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Some Identities of Ordinary and Degenerate Bernoulli Numbers and Polynomials

Dmitry V. Dolgy [1], Dae San Kim [2], Jongkyum Kwon [3,*] and Taekyun Kim [4]

1. Hanrimwon, Kwangwoon University, Seoul 139-701, Korea
2. Department of Mathematics, Sogang University, Seoul 121-742, Korea
3. Department of Mathematics Education and ERI, Gyeongsang National University, Jinju, Gyeongsangnamdo 52828, Korea
4. Department of Mathematics, Kwangwoon University, Seoul 139-701, Korea

* Correspondence: mathkjk26@gnu.ac.kr

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Abstract: In this paper, we investigate some identities on Bernoulli numbers and polynomials and those on degenerate Bernoulli numbers and polynomials arising from certain p-adic invariant integrals on $\mathbb{Z}_p$. In particular, we derive various expressions for the polynomials associated with integer power sums, called integer power sum polynomials and also for their degenerate versions. Further, we compute the expectations of an infinite family of random variables which involve the degenerate Stirling polynomials of the second and some value of higher-order Bernoulli polynomials.

Keywords: Bernoulli polynomials; degenerate Bernoulli polynomials; random variables; p-adic invariant integral on $\mathbb{Z}_p$; integer power sums polynomials; Stirling polynomials of the second kind; degenerate Stirling polynomials of the second kind

1. Introduction

We begin this section by reviewing some known facts. In more detail, we recall the integral equation for the p-adic invariant integral of a uniformly differentiable function on $\mathbb{Z}_p$ and its generalizations, the expression in terms of some values of Bernoulli polynomials for the integer power sums, and the p-adic integral representaions of Bernoulli polynomials and of their generating functions.

Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. The p-adic norm is normalized as $|p|_p = \frac{1}{p}$. Let f be a uniformly differentiable function on $\mathbb{Z}_p$. Then the p-adic invariant integral of f (also called the Volkenborn integral of f) on $\mathbb{Z}_p$ is defined by

$$\begin{aligned} I_0(f) &= \int_{\mathbb{Z}_p} f(x)d\mu_0(x) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x) \\ &= \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)\mu_0(x+p^N\mathbb{Z}_p). \end{aligned} \tag{1}$$

Here we note that $\mu_0(x+p^N\mathbb{Z}_p) = \frac{1}{p^N}$ is a distribution but not a measure. The existence of such integrals for uniformly differentiable functions on $\mathbb{Z}_p$ is detailed in [1,2]. It can be seen from (1) that

$$I_0(f_1) = I_0(f) + f'(0), \tag{2}$$

where $f_1(x) = f(x+1)$, and $f'(0) = \frac{df(x)}{dx}|_{x=0}$, (see [1,2]).

In general, by induction and with $f_n(x) = f(x+n)$, we can show that

$$I_0(f_n) = I_0(f) + \sum_{k=0}^{n-1} f'(k), \quad (n \in \mathbb{N}), \tag{3}$$

As is well known, the Bernoulli polynomials are given by the generating function (see [3–5])

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \tag{4}$$

When $x = 0$, $B_n = B_n(0)$ are called the Bernoulli numbers.
From (4), we note that (see [3–5])

$$B_n(x) = \sum_{l=0}^{n} \binom{n}{l} B_l x^{n-l}, \quad (n \geq 0), \tag{5}$$

and

$$B_0 = 1, \quad \sum_{k=0}^{n} \binom{n}{k} B_k - B_n = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n > 1, \end{cases}$$

Let (see [6–13])

$$S_p(n) = \sum_{k=1}^{n} k^p, \quad (n, p \in \mathbb{N}). \tag{6}$$

The generating function of $S_p(n)$ is given by

$$\begin{aligned} \sum_{p=0}^{\infty} S_p(n) \frac{t^p}{p!} &= \sum_{k=1}^{n} e^{kt} = \frac{1}{t} \left(\frac{t}{e^t - 1} \left(e^{(n+1)t} - e^t \right) \right) \\ &= \sum_{p=0}^{\infty} \left(\frac{B_{p+1}(n+1) - B_{p+1}(1)}{p+1} \right) \frac{t^p}{p!}. \end{aligned} \tag{7}$$

Thus, by (7), we get

$$S_p(n) = \frac{B_{p+1}(n+1) - B_{p+1}(1)}{p+1}, \quad (n, p \in \mathbb{N}). \tag{8}$$

