^{\mu\nu}}. \tag{26}$$

The variational procedure that generates the stress tensor source for gravity from (15) requires that we probe the full metric dependence of the electromagnetic action. In the Weyl method, we probe only a subset and evidently, that subset is too small to reproduce the correct stress tensor structure. Indeed, the electromagnetic piece of the Lagrangian,

$$\mathcal{L}_{\text{EM}} = \sqrt{-g} F_{\mu\nu} F^{\mu\nu} = 2r^2 b(r) \sin\theta \left(B^2 - b(r)^2 E^2\right) \tag{27}$$

depends on only one of the metric's two independent functions, a clear warning sign that we will be looking at only a portion of the stress tensor defined by (26).

The situation is similar to the failures described in Section 3 but in those cases, only the gravitational piece was relevant. We know that the line element ansatz (6) used here is enough to capture the spherically symmetric vacuum solution but it is not enough to provide the correct source term. It is conceivable that we could introduce Lagrange multipliers to restore the procedure, as with Birkhoff's theorem or the axially symmetric Weyl metrics. But the ease with which we obtain the correct solution from the full field equations makes the task of finding such fixes unnecessary.

5. Conclusions

Symmetry is a powerful simplifying tool in many settings. In general relativity, the Weyl method makes good use of symmetry observations by reducing the number of degrees of freedom in the Einstein-Hilbert action. The method does not always work and it is important to test solutions obtained in this way by running them through the Einstein field equations. While the Weyl method can be used to correctly obtain the Reissner-Nordström spacetime outside of an electrically charged spherical central body, it fails to produce the correct spacetime when the central body is magnetically charged.

For most applications, the problem with the Weyl method is similar in spirit to the extreme example from Section 3, where the degrees of freedom in the action have been over-reduced, leaving us with no information. That's what happens for the spherically symmetric starting point (11) and the same deficiency occurs when trying to prove Birkhoff's theorem and establish the Weyl class of metrics starting from (14). In each of these cases, the hallmark is a lack of information, the field variables are unconstrained in some way that they should be, according to Einstein's equation. We are left with no information and that lack of information tells us that the error has occurred and suggests a fix.

The monopole case discussed here is different. We are not simply missing information that we suspect should be there. Rather, the information we have is incorrect. The symptom is different but the deficiency is the same, a lack of ability to probe the starting action's full field degrees of freedom. This time, it is the "source" term, the electromagnetic action, rather than the gravitational one that is the culprit.

Conflicts of Interest: The author declares no conflict of interest.

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