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Some Novel Picture 2-Tuple Linguistic Maclaurin Symmetric Mean Operators and Their Application to Multiple Attribute Decision Making

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Abstract: When solving multiple attribute decision making (MADM) problems, the 2-tuple linguistic variable is an effective tool that can not only express complex cognitive information but also prevent loss of information in calculation. The picture fuzzy set (PFS) has three degrees and has more freedom to express cognitive information. In addition, Archimedean *t*-conorm and *t*-norm (ATT) can generalize most existing *t*-conorms and *t*-norms and Maclaurin symmetric mean (MSM) operators can catch the relationships among the multi-input parameters. Therefore, we investigate several novel aggregation operators, such as the picture 2-tuple linguistic MSM (2TLMSM) operator based on the ATT (ATT-P2TLMSM) and the picture 2-tuple linguistic generalized MSM (2TLGMSM) operator based on ATT (ATT-P2TLGMSM). Considering that the input parameters have different importance, we proposed picture 2-tuple linguistic weighted MSM (2TLWMSM) operators based on ATT (ATT-P2TLWMSM) and picture 2-tuple linguistic weighted generalized MSM (2TLWGMSM) operators based on ATT (ATT-P2TLWGMSM). Finally, a MADM method is introduced, and an expositive example is presented to explain the availability and applicability of the developed operators and methods.

Keywords: picture fuzzy set; 2-tuple linguistic variable; MSM operators; ATT; MADM

1. Introduction

The multiple attribute decision making (MADM) problem is a significant area of decision science, whose theories and methods are widely used in engineering, economics, management, the military and many other fields. Generally, decision makers will provide an evaluation of each alternative for every attribute or criterion according to their own cognitive beliefs. The main task of solving MADM problems is sorting a group of choices and finding the best one based on decision information provided by decision makers.

Since being proposed by Zadeh [1] in 1965, fuzzy theory has been widely used with applications in various areas. However, the fuzzy theory can only express membership—Non-membership cannot be represented. The intuitionistic fuzzy set (IFS) presented by Atanassov [2], which is an important extension of traditional fuzzy sets, contains membership degree and non-membership degree. Because of the ambiguity of objects and the uncertainty of human thought and cognition, decision makers have difficulty using crisp numbers to evaluate relevant decision making problems, such as student assessment and car performance evaluation. Decision makers are more accustomed to making evaluations directly in linguistic terms, such as good, generally, and not good. Therefore, many methods and models have been developed to solve real problems based on linguistic variables [3–6]. For example, since the type 2 fuzzy sets (T2FSs) could better represent the indeterminacy and simplify the calculation process, the interval type 2 hesitant fuzzy sets (IT2HFSs) can reflect the uncertainty of inaccurate information more effectively. Deveci et al. [7] proposed a method including T2FSs and the IT2HFSs to access airlines' service quality. Finally, accurate data results and practical implications were obtained.

Herrera and Martinez [8] proposed the concept of 2-tuples made up of a linguistic variable and a numerical value to prevent loss of information when addressing MADM problems. Subsequently, many operators and methods based on 2-tuples have been proposed. On the basis of the power average (PA) operator, Xu and Wang [9] studied several 2-tuple linguistic power average (2TLPA) operators which could alleviate the impact of partial arguments on the aggregated consequences. Furthermore, the method proposed in the paper considered all the decision parameters and the interrelationships of each other. However, there is a slight disadvantage in that it ignores the relationship between the two parameters. In addition, considering the significance of different parameters, a 2-tuple linguistic weighted PA (2TLWPA) operator was proposed. Wei and Zhao [10] came up with series aggregation operators according to 2-tuple linguistic information and a dependent operator that eliminates the influence of unjust 2-tuple linguistic parameters on the aggregation results. Jiang and Wei [11] developed a 2-tuple linguistic Bonferroni mean (2TLBM) operator based on the Bonferroni mean (BM)operator and a 2-tuple linguistic weighted BM (2TLWBM) operator to account for the different importance of the input parameters. Merigó et al. [12] introduced some aggregation operators based on 2-tuple linguistic information that provide a more complete understanding of the situation being considered. Moreover, the authors also studied the applicability of the novel method in different fields. A modified composite scale that can enhance the precision of decision making was developed by Wang et al. [13] to overcome the limitation of the 2-tuple linguistic representation model. Qin and Liu [14] proposed several operators based on 2-tuple linguistic information and the Muirhead mean (MM) operator. It is known as a mean type aggregation operator that can utilize the intact relation between the multi-input parameters. Meanwhile, they applied the method proposed in the paper for supplier selection.

As the decision environment and content become increasingly complex, the use of 2-tuple linguistic variables alone fails to accurately describe ambiguous and fragmentary cognitive information. Cuong [15] developed the picture fuzzy set (PFS) to express uncertain cognitive information characterized by three degrees: A positive membership degree $\mu(x)$, a neutral membership degree $\nu(x)$ and a negative membership degree $\eta(x)$. Therefore, PFS allows several types of answers when solving decision making problems, such as yes, abstain, no, and refusal. Many research achievements have been made in the field of PFS theory. Singh [16] applied the correlation coefficient to clustering analysis where the attribute values are in the form of PFS. Because the PFS contains more information about people's evaluation than IFS, the proposed correlation coefficients are a further generalization of IFSs. Yang et al. [17] proposed picture fuzzy soft sets and studied their relevant properties. In particular, there is a method based on adjustable soft discernibility matrix which could obtain a sequential relationship between all objects. Son [18] proposed a generalized distance measure for pictures and the method of hierarchical picture clustering (HPC). Wei [19] proposed picture fuzzy cross entropy to address the MADM problem which can reflect the fuzziness of subjective judgment easily. Thong and Son [20] developed a novel hybrid model including picture fuzzy clustering and intuitionistic fuzzy recommender systems that are applicable to health care support systems. These models not only improve the accuracy of medical diagnosis but also guarantee the development of a medical security system. But the limitations of these models are the time complexity and the capability of the model when new patients are added to the system.