From (2), we have

$$\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_0(y) = \frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}. \tag{9}$$

By (9), we get (see [11,12])

$$\int_{\mathbb{Z}_p} (x+y)^n d\mu_0(y) = B_n(x), \quad (n \geq 0), \tag{10}$$

From (8) and (10), we can derive the following equation.

$$\int_{\mathbb{Z}_p} (x+k+1)^{p+1} d\mu_0(x) - \int_{\mathbb{Z}_p} x^{p+1} d\mu_0(x) = (p+1) \sum_{n=1}^{k} n^p, \quad (p \in \mathbb{N}). \tag{11}$$

Thus, by (6) and (11), and for $p \in \mathbb{N}$, we get

$$S_p(k) = \frac{1}{p+1} \left\{ \int_{\mathbb{Z}_p} (x+k+1)^{p+1} d\mu_0(x) - \int_{\mathbb{Z}_p} x^{p+1} d\mu_0(x) \right\}. \tag{12}$$

The purpose of this paper is to investigate some identities on Bernoulli numbers and polynomials and those on degenerate Bernoulli numbers and polynomials arising from certain p-adic invariant integrals on $\mathbb{Z}_p$.

The outline of this paper is as in the following. After reviewing well- known necessary results in Section 1, we will derive some identities on Bernoulli polynomials and numbers in Section 2. In particular, we will introduce the integer power sum polynomials and derive several expressions for them. In Section 3, we will obtain some identities on degenerate Bernoulli numbers and polynomials. Especially, we will introduce the degenerate integer power sum polynomials, a degenerate version of the integer power sum polynomials and deduce various representations of them. In the final Section 4, we will consider an infinite family of random variables and compute their expectations to see that they involve the degenerate Stirling polynomials of the second and some value of higher-order Bernoulli polynomials.

2. Some Identities of Bernoulli Numbers and Polynomials

For $p \in \mathbb{N}$, we observe that

$$\begin{aligned}(j+1)^{p+1} - j^{p+1} &= \sum_{i=0}^{p+1} \binom{p+1}{i} j^i - j^{p+1} \\ &= (p+1)j^p + \sum_{i=1}^{p-1} \binom{p+1}{i} j^i + 1.\end{aligned} \tag{13}$$

Thus, we get

$$(n+1)^{p+1} = \sum_{j=0}^{n} \left\{ (j+1)^{p+1} - j^{p+1} \right\} = (p+1)\sum_{j=0}^{n} j^p + \sum_{i=1}^{p-1} \binom{p+1}{i} \sum_{j=0}^{n} j^i + (n+1). \tag{14}$$

From (14), we have

$$S_p(n) = \frac{1}{p+1}\left\{ (n+1)^{p+1} - (n+1) - \sum_{i=1}^{p-1} \binom{p+1}{i} S_i(n) \right\}. \tag{15}$$

Therefore, by (15), we obtain the following lemma.

Lemma 1. *For $n, p \in \mathbb{N}$, we have*

$$\begin{aligned}&\int_{\mathbb{Z}_p} (x+n+1)^{p+1} d\mu_0(x) - \int_{\mathbb{Z}_p} x^{p+1} d\mu_0(x) \\ =&(n+1)^{p+1} - (n+1) - \sum_{i=1}^{p-1} \binom{p+1}{i} \frac{1}{i+1} \\ &\times \left\{ \int_{\mathbb{Z}_p} (x+n+1)^{i+1} d\mu_0(x) - \int_{\mathbb{Z}_p} x^{i+1} d\mu_0(x) \right\}.\end{aligned} \tag{16}$$

From Lemma 1, we note the following.

Corollary 1. *For $n, p \in \mathbb{N}$, we have*

$$B_{p+1}(n+1) - B_{p+1} = (n+1)^{p+1} - (n+1) - \sum_{i=1}^{p-1} \binom{p+1}{i} \frac{1}{i+1} \Big(B_{i+1}(n+1) - B_{i+1} \Big). \tag{17}$$

For $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, by (1), we get

$$\int_{\mathbb{Z}_p} \left(y + 1 - x\right)^n d\mu_0(y) = (-1)^n \int_{\mathbb{Z}_p} (y + x)^n d\mu_0(y). \tag{18}$$

From (18), we note that

$$B_n(1 - x) = (-1)^n B_n(x), \ \ (n \geq 0). \tag{19}$$

Now, we observe that, for $n \geq 1$,

$$\begin{aligned} B_n(2) &= \sum_{l=0}^{n} \binom{n}{l} B_l(1) = B_0 + \binom{n}{1} B_1(1) + \sum_{l=2}^{n} \binom{n}{l} B_l(1) \\ &= B_0 + \binom{n}{1} B_1 + n + \sum_{l=2}^{n} \binom{n}{l} B_l = n + \sum_{l=0}^{n} \binom{n}{l} B_l \\ &= n + B_n(1). \end{aligned} \tag{20}$$

Thus we have completed the proof for the next lemma.

Lemma 2. *For any $n \in \mathbb{N}_0$, the following identity is valid:*

$$B_n(2) = n + B_n + \delta_{n,1}, \tag{21}$$

where $\delta_{n,1}$ is the Kronecker's delta.
For any $n, m \in \mathbb{N}$ with $n, m \geq 2$, we have

$$\begin{aligned} \int_{\mathbb{Z}_p} x^m(-1 + x)^n d\mu_0(x) &= \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} \int_{\mathbb{Z}_p} x^{m+i} d\mu_0(x) \\ &= \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} B_{m+i} \\ &= (-1)^{n-m} \sum_{i=0}^{n} \binom{n}{i} B_{m+i}. \end{aligned} \tag{22}$$

On the other hand,

$$\begin{aligned} \int_{\mathbb{Z}_p} x^m (x - 1)^n d\mu_0(x) &= \sum_{i=0}^{m} \binom{m}{i} \int_{\mathbb{Z}_p} (x - 1)^{n+i} d\mu_0(x) \\ &= \sum_{i=0}^{m} \binom{m}{i} (-1)^{n+i} \int_{\mathbb{Z}_p} (x + 2)^{n+i} d\mu_0(x) \\ &= \sum_{i=0}^{m} \binom{m}{i} (-1)^{n+i} \left(B_{n+i} + n + i \right) \\ &= \sum_{i=0}^{m} \binom{m}{i} (-1)^{n+i} B_{n+i} \\ &= \sum_{i=0}^{m} \binom{m}{i} B_{n+i}. \end{aligned} \tag{23}$$

Therefore, by (22) and (23), we obtain the following theorem.

Theorem 1. *For any $m, n \in \mathbb{N}$ with $m, n \geq 2$, the following symmetric identity holds:*

$$(-1)^n \sum_{i=0}^{n} \binom{n}{i} B_{m+i} = (-1)^m \sum_{i=0}^{m} \binom{m}{i} B_{n+i}. \tag{24}$$

From (5), we note that

$$B_n(1) = \sum_{l=0}^{n} \binom{n}{l} B_l, \quad (n \geq 0).$$

For $n \geq 2$, we have

$$B_n = B_n(1) = \sum_{l=0}^{n} \binom{n}{l} B_l = \sum_{l=0}^{n} \binom{n}{l} B_{n-l}. \tag{25}$$

Now, we define the *integer power sum polynomials* by

$$S_p(n|x) = \sum_{k=0}^{n} (k+x)^p, \quad (n, p \in \mathbb{N}_0). \tag{26}$$

Note that $S_p(n|0) = S_p(n), \ (n \in \mathbb{N}_0, p \in \mathbb{N})$.
For $N \in \mathbb{N}_0$, we have

$$t \sum_{k=0}^{N} e^{(k+x)t} = \int_{\mathbb{Z}_p} e^{(N+1+x+y)t} d\mu_0(y) - \int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_0(y). \tag{27}$$

Then it is immediate to see from (27) that we have

$$\sum_{k=0}^{N} e^{(k+x)t} = \sum_{n=0}^{\infty} \frac{1}{n+1} \left\{ \int_{\mathbb{Z}_p} (N+1+x+y)^{n+1} d\mu_0(y) - \int_{\mathbb{Z}_p} (x+y)^{n+1} d\mu_0(y) \right\} \frac{t^n}{n!}. \tag{28}$$

Now, we see that (28) is equivalent to the next theorem.

Theorem 2. *For $n, N \in \mathbb{N}_0$, we have*

$$S_n(N|x) = \frac{1}{n+1} \left\{ B_{n+1}(x+N+1) - B_{n+1}(x) \right\}. \tag{29}$$

Let $\triangle$ denote the difference operator given by

$$\triangle f(x) = f(x+1) - f(x). \tag{30}$$

Then, by (30) and induction, we get

$$\triangle^n f(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(x+k), \quad (n \geq 0). \tag{31}$$

Now, we can deduce the Equation (32) from (27) as in the following:

$$\begin{aligned}
\sum_{k=0}^{N} e^{(k+x)t} &= \frac{1}{t}e^{xt}(e^{(N+1)t}-1)\int_{\mathbb{Z}_p} e^{yt}d\mu_0(y) \\
&= \frac{1}{e^t-1}\left(\sum_{m=0}^{N+1}\binom{N+1}{m}(e^t-1)^m-1\right)e^{xt} \\
&= \frac{1}{e^t-1}\sum_{m=1}^{N+1}\binom{N+1}{m}(e^t-1)^m e^{xt} \\
&= \sum_{m=0}^{N}\binom{N+1}{m+1}(e^t-1)^m e^{xt} \\
&= \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{N}\binom{N+1}{m+1}\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}(k+x)^n\right\}\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left\{\sum_{k=0}^{N}\sum_{m=k}^{N}\binom{N+1}{m+1}\binom{m}{k}(-1)^{m-k}(k+x)^n\right\}\frac{t^n}{n!}.
\end{aligned} \tag{32}$$

Therefore, (31) and (32) together yield the next theorem.

Theorem 3. *For $n, N \geq 0$, we have*

$$S_n(N|x) = \sum_{m=0}^{N}\binom{N+1}{m+1}\triangle^m x^n = \sum_{k=0}^{N}(k+x)^n T(N,k), \tag{33}$$

where $T(N,k) = \sum_{m=k}^{N}\binom{N+1}{m+1}\binom{m}{k}(-1)^{m-k}$.

In particular, we have

$$S_0(N|x) = \sum_{k=0}^{N} T(N,k) = N+1.$$

We recall here that the Stirling polynomials of the second kind $S_2(n,k|x)$ are given by (see [14])

$$\frac{1}{k!}(e^t-1)^k e^{xt} = \sum_{n=k}^{\infty} S_2(n,k|x)\frac{t^n}{n!}. \tag{34}$$

Note here that $S_2(n,k|0) = S_2(n,k)$ are Stirling numbers of the second kind. Then, we can show that, for integers $n, m \geq 0$, we have

$$\frac{1}{m!}\triangle^m x^n = \begin{cases} S_2(n,m|x), & \text{if } n \geq m, \\ 0, & \text{if } n < m. \end{cases} \tag{35}$$

We can see this, for example, by taking $\lambda \to 0$ in (51).

Remark 1. *Combing (33) and (35), we obtain*

$$S_n(N|x) = \sum_{m=0}^{min\{N,n\}}\binom{N+1}{m+1}m!S_2(n,m|x).$$

For any $m,k \in \mathbb{N}$ with $m-k \geq 2$, we observe that

$$\begin{aligned}\int_{\mathbb{Z}_p} x^{m-k} d\mu_0(x) &= \int_{\mathbb{Z}_p} (x+1)^{m-k} d\mu_0(x) \\ &= \sum_{j=0}^{m-k} \binom{m-k}{m-k-j} \int_{\mathbb{Z}_p} x^{m-k-j} d\mu_0(x) \\ &= \sum_{j=k}^{m} \binom{m-k}{m-j} \int_{\mathbb{Z}_p} x^{m-j} d\mu_0(x) \\ &= \frac{1}{\binom{m}{k}} \sum_{j=k}^{m} \binom{m}{j}\binom{j}{k} \int_{\mathbb{Z}_p} x^{m-j} d\mu_0(x).\end{aligned} \tag{36}$$

Thus we have shown the following result.

Theorem 4. *For any $m,k \in \mathbb{N}$ with $m-k \geq 2$, the following holds true:*

$$\binom{m}{k} \int_{\mathbb{Z}_p} x^{m-k} d\mu_0(x) = \sum_{j=k}^{m} \binom{m}{j}\binom{j}{k} \int_{\mathbb{Z}_p} x^{m-j} d\mu_0(x). \tag{37}$$

From (10) and (37), we derive the following corollary.

Corollary 2. *For $m,k \in \mathbb{N}$ with $m-k \geq 2$, we have*

$$\binom{m}{k} B_{m-k} = \sum_{j=k}^{m} \binom{m}{j}\binom{j}{k} B_{m-j}. \tag{38}$$

3. Some Identities of Degenerate Bernoulli Numbers and Polynomials

In this section, we assume that $0 \neq \lambda \in \mathbb{C}_p$ with $|\lambda|_p < p^{-\frac{1}{p-1}}$. The degenerate exponential function is defined as (see [3,13])

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}}.$$

Note that $\lim_{\lambda\to 0} e_\lambda^x(t) = e^{xt}$. In addition, we denote $(1+\lambda t)^{\frac{1}{\lambda}} = e_\lambda^1(t)$ simply by $e_\lambda(t)$.

As is well known, the degenerate Bernoulli polynomials are defined by Carlitz as

$$\frac{t}{e_\lambda(t)-1} e_\lambda^x(t) = \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_{n,\lambda}(x)\frac{t^n}{n!}. \tag{39}$$

When $x=0$, $\beta_{n,\lambda} = \beta_{n,\lambda}(0)$ are called the degenerate Bernoulli numbers, (see [3,15]).

From (39), we note that (see [3])

$$\beta_{n,\lambda}(x) = \sum_{l=0}^{n} \binom{n}{l} (x)_{n-l,\lambda}\beta_{l,\lambda}, \tag{40}$$

where $(x)_{0,\lambda} = 1$, $(x)_{n,\lambda} = x(x-\lambda)\cdots(x-(n-1)\lambda)$, $(n \geq 1)$.

By (39) and (40), we get

$$\beta_{n,\lambda}(1) - \beta_{n,\lambda} = \delta_{n,1}. \tag{41}$$

Now, we observe that

$$\begin{aligned}\sum_{k=0}^{N} e_{\lambda}^{k+x}(t) &= \frac{e_{\lambda}^{N+1}(t)-1}{e_{\lambda}(t)-1} e_{\lambda}^{x}(t) = \frac{1}{t}\left\{\frac{t}{e_{\lambda}(t)-1}\left(e_{\lambda}^{N+1+x}(t)-e_{\lambda}^{x}(t)\right)\right\} \\ &= \frac{1}{t}\sum_{n=0}^{\infty}\left(\beta_{n,\lambda}(N+1+x)-\beta_{n,\lambda}(x)\right)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\frac{\beta_{n+1,\lambda}(N+1+x)-\beta_{n+1,\lambda}(x)}{n+1}\right)\frac{t^n}{n!}, \quad (n\in\mathbb{N}_0).\end{aligned} \tag{42}$$

On the other hand,

$$\sum_{k=0}^{N} e_{\lambda}^{k+x}(t) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{N}(k+x)_{n,\lambda}\right)\frac{t^n}{n!}. \tag{43}$$

Let us define a degenerate version of the integer power sum polynomials, called the *degenerate integer power sum polynomials*, by

$$S_{p,\lambda}(n|x) = \sum_{k=0}^{n}(k+x)_{p,\lambda}, \quad (n\geq 0). \tag{44}$$

Note that $\lim_{\lambda\to 0} S_{p,\lambda}(n|x) = S_p(n|x), \quad (n\geq 0)$.
Therefore, by (42) and (43), we obtain the following theorem.

Theorem 5. *For $n, N\in\mathbb{N}_0$, we have*

$$S_{n,\lambda}(N|x) = \frac{1}{n+1}\left(\beta_{n+1,\lambda}(N+1+x)-\beta_{n+1,\lambda}(x)\right). \tag{45}$$

Now, we observe that

$$\begin{aligned}\sum_{k=0}^{N} e_{\lambda}^{x+k}(t) &= \frac{1}{e_{\lambda}(t)-1}\left(e_{\lambda}^{N+1}(t)-1\right)e_{\lambda}^{x}(t) \\ &= \frac{1}{e_{\lambda}(t)-1}\left((e_{\lambda}(t)-1+1)^{N+1}-1\right)e_{\lambda}^{x}(t) \\ &= \frac{1}{e_{\lambda}(t)-1}\sum_{m=1}^{N+1}\binom{N+1}{m}(e_{\lambda}(t)-1)^m e_{\lambda}^{x}(t) \\ &= \sum_{m=0}^{N}\binom{N+1}{m+1}(e_{\lambda}(t)-1)^m e_{\lambda}^{x}(t) \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{N}\binom{N+1}{m+1}\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}(k+x)_{n,\lambda}\right)\frac{t^n}{n!}. \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{N}\sum_{m=k}^{N}\binom{N+1}{m+1}\binom{m}{k}(-1)^{m-k}(k+x)_{n,\lambda}\right)\frac{t^n}{n!}.\end{aligned} \tag{46}$$

Therefore, (31) and (46) together give the next result.

Theorem 6. *For any $n, N\in\mathbb{N}_0$, the following identity holds:*

$$S_{n,\lambda}(N|x) = \sum_{m=0}^{N}\binom{N+1}{m+1}\triangle^m\,(x)_{n,\lambda} = \sum_{k=0}^{N}(k+x)_{n,\lambda}T(N,k), \tag{47}$$

where $T(N,k) = \sum_{m=k}^{N} \binom{N+1}{m+1}\binom{m}{k}(-1)^{m-k}$.

As is known, the degenerate Stirling polynomials of the second kind are defined by Kim as (see [14])

$$(x+y)_{n,\lambda} = \sum_{k=0}^{n} S_{2,\lambda}(n,k|x)(y)_k, \tag{48}$$

where $(x)_0 = 1, (x)_n = x(x-1)\cdots(x-n+1), \ \ (n \geq 1)$.

From (48), we can derive the generating function for $S_{2,\lambda}(n,k|x), \ \ (n,k \geq 0)$, as follows:

$$\frac{1}{k!}(e_\lambda(t)-1)^k e_\lambda^x(t) = \sum_{n=k}^{\infty} S_{2,\lambda}(n,k|x)\frac{t^n}{n!}. \tag{49}$$

When $x = 0$, $S_{2,\lambda}(n,k|0) = S_{2,\lambda}(n,k)$ are called the degenerate Stirling numbers of the second kind.

By (49), we get

$$\begin{aligned} \sum_{n=m}^{\infty} S_{2,\lambda}(n,m|x)\frac{t^n}{n!} &= \frac{1}{m!}(e_\lambda(t)-1)^m e_\lambda^x(t) \\ &= \frac{1}{m!}\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}e_\lambda^{k+x}(t) \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{m!}\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}(x+k)_{n,\lambda}\right)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{m!}\triangle^m (x)_{n,\lambda}\right)\frac{t^n}{n!}. \end{aligned} \tag{50}$$

Now, comparison of the coefficients on both sides of (50) yield following theorem.

Theorem 7. *For any $n, m \geq 0$, the following identity holds:*

$$\frac{1}{m!}\triangle^m (x)_{n,\lambda} = \begin{cases} S_{2,\lambda}(n,m|x), & \text{if } n \geq m, \\ 0, & \text{if } n < m. \end{cases} \tag{51}$$

Remark 2. *Combing (47) and (51), we obtain*

$$S_{n,\lambda}(N|x) = \sum_{m=0}^{\min\{N,n\}}\binom{N+1}{m+1}m!S_{2,\lambda}(n,m|x).$$

From (30) and proceeding by induction, we have

$$(1+\triangle)^k f(x) = \sum_{m=0}^{k}\binom{k}{m}\triangle^m f(x) = f(x+k), \ \ (k \geq 0). \tag{52}$$

By (52), we get

$$\sum_{k=0}^{N}(x+k)_{n,\lambda} = \sum_{k=0}^{N}(1+\triangle)^k (x)_{n,\lambda}. \tag{53}$$

It is known that Daehee numbers are given by the generating function

$$\frac{\log(1+t)}{t} = \sum_{n=0}^{\infty} D_n\frac{t^n}{n!}, \ \ \text{(see [1,4,6])}. \tag{54}$$

From (2), we have

$$\begin{aligned}\int_{\mathbb{Z}_p} e_\lambda^{x+y}(t)d\mu_0(y) &= \frac{\frac{1}{\lambda}\log(1+\lambda t)}{e_\lambda(t)-1}e_\lambda^x(t) \\ &= \frac{\log(1+\lambda t)}{\lambda t}\frac{t}{e_\lambda(t)-1}e_\lambda^x(t) \\ &= \sum_{l=0}^{\infty} D_l \frac{\lambda^l t^l}{l!}\sum_{m=0}^{\infty}\beta_{m,\lambda}(x)\frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\lambda^l D_l \beta_{n-l,\lambda}(x)\right)\frac{t^n}{n!}.\end{aligned} \tag{55}$$

From (55), we have

$$\int_{\mathbb{Z}_p}(x+y)_{n,\lambda}d\mu_0(y) = \sum_{l=0}^{n}\binom{n}{l}\lambda^l D_l \beta_{n-l,\lambda}(x), \ \ (n \geq 0).$$

4. Further Remark

A random variable X is a real-valued function defined on a sample space. We say that X is a continuous random variable if there exists a nonnegative function f, defined on $(-\infty,\infty)$, having the property that for any set B of real numbers (see [16,17])

$$P\{X \in B\} = \int_B f(x)dx. \tag{56}$$

The function f is called the probability density function of random variable X.

Let X be a uniform random variable on the interval (α,β). Then the probability density function f of X is given by

$$f(x) = \begin{cases} \frac{1}{\beta-\alpha}, & \text{if } \alpha < x < \beta, \\ 0, & \text{otherwise.}\end{cases} \tag{57}$$

Let X be a continuous random variable with the probability density function f. Then the expectation of X is defined by

$$E[X] = \int_{-\infty}^{\infty} xf(x)dx.$$

For any real-valued function $g(x)$, we have (see [16])

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)dx. \tag{58}$$

Assume that $X_1, X_2, \cdots, X_k$ are independent uniform random variables on $(0,1)$. Then we have

$$\begin{aligned}
&E[e_\lambda^{x+X_1+X_2+\cdots+X_k}(t)] = e_\lambda^x(t)E[e_\lambda^{X_1}(t)]E[e_\lambda^{X_2}(t)]\cdots E[e_\lambda^{X_k}(t)] \\
&= e_\lambda^x(t)\underbrace{\frac{\lambda}{\log(1+\lambda t)}(e_\lambda(t)-1)\times\cdots\times\frac{\lambda}{\log(1+\lambda t)}(e_\lambda(t)-1)}_{k-times} \\
&= \left(\frac{\lambda t}{\log(1+\lambda t)}\right)^k \frac{k!}{t^k}\frac{1}{k!}(e_\lambda(t)-1)^k e_\lambda^x(t) \\
&= \frac{k!}{t^k}\sum_{l=0}^{\infty} B_l^{(l-k+1)}(1)\lambda^l\frac{t^l}{l!}\sum_{m=k}^{\infty} S_{2,\lambda}(m,k\mid x)\frac{t^m}{m!} \\
&= \frac{k!}{t^k}\sum_{n=k}^{\infty}\left(\sum_{m=k}^{n}\binom{n}{m}S_{2,\lambda}(m,k\mid x)B_{n-m}^{(n-m-k+1)}(1)\lambda^{n-m}\right)\frac{t^n}{n!},
\end{aligned} \tag{59}$$

where $B_n^{(\alpha)}(x)$ are the Bernoulli polynomials of order α, given by (see [4,7,8])

$$\left(\frac{t}{e^t-1}\right)^\alpha e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{t^n}{n!}, \tag{60}$$

and we used the well-known formula

$$\left(\frac{t}{\log(1+t)}\right)^n (1+t)^{x-1} = \sum_{k=0}^{\infty} B_k^{(k-n+1)}(x)\frac{t^k}{k!}. \tag{61}$$

From (59), we note that

$$\begin{aligned}
&\binom{n}{k}E[(x+X_1+X_2+\cdots+X_k)_{n-k,\lambda}] \\
&\quad = \sum_{m=k}^{n}\binom{n}{m}S_{2,\lambda}(m,k\mid x)B_{n-m}^{(n-m-k+1)}(1)\lambda^{n-m}.
\end{aligned} \tag{62}$$

5. Conclusions

It is well-known and classical that the first n positive integer power sums can be given by an expression involving some values of Bernoulli polynomials. Here we investigated some identities on Bernoulli numbers and polynomials and those on degenerate Bernoulli numbers and polynomials, which can be deduced from certain p-adic invariant integrals on $\mathbb{Z}_p$.

In particular, we introduced the integer power sum polynomials associated with integer power sums and obtained various expressions of them. Namely, they can be given in terms of Bernoulli polynomials, difference operators, and of the Stirling polynomials of the second kind. In addition, we introduced a degenerate version of the integer power sum polynomials, called the degenerate integer power sum polynomials and were able to find several representations of them. In detail, they can be represented in terms of Carlitz degenerate Bernoulli polynomials, difference operators, and of the degenerate Stirling numbers of the second kind.

In the final section, we considered an infinite family of random variables and proved that the expectations of them are expressed in terms of the degenerate Stirling polynomials of the second and some value of higher-order Bernoulli polynomials.

Most of the results in Sections 1 and 2 are reviews of known results, other than that, we demonstrated the usefulness of the p-adic invariant integrals in the study of integer power sum polynomials. However, we emphasize that the results in Sections 3 and 4 are new. In particular, we showed that the degenerate Stirling polynomials of the second kind, introduced as a degenerate version of the Stirling polynomials of the second kind, appear naturally and meaningfully in the context of calculations of an infinite family of random variables (see (62)). We also showed that they appear in an expression of the degenerate integer power sum polynomials (Remark 2) which is a degenerate version of the integer power sum polynomials (see (26)).

We have witnessed in recent years that studying various degenerate versions of some old and new polynomials, initiated by Carlitz in the classical papers [3,15], is very productive and promising (see [3,5,14,15,18,19] and references therein). Lastly, we note that this idea of considering degenerate versions of some polynomials extended even to transcendental functions like the gamma functions (see [19]).

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References

1. Kim, T. q-Volkenborn integration. *Russ. J. Math. Phys.* **2002**, *9*, 288–299.
2. Schikhof, W.H. *Ultrametric Calculus: An Introduction to p-Adic Analysis*; Cambridge Studies in Advanced Mathematics, 4; Cambridge University Press: Cambridge, UK, 1984.
3. Carlitz, L. Degenerate Stirling, Bernoulli and Eulerian numbers. *Utilitas Math. J.* **1979**, *15*, 51–88.
4. El-Desoulky, B.S.; Mustafa, A. New results on higher-order Daehee and Bernoulli numbers and polynomials. *Adv. Differ. Equ.* **2016**, *2016*, 32. [CrossRef]
5. Kim, T.; Kim, D.S. Identities for degenerate Bernoulli polynomials and Korobov polynomials of the first kind. *Sci. China Math.* **2019**, *62*, 999–1028. [CrossRef]
6. Araci, S.; Özer, O. Extened q-Dedekind-type Daehee-Changhee sums associated with q-Euler polynomials. *Adv. Differ. Equ.* **2015**, *2015*, 272. [CrossRef]
7. Kim, T. Symmetry of power sum polynomials and multivariate fermionic p-adic invariant integral on $\mathbb{Z}_p$. *Russ. J. Math. Phys.* **2009**, *16*, 93–96. [CrossRef]
8. Kim, T. Sums of powers of consecutive q-integers. *Adv. Stud. Contemp. Math. (Kyungshang)* **2004**, *9*, 15–18.
9. Kim, T. A note on exploring the sums of powers of consecutive q-integers. *Adv. Stud. Contemp. Math. (Kyungshang)* **2005**, *11*, 137–140.
10. Kim, T. On the alternating sums of powers of consecutive integers. *J. Anal. Comput.* **2005**, *1*, 117–120.
11. Rim, S.-H.; Kim, T.; Ryoo, C.S. On the alternating sums of powers of consecutive q-integers. *Bull. Korean Math. Soc.* **2006**, *43*, 611–617. [CrossRef]
12. Ryoo, C.S.; Kim, T. Exploring the q-analogues of the sums of powers of consecutive integers with Mathematica. *Adv. Stud. Contemp. Math. (Kyungshang)* **2009**, *18*, 69–77.
13. Simsek, Y.; Kim, D.S.; Kim, T.; Rim, S.-H. A note on the sums of powers of consecutive q-integers. *J. Appl. Funct. Differ. Equ.* **2016**, *1*, 81–88.
14. Kim, T. A note on degenerate Stirling polynomials of the second kind. *Proc. Jangjeon Math. Soc.* **2017**, *20*, 319–331.
15. Carlitz, L. A degenerate Staudt-Clausen theorem. *Arch. Math. (Basel)* **1956**, *7*, 28–33. [CrossRef]
16. Kim, T.; Yao, Y.; Kim, D.S.; Kwon, H.I. Some identities involving special numbers and moments of random variables. *Rocky Mt. J. Math.* **2019**, *49*, 521–538. [CrossRef]
17. Liu, C.; Bao, W. Application of probabilistic method on Daehee sequences. *Eur. J. Pure Appl. Math.* **2018**, *11*, 69–78.

[CrossRef]

18. Kim, T.; Yao, Y.; Kim, D.S.; Jang, G.-W. Degenerate *r*-Stirling numbers and *r*-Bell polynomials. *Russ. J. Math. Phys.* **2018**, *25*, 44–58. [CrossRef]
19. Kim, T.; Kim, D.S. Degenerate Laplace transform and degenerate gamma function. *Russ. J. Math. Phys.* **2017**, *24*, 241–248. [CrossRef]