Archimedean t-norms and t-conorms (ATT) are types of t-norms and t-conorms that have become important tools for explaining the conjunction, and the operational rules have been defined. Beliakov et al. [21] used ATT to calculate the IFS, thus simplifying and extending the existing constructions. Liu [22] developed single-valued neutrosophic number operators based on ATT which are able to extend to most of the existing t-norms and t-conorms and single-valued neutrosophic numbers (SVNNs). Liu [23] developed some operators based on ATT and PFS and studied several properties and particular cases of the operators.

The aggregation operator is a crucial tool for addressing MADM problems. Many effective aggregation operators have been developed for situations where the input arguments have some relations. Yager [24] introduced the PA operator. In the process of aggregation, parameter values support each other. Tan and Chen [25] investigated the induced Choquet ordered averaging (I-COA) operator and demonstrated its relationship to the induced ordered weighted averaging operator. Bonferroni [26] developed the Bonferroni mean (BM) operator, which can effectively address the relationships among input parameters. Liu et al. [27] presented several intuitionistic uncertain linguistic Bonferroni OWA (IULBOWA) operators that can aggregate max and min operators and introduced relevant score functions, accuracy functions, and comparative methods. Li and Liu [28] proposed novel aggregation operators according to the Heronian mean (HM) operator that considered the interrelationships of attribute values. BM and HM operators can only account for relationships between input arguments and not the correlation between multiple arguments. To overcome this limitation, Maclaurin [29] proposed a Maclaurin symmetric mean (MSM) operator to capture the relationships among multiple input arguments. Qin and Liu [30] solved MADM problems based on MSM operators under a hesitant fuzzy environment. Wang et al. [31] extended MSM aggregation operators with single-valued neutrosophic linguistic variables and developed methods for multiple-criteria decision making (MCDM). Wei and Lu [32] proposed the Pythagorean fuzzy MSM (PFMSM) and Pythagorean fuzzy weighted MSM (PFWMSM) operators and discussed their desirable properties. Liu and Zhang [33] extended MSM operators with the single-valued trapezoidal neutrosophic number (SVTNNs) to not only account for the correlation between multi-input arguments but also conveniently depict uncertain information in the decision making process.

Inspired by the above analysis, although the PFS can address complex and uncertain problem flexibly, PFS has difficulty expressing cognitive information. Therefore, we use the picture 2-tuple linguistic set based on PFS and 2-tuple linguistic information to address MADM problems, thereby overcoming the above limitation and preventing loss of information in the calculation and aggregation processes. In addition, we apply the ATT to address MADM problems described by picture 2-tuple linguistic numbers (P2TLNs). Then, we extend MSM operators under a P2TLN environment, such as ATT-P2TLMSM and ATT-P2TLGMSM, to capture the interrelationships among multiple input parameters. In cases where the input parameters have different significances, ATT-P2TLWMSM and ATT-P2TLGWMSM are proposed. Based on the above operators, we propose a method to handle MADM problems.

The framework of this paper is as follows. In Section 2, we explain several basic concepts and theories. In Section 3, a novel operation for picture 2-tuple linguistic sets based on ATT is proposed. In Section 4, we develop novel P2TLMSM operators. In Section 5, we present models based on the ATT-P2TLWMSM operator and the ATT-P2TLGWMSM operator to solve the MADM problems. Finally, an expositive instance is provided in Section 6.

2. Preliminaries

2.1. 2-Tuple Linguistic Term Sets

Let $S = \{s_i | i = 1, 2, \ldots, l\}$ be a linguistic term set, where l is an odd number; for instance, when $l = 7$, S is defined as $S = \{s_1 = particularly\ bad, s_2 = bad, s_3 = slightly\ bad, s_4 = medium, s_5 = slightly\ good, s_6 = good\}, s_7 = particularly$ *good*. Every label s_i of S is a feasible value of a linguistic variable, and the following characteristics should be satisfied [6,28,34]:

(1) If and only if $i < j$, then $s_i < s_j$.
(2) $neg(s_i) = s_{l-i-1}$ is the negation operator.
(3) If $i \geq j$, then $\max(s_i, s_j) = s_i$.
(4) If $i \leq j$, then $\min(s_i, s_j) = s_i$.

To process fuzzy linguistic information effectively, Herrera and Martinez [8] developed a 2-tuple fuzzy linguistic representation model via symbolic translation to perform computations with words. Next, we provide several relevant definitions.

Definition 1. *[8,28,35] Let $S = (s_0, s_1, \ldots, s_{l-1})$ be a linguistic term set and β be a real number in $[0, l-1]$ based on the calculation result of a symbolic aggregation operation. Then, the function Δ used to obtain the 2-tuple corresponding to the elements in S is defined as:*

$$\Delta : [0, l-1] \to S \times [-0.5, 0.5)$$
$$\Delta(\beta) = (s_i, \alpha) \tag{1}$$

where $i = round(\beta)$, $\alpha = \beta - i$, $\alpha \in [-0.5, 0.5)$, and $round(.)$ is the usual rounding operation.

Definition 2. *[8,28,35] Let $S = (s_0, s_1, \ldots, s_{l-1})$ be a linguistic term set and (s_i, α_i) be a 2-tuple. There is a function Δ^{-1} that can obtain $\beta \in [0, l-1]$ based on the 2-tuple, that is:*

$$\Delta^{-1} : S \times [-0.5, 0.5) \to [0, l-1]$$
$$\Delta^{-1}(s_i, \alpha) = i + \alpha = \beta \tag{2}$$

From the above definitions, it is clear that a value of 0 must be added as symbolic translation to convert a linguistic term into a linguistic 2-tuple:

$$\Delta(s_i) = (s_i, 0) \tag{3}$$

2.2. Picture Fuzzy Set

Definition 3. *[15] Let X be a fixed universe; a picture fuzzy set (PFS) P on X is defined as $P = \{\langle x, \mu_P(x), \eta_P(x), \nu_P(x)\rangle \big| x \in X\}$, where $\mu_P(x) \in [0,1]$, $\eta_P(x) \in [0,1]$, and $\nu_P(x) \in [0,1]$ are the positive membership, neutral membership, and negative membership of P, respectively, which satisfy $0 \le \mu_P(x) + \eta_P(x) + \nu_P(x) \le 1$. Furthermore, $\pi(x) = 1 - \mu_P(x) - \eta_P(x) - \nu_P(x)$ is the degree of refusal membership of P for all x. For convenience, the set $P = \langle \mu_P, \eta_P, \nu_P \rangle$ is called a picture fuzzy number (PFN).*

Cuong [15] defined several operations as follows.

Definition 4. *[15] Suppose P and G are two PFNs; then,*

(1) $P \subseteq G$ *if* $\mu_P(x) \le \mu_G(x)$, $\eta_P(x) \le \eta_G(x)$, $\nu_P(x) \ge \nu_G(x)$, $\forall x \in X$;
(2) $P \cup G = \{(x, \max(\mu_P(x), \mu_G(x)), \min(\eta_P(x), \eta_G(x)), \min(\nu_P(x), \nu_G(x))) \big| x \in X\}$;
(3) $P \cap G = \{(x, \min(\mu_P(x), \mu_G(x)), \max(\eta_P(x), \eta_G(x)), \max(\nu_P(x), \nu_G(x))) \big| x \in X\}$;
(4) $\overline{P} = \{(x, \nu_P(x), \eta_P(x), \mu_P(x)) \big| x \in X\}$.

Based on Definition 5, Wei [36] proposed the following operational rules for PFNs.

Definition 5. *[36] Suppose P and G are two PFNs; then,*

(1) $P \oplus G = (\mu_P + \mu_G - \mu_P\mu_G,\ \eta_P\eta_G,\ \nu_P\nu_G)$;
(2) $P \otimes G = (\mu_P\mu_G,\ \eta_P + \eta_G - \eta_P\eta_G,\ \nu_P + \nu_G - \nu_P\nu_G)$;
(3) $\lambda P = \left(1 - (1-\mu_P)^\lambda,\ {\eta_P}^\lambda,\ {\nu_P}^\lambda\right)$, $\lambda > 0$;
(4) $P^\lambda = \left({\mu_P}^\lambda, 1 - (1-\eta_P)^\lambda, 1 - (1-\nu_P)^\lambda\right)$, $\lambda > 0$.

2.3. Archimedean T-Norm and T-Conorm

Definition 6. *[37,38]. T: $[0,1]\times[0,1]\rightarrow[0,1]$ is called a t-norm if T satisfies the following four axioms:*

(1) $T(x,0)=0$ *and* $T(1,x)=x$ *for all x;*
(2) $T(x,y)=T(y,x)$ *for all x and y;*
(3) $T(x,y)\leq T(x',y')$ *if* $x\leq x'$ *and* $y\leq y'$*;*
(4) $T(T(x,y),z)=T(x,T(y,z))$ *for all x, y and z.*

Definition 7. *[37,38]* T^**:* $[0,1]\times[0,1]\rightarrow[0,1]$ *is called a t-conorm if* T^* *satisfies the following four axioms:*

(1) $T^*(0,x)=x$ *and* $T^*(1,x)=1$ *for all x;*
(2) $T^*(x,y)=T^*(y,x)$ *for all x and y;*
(3) $T^*(x,y)\leq T^*(x',y')$ *if* $x\leq x'$ *and* $y\leq y'$*;*
(4) $T^*(T^*(x,y),z)=T^*(x,T^*(y,z))$ *for all x, y and z.*

If $T(x,y)$ and $T^*(x,y)$ are continuous, and $T(x,x)<x$ and $T^*(x,x)>x$ for all $x\in(0,1)$, then we call $T(x,y)$ and $T^*(x,y)$ the Archimedean t-norm and t-conorm, respectively. We can obtain a strict Archimedean t-norm by $g(x)$ as $T(x,y)=g^{-1}(g(x)+g(y))$, where $g(x)$ is a strictly decreasing function $g:[0,1]\rightarrow[0,\infty]$ such that $g(1)=0$. Similarly, we can obtain $T^*(x,y)=f^{-1}(f(x)+f(y))$ and $f(x)=g(1-x)$.

Based on the above definitions, Tao et al. [39] developed several algebraic operations.

Definition 8. *[40] Let x, y∈ $[0,1]$ and $\lambda>0$ be a scalar; then,*

(1) $x\oplus y=T^*(x,y)=f^{-1}(f(x)+f(y))$*;*
(2) $x\otimes y=T(x,y)=g^{-1}(g(x)+g(y))$*;*
(3) $\lambda\odot x=f^{-1}(\lambda f(x))$*;*
(4) $x^{\lambda}=g^{-1}(\lambda g(x))$*.*

Some special cases exist when the generator function is set to different values:

(1) Let $g(t)=-\log t$, $f(t)=-\log(1-t)$, $g^{-1}(t)=e^{-t}$, and $f^{-1}(t)=1-e^{-t}$. Then, the algebraic t-norm and t-conorm are obtained: $T_A(x,y)=xy$, ${T_A}^*(x,y)=x+y-xy$.
(2) Let $g(t)=\log\left(\frac{2-t}{t}\right)$, $f(t)=\log\left(\frac{2-(1-t)}{1-t}\right)$, $g^{-1}(t)=\frac{2}{e^t+1}$, and $f^{-1}(t)=1-\frac{2}{e^t+1}$. Then, the Einstein t-norm and t-conorm are obtained: $T_E(x,y)=\frac{xy}{1+(1-x)(1-y)}$, ${T_E}^*(x,y)=\frac{x+y}{1+xy}$.
(3) Let $g(t)=\log\left(\frac{\gamma-(1-\gamma)t}{t}\right)$, $f(t)=\log\left(\frac{\gamma-(1-\gamma)(1-t)}{1-t}\right)$, $g^{-1}(t)=\frac{\gamma}{e^t+\gamma-1}$, $f^{-1}(t)=1-\frac{\gamma}{e^t+\gamma-1}$, and $\gamma>0$. Then, the Hamacher t-norm and t-conorm are obtained: $T_H(x,y)=\frac{xy}{\gamma+(1-\gamma)(x+y-xy)}$, ${T_H}^*(x,y)=\frac{x+y-xy-(1-\gamma)xy}{1-(1-\gamma)xy}$, $\gamma>0$.

2.4. MSM Operators

Definition 9. *[29,31] Let $x_i(i=1,2,\ldots,n)$ be the collection of nonnegative real numbers. An MSM operator of dimension n is a mapping* $MSM^{(m)}:(R^+)^n\rightarrow R^+$ *such that:*

$$MSM^{(m)}(x_1,\ldots,x_n)=\left(\frac{\sum_{1\leq i_1<\ldots<i_m\leq n}\prod_{j=1}^{m}x_{i_j}}{C_n^m}\right)^{\frac{1}{m}} \tag{4}$$

where $(i_1, i_2, \ldots, i_m)$ *traverses all m-tuple combinations of* $(1, 2, \ldots, n)$ *and* $C_n^m = \frac{n!}{m!(n-m)!}$ *is the binomial coefficient. In the following analysis, assume that* $i_1 < i_2 < \ldots < i_m$*. Furthermore,* x_{i_j} *represents the* i_j *th element of a specific arrangement.*

Clearly, the $MSM^{(m)}$ operator has the following important attributes.

(1) Idempotency. If $x_i = x$ for each i, $MSM^{(m)}(x, x, \ldots, x) = x$;
(2) Monotonicity. If $x_i \le y_i$ for all i, $MSM^{(m)}(x_1, x_2, \ldots, x_n) \le MSM^{(m)}(y_1, y_2, \ldots, y_n)$;
(3) Boundedness. Min $\{x_1, x_2, \ldots, x_n\} \le MSM^{(m)}\{x_1, x_2, \ldots, x_n\} \le$ max $\{x_1, x_2, \ldots, x_n\}$.

Definition 10. *[31] Let* $x_i (i = 1, 2, \ldots, n)$ *be the collection of nonnegative real numbers and* $p_1, p_2, \ldots, p_m \ge 0$*. A generalized MSM operator of dimension n is a mapping* $GMSM^{(m,p1,p2,\ldots,pm)} : (R^+)^n \to R^+$ *such that*

$$GMSM^{(m,p_1,p_2,\ldots,p_m)}(x_1, \ldots, x_n) = \left(\frac{\sum_{1 \le i_1 < \ldots < i_m \le n} \prod_{j=1}^{m} x_{i_j}^{p_j}}{C_n^m} \right)^{\frac{1}{p_1+p_2+\ldots p_m}} \quad (5)$$

where $(i_1, i_2, \ldots, i_m)$ *traverses all m-tuple combinations of* $(1, 2, \ldots, n)$ *and* $C_n^m = \frac{n!}{m!(n-m)!}$ *is the binomial coefficient.*

It can easily be seen that the $GMSM^{(m,p_1,p_2,\ldots,p_m)}$ operator has the following desirable properties.

(1) Idempotency. If $x_i = x$ for each i, then $GMSM^{(m,p_1,p_2,\ldots,p_m)}(x, x, \ldots, x) = x$;
(2) Monotonicity. If $x_i \le y_i$ for all i, $GMSM^{(m,p_1,p_2,\ldots,p_m)}(x_1, x_2, \ldots, x_n) \le GMSM^{(m,p_1,p_2,\ldots,p_m)}(y_1, y_2, \ldots, y_n)$;
(3) Boundedness. $\min\{x_1, x_2, \ldots x_n\} \le GMSM^{(m,p_1,p_2,\ldots,p_m)}(x_1, x_2, \ldots x_n) \le \max\{x_1, x_2, \ldots x_n\}$.

Definition 11. *[31] Let* $x_i (i = 1, 2, \ldots, n)$ *be the collection of nonnegative real numbers and* $p_1, p_2, \ldots, p_m \ge 0$*. A geometric MSM operator of dimension n is a mapping* $G_{eo}MSM^{(m,p_1,p_2,\ldots,p_m)} : (R^+)^n \to R^+$ *such that*

$$G_{eo}MSM^{(m,p_1,p_2,\ldots,p_m)}(x_1, \ldots, x_n) = \frac{1}{(p_1+p_2+\ldots+p_m)} \left(\prod_{1 \le i_1 < \ldots < i_m \le n} \left(p_1 x_{i_1} + p_2 x_{i_2} + \ldots + p_m x_{i_m} \right) \right)^{\frac{1}{C_n^m}} \quad (6)$$

where $(i_1, i_2, \ldots, i_m)$ *traverses all m-tuple combinations of* $(1, 2, \ldots, n)$ *and* $C_n^m = \frac{n!}{m!(n-m)!}$ *is the binomial coefficient.*

Clearly, the $G_{eo}MSM^{(m,p_1,p_2,\ldots,p_m)}$ operator has the following desirable properties.

(1) Idempotency. If $x > 0$ and $x_i = x$ for each i, then $G_{eo}MSM^{(m,p_1,p_2,\ldots,p_m)}(x, x, \ldots, x) = x$;
(2) Monotonicity. If $x_i \le y_i$ for all i, $G_{eo}MSM^{(m,p_1,p_2,\ldots,p_m)}(x_1, x_2 \ldots, x_n) \le G_{eo}MSM^{(m,p_1,p_2,\ldots,p_m)}(y_1, y_2 \ldots, y_n)$;
(3) Boundedness. $\min\{x_1, x_2, \ldots x_n\} \le G_{eo}MSM^{(m,p_1,p_2,\ldots,p_m)}(x_1, x_2, \ldots x_n) \le \max\{x_1, x_2, \ldots x_n\}$.

3. Picture 2-Tuple Linguistic Sets and a New Operation

3.1. Picture 2-Tuple Linguistic Sets

In this section, we introduce several concepts of picture 2-tuple linguistics based on PFS and 2-tuple linguistic term sets.

Definition 12. *[36,41] $A = \{(s_i, \alpha), (\mu_A(x), \eta_A(x), \nu_A(x)), x \in X\}$ is called a picture 2-tuple linguistic set if $\mu_A(x) \in [0,1]$, $\eta_A(x) \in [0,1]$, $\nu_A(x) \in [0,1]$, $0 \le \mu_A(x) + \eta_A(x) + \nu_A(x) \le 1$, $s_i \in S$ and $\alpha \in [-0.5, 0.5)$. $\mu_A(x), \eta_A(x), \nu_A(x)$ are the degrees of positive membership, neutral membership and negative membership, respectively of element x to (s_i, α). Then, $\pi(x) = 1 - \mu_P(x) - \eta_P(x) - \nu_P(x)$ is the degree of refusal membership of element x to (s_i, α) for all x.*

For convenience, we call $\Phi = \left\langle (s_i, \alpha), \left(\mu_\varphi, \eta_\varphi, \nu_\varphi\right) \right\rangle$ a picture 2-tuple linguistic number (P2LTN), and $\mu_\varphi \in [0,1]$, $\eta_\varphi \in [0,1]$, $\nu_\varphi \in [0,1]$, $\mu_\varphi + \eta_\varphi + \nu_\varphi \le 1$, $s_i \in S$ and $\alpha \in [-0.5, 0.5)$.

Definition 13. *[36,41] Let $\Phi = \left\langle (s_i, \alpha), \left(\mu_\varphi, \eta_\varphi, \nu_\varphi\right) \right\rangle$ be a P2TLN; the score function S of a P2TLN can be defined as:*

$$S(\Phi) = \Delta\left(\Delta^{-1}(s_i, \alpha) \cdot \frac{1 + \mu_\varphi - \nu_\varphi}{2}\right), \Delta^{-1}(S(\Phi)) \in [0, l-1] \tag{7}$$

Definition 14. *[36,41] Let $\Phi = \left\langle (s_i, \alpha), \left(\mu_\varphi, \eta_\varphi, \nu_\varphi\right) \right\rangle$ be a P2TLN; the accuracy function H of a P2TLN can be defined as:*

$$H(\Phi) = \Delta\left(\Delta^{-1}(s_i, \alpha) \cdot \frac{\mu_\varphi + \eta_\varphi + \nu_\varphi}{2}\right), \Delta^{-1}(H(\Phi)) \in [0, l-1] \tag{8}$$

Based on Definitions 13 and 14, Wei [36,41] presented a related comparison of P2TLN as follows.

Definition 15. *[36,41] Let $\Phi_1 = \left\langle (s_{i1}, \alpha_1), \left(\mu_{\varphi 1}, \eta_{\varphi 1}, \nu_{\varphi 1}\right) \right\rangle$ and $\Phi_2 = \left\langle (s_{i2}, \alpha_2), \left(\mu_{\varphi 2}, \eta_{\varphi 2}, \nu_{\varphi 2}\right) \right\rangle$ be two P2TLNs, $S(\Phi_1)$ and $S(\Phi_2)$ be the scores of Φ_1 and Φ_2, and $H(\Phi_1)$ and $H(\Phi_2)$ be the accuracy degrees of Φ_1 and Φ_2, respectively.*

If $S(\Phi_1) \prec S(\Phi_2)$, then Φ_1 is smaller than Φ_2, denoted by $\Phi_1 \prec \Phi_2$.
If $S(\Phi_1) = S(\Phi_2)$,

(a) If $H(\Phi_1) \prec H(\Phi_2)$, Φ_1 is smaller than Φ_2, denoted by $\Phi_1 \prec \Phi_2$.
(b) If $H(\Phi_1) = H(\Phi_2)$, Φ_1 is the same as Φ_2, denoted by $\Phi_1 = \Phi_2$.
(c) If $H(\Phi_1) \succ H(\Phi_2)$, Φ_1 is larger than Φ_2, denoted by $\Phi_1 \succ \Phi_2$.

3.2. New Operations for Picture 2-Tuple Linguistic Sets Based on ATT

In this section, inspired by the new operational laws of 2-tuple linguistic and picture fuzzy linguistic sets, we introduce the operational rules of P2TLNs based on ATT.

Definition 16. *Let $\Phi_1 = \left\langle (s_{i1}, \alpha_1), \left(\mu_{\varphi 1}, \eta_{\varphi 1}, \nu_{\varphi 1}\right) \right\rangle$ and $\Phi_2 = \left\langle (s_{i2}, \alpha_2), \left(\mu_{\varphi 2}, \eta_{\varphi 2}, \nu_{\varphi 2}\right) \right\rangle$ be two P2TLNs; then,*

(1) $\Phi_1 \oplus \Phi_2 = \left\langle \Delta\left((l-1) f^{-1}\left(f\left(\frac{\Delta^{-1}(s_{i1},\alpha_1)}{l-1}\right) + f\left(\frac{\Delta^{-1}(s_{i2},\alpha_2)}{l-1}\right)\right)\right), \left(\begin{array}{l} f^{-1}\left(f\left(\mu_{\varphi_1}\right) + f\left(\mu_{\varphi_2}\right)\right), g^{-1}\left(g\left(\eta_{\varphi_1}\right) + g\left(\eta_{\varphi_2}\right)\right), \\ g^{-1}\left(g\left(\nu_{\varphi_1}\right) + g\left(\nu_{\varphi_2}\right)\right) \end{array}\right) \right\rangle$;

(2) $\Phi_1 \otimes \Phi_2 = \left\langle \Delta\left((l-1) g^{-1}\left(g\left(\frac{\Delta^{-1}(s_{i1},\alpha_1)}{l-1}\right) + g\left(\frac{\Delta^{-1}(s_{i2},\alpha_2)}{l-1}\right)\right)\right), \left(\begin{array}{l} g^{-1}\left(g\left(\mu_{\varphi_1}\right) + g\left(\mu_{\varphi_2}\right)\right), f^{-1}\left(f\left(\eta_{\varphi_1}\right) + f\left(\eta_{\varphi_2}\right)\right), \\ f^{-1}\left(f\left(\nu_{\varphi_1}\right) + f\left(\nu_{\varphi_2}\right)\right) \end{array}\right) \right\rangle$;

(3) $\lambda \odot \Phi_1 = \left\langle \Delta\left((l-1) f^{-1}\left(\lambda f\left(\frac{\Delta^{-1}(s_{i1},\alpha_1)}{l-1}\right)\right)\right), \left(f^{-1}\left(\lambda f\left(\mu_{\varphi_1}\right)\right), g^{-1}\left(\lambda g\left(\eta_{\varphi_1}\right)\right), g^{-1}\left(\lambda g\left(\nu_{\varphi_1}\right)\right)\right) \right\rangle$;

(4) $\Phi_1{}^\lambda = \left\langle \Delta\left((l-1) g^{-1}\left(\lambda g\left(\frac{\Delta^{-1}(s_{i1},\alpha_1)}{l-1}\right)\right)\right), \left(g^{-1}\left(\lambda g\left(\mu_{\varphi_1}\right)\right), f^{-1}\left(\lambda f\left(\eta_{\varphi_1}\right)\right), f^{-1}\left(\lambda f\left(\nu_{\varphi_1}\right)\right)\right) \right\rangle$.

If $g(x)$ takes different forms, we obtain some special cases.

Case 1. *If* $g(x) = -\log x$, *then we have:*

(1) $$\Phi_1 \oplus_A \Phi_2 = \left\langle \Delta\left(\Delta^{-1}(s_{i1}, \alpha_1) + \Delta^{-1}(s_{i2}, \alpha_2) - \frac{\Delta^{-1}(s_{i1},\alpha_1)\cdot\Delta^{-1}(s_{i2},\alpha_2)}{l-1}\right), \left(\mu_{\varphi_1} + \mu_{\varphi_2} - \mu_{\varphi_1}\mu_{\varphi_2}, \eta_{\varphi_1}\eta_{\varphi_2}, \nu_{\varphi_1}\nu_{\varphi_2}\right)\right\rangle;$$

(2) $$\Phi_1 \otimes_A \Phi_2 = \left\langle \Delta\left(\frac{\Delta^{-1}(s_{i1},\alpha_1)\cdot\Delta^{-1}(s_{i2},\alpha_2)}{l-1}\right), \left(\mu_{\varphi_1}\mu_{\varphi_2}, \eta_{\varphi_1} + \eta_{\varphi_2} - \eta_{\varphi_1}\eta_{\varphi_2}, \nu_{\varphi_1} + \nu_{\varphi_2} - \nu_{\varphi_1}\nu_{\varphi_2}\right)\right\rangle;$$

(3) $$\lambda \odot_A \Phi_1 = \left\langle \Delta\left((l-1)\left(1 - \left(1 - \left(\frac{\Delta^{-1}(s_{i1},\alpha_1)}{l-1}\right)^{\lambda}\right)\right)\right), \left(1 - \left(1 - \mu_{\varphi_1}\right)^{\lambda}, \left(\eta_{\varphi_1}\right)^{\lambda}, \left(\nu_{\varphi_1}\right)^{\lambda}\right)\right\rangle;$$

(4) $$\Phi_1{}^{\lambda} = \left\langle \Delta\left((l-1)\left(\frac{\Delta^{-1}(s_{i1},\alpha_1)}{l-1}\right)^{\lambda}\right), \left(\left(\mu_{\varphi_1}\right)^{\lambda}, 1 - \left(1 - \eta_{\varphi_1}\right)^{\lambda}, 1 - \left(1 - \nu_{\varphi_1}\right)^{\lambda}\right)\right\rangle.$$

Case 2. *If* $g(x) = \log\left(\frac{2-x}{x}\right)$, *then we obtain the following formulas:*

(1) $$\Phi_1 \oplus_{\mathrm{E}} \Phi_2 = \left\langle \Delta\left((l-1)^2 \frac{\Delta^{-1}(s_{i1},\alpha_1)+\Delta^{-1}(s_{i2},\alpha_2)}{(l-1)^2+\Delta^{-1}(s_{i1},\alpha_1)\cdot\Delta^{-1}(s_{i2},\alpha_2)}\right), \left(\frac{\mu_{\varphi_1}+\mu_{\varphi_2}}{1+\mu_{\varphi_1}\mu_{\varphi_2}}, \frac{\eta_{\varphi_1}\eta_{\varphi_2}}{1+\left(1-\eta_{\varphi_1}\right)\left(1-\eta_{\varphi_2}\right)}, \frac{\nu_{\varphi_1}\nu_{\varphi_2}}{1+\left(1-\nu_{\varphi_1}\right)\left(1-\nu_{\varphi_2}\right)}\right)\right\rangle;$$

(2) $$\Phi_1 \otimes_{\mathrm{E}} \Phi_2 = \left\langle \Delta\left((l-1) \frac{\Delta^{-1}(s_{i1},\alpha_1)\cdot\Delta^{-1}(s_{i2},\alpha_2)}{(l-1)^2+\left((l-1)-\Delta^{-1}(s_{i1},\alpha_1)\right)\left((l-1)-\Delta^{-1}(s_{i2},\alpha_2)\right)}\right), \left(\frac{\mu_{\varphi_1}\mu_{\varphi_2}}{1+\left(1-\mu_{\varphi_1}\right)\left(1-\mu_{\varphi_2}\right)}, \frac{\eta_{\varphi_1}+\eta_{\varphi_2}}{1+\eta_{\varphi_1}\eta_{\varphi_2}}, \frac{\nu_{\varphi_1}+\nu_{\varphi_2}}{1+\nu_{\varphi_1}\nu_{\varphi_2}}\right)\right\rangle;$$

(3) $$\lambda \odot \Phi_1 = \left\langle \begin{array}{l} \Delta\left((l-1) \frac{\left((l-1)+\Delta^{-1}(s_{i1},\alpha_1)\right)^{\lambda}-\left((l-1)-\Delta^{-1}(s_{i1},\alpha_1)\right)^{\lambda}}{\left((l-1)+\Delta^{-1}(s_{i1},\alpha_1)\right)^{\lambda}+\left((l-1)-\Delta^{-1}(s_{i1},\alpha_1)\right)^{\lambda}}\right), \\ \left(\frac{\left(1+\mu_{\varphi_1}\right)^{\lambda}-\left(1-\mu_{\varphi_1}\right)^{\lambda}}{\left(1+\mu_{\varphi_1}\right)^{\lambda}+\left(1-\mu_{\varphi_1}\right)^{\lambda}}, \frac{2\left(\eta_{\varphi_1}\right)^{\lambda}}{\left(2-\eta_{\varphi_1}\right)^{\lambda}+\left(\eta_{\varphi_1}\right)^{\lambda}}, \frac{2\left(\nu_{\varphi_1}\right)^{\lambda}}{\left(2-\nu_{\varphi_1}\right)^{\lambda}+\left(\nu_{\varphi_1}\right)^{\lambda}}\right) \end{array} \right\rangle;$$

(4) $$\Phi_1{}^{\lambda} = \left\langle \Delta\left((l-1) \frac{2\left(\Delta^{-1}(s_{i1},\alpha_1)\right)^{\lambda}}{\left(2(l-1)-\Delta^{-1}(s_{i1},\alpha_1)\right)^{\lambda}+\left(\Delta^{-1}(s_{i1},\alpha_1)\right)^{\lambda}}\right), \left(\frac{2\left(\mu_{\varphi_1}\right)^{\lambda}}{\left(2-\mu_{\varphi_1}\right)^{\lambda}+\left(\mu_{\varphi_1}\right)^{\lambda}}, \frac{\left(1+\eta_{\varphi_1}\right)^{\lambda}-\left(1-\eta_{\varphi_1}\right)^{\lambda}}{\left(1+\eta_{\varphi_1}\right)^{\lambda}+\left(1-\eta_{\varphi_1}\right)^{\lambda}}, \frac{\left(1+\nu_{\varphi_1}\right)^{\lambda}-\left(1-\nu_{\varphi_1}\right)^{\lambda}}{\left(1+\nu_{\varphi_1}\right)^{\lambda}+\left(1-\nu_{\varphi_1}\right)^{\lambda}}\right)\right\rangle.$$

Case 3. *If* $g(x) = \log\left(\frac{\xi+(1-\xi)x}{x}\right)$ *and* $\xi > 0$, *then we have the following operational rules:*

(1) $$\Phi_1 \oplus_{\mathrm{H}} \Phi_2 = \left\langle \begin{array}{l} \Delta\left((l-1) \frac{(l-1)\Delta^{-1}(s_{i1},\alpha_1)+(l-1)\Delta^{-1}(s_{i2},\alpha_2)-(1-\xi)\cdot\Delta^{-1}(s_{i1},\alpha_1)\cdot\Delta^{-1}(s_{i2},\alpha_2)}{(l-1)^2-(1-\xi)\cdot\Delta^{-1}(s_{i1},\alpha_1)\cdot\Delta^{-1}(s_{i2},\alpha_2)}\right), \\ \left(\frac{\mu_{\varphi_1}+\mu_{\varphi_2}-(1-\xi)\mu_{\varphi_1}\mu_{\varphi_2}}{1-(1-\xi)\mu_{\varphi_1}\mu_{\varphi_2}}, \frac{\eta_{\varphi_1}\eta_{\varphi_2}}{\xi+(1-\xi)\eta_{\varphi_1}+\eta_{\varphi_2}-\eta_{\varphi_1}\eta_{\varphi_2}}, \frac{\nu_{\varphi_1}\nu_{\varphi_2}}{\xi+(1-\xi)\nu_{\varphi_1}+\nu_{\varphi_2}-\nu_{\varphi_1}\nu_{\varphi_2}}\right) \end{array} \right\rangle;$$

(2) $$\Phi_1 \otimes_{\mathrm{H}} \Phi_2 = \left\langle \begin{array}{l} \Delta\left((l-1) \frac{\Delta^{-1}(s_{i1},\alpha_1)\cdot\Delta^{-1}(s_{i2},\alpha_2)}{\xi(l-1)^2+(1-\xi)((l-1)\Delta^{-1}(s_{i1},\alpha_1)+(l-1)\Delta^{-1}(s_{i2},\alpha_2)-\Delta^{-1}(s_{i1},\alpha_1)\cdot\Delta^{-1}(s_{i2},\alpha_2))}\right), \\ \left(\frac{\mu_{\varphi_1}\mu_{\varphi_2}}{\xi+(1-\xi)\mu_{\varphi_1}+\mu_{\varphi_2}-\mu_{\varphi_1}\mu_{\varphi_2}}, \frac{\eta_{\varphi_1}+\eta_{\varphi_2}-(1-\xi)\eta_{\varphi_1}\eta_{\varphi_2}}{1-(1-\xi)\eta_{\varphi_1}\eta_{\varphi_2}}, \frac{\nu_{\varphi_1}+\nu_{\varphi_2}-(1-\xi)\nu_{\varphi_1}\nu_{\varphi_2}}{1-(1-\xi)\nu_{\varphi_1}\nu_{\varphi_2}}\right) \end{array} \right\rangle;$$

(3) $$\lambda \odot_H \Phi_1 = \left\langle \begin{array}{l} \Delta\left((l-1) \frac{\left((l-1)+(\xi-1)\cdot\Delta^{-1}(s_{i1},\alpha_1)\right)^{\lambda}-\left((l-1)-\Delta^{-1}(s_{i1},\alpha_1)\right)^{\lambda}}{((l-1)+(\xi-1)\cdot\Delta^{-1}(s_{i1},\alpha_1))^{\lambda}+(\xi-1)((l-1)-\Delta^{-1}(s_{i1},\alpha_1))^{\lambda}}\right), \\ \left(\begin{array}{l} \frac{\left(1+(\xi-1)\mu_{\varphi_1}\right)^{\lambda}-\left(1-\mu_{\varphi_1}\right)^{\lambda}}{\left(1+(\xi-1)\mu_{\varphi_1}\right)^{\lambda}+(\xi-1)\left(1-\mu_{\varphi_1}\right)^{\lambda}}, \frac{\xi\left(\eta_{\varphi_1}\right)^{\lambda}}{\left(1+(\xi-1)\left(1-\eta_{\varphi_1}\right)\right)^{\lambda}+(\xi-1)\left(\eta_{\varphi_1}\right)^{\lambda}}, \\ \frac{\xi\left(\nu_{\varphi_1}\right)^{\lambda}}{\left(1+(\xi-1)\left(1-\nu_{\varphi_1}\right)\right)^{\lambda}+(\xi-1)\left(\nu_{\varphi_1}\right)^{\lambda}} \end{array} \right) \end{array} \right\rangle;$$

(4) $$\Phi_1{}^{\lambda} = \left\langle \begin{array}{l} \Delta\left((l-1) \frac{\xi\left(\Delta^{-1}(s_{i1},\alpha_1)\right)^{\lambda}}{((l-1)+(\xi-1)\cdot((l-1)-\Delta^{-1}(s_{i1},\alpha_1)))^{\lambda}+(\xi-1)(\Delta^{-1}(s_{i1},\alpha_1))^{\lambda}}\right), \\ \left(\begin{array}{l} \frac{\xi\left(\mu_{\varphi_1}\right)^{\lambda}}{\left(1+(\xi-1)\left(1-\mu_{\varphi_1}\right)\right)^{\lambda}+(\xi-1)\left(\mu_{\varphi_1}\right)^{\lambda}}, \frac{\left(1+(\xi-1)\left(\eta_{\varphi_1}\right)\right)^{\lambda}}{\left(1+(\xi-1)\eta_{\varphi_1}\right)^{\lambda}+(\xi-1)\left(1-\eta_{\varphi_1}\right)^{\lambda}}, \\ \frac{\left(1+(\xi-1)\left(\nu_{\varphi_1}\right)\right)^{\lambda}}{\left(1+(\xi-1)\nu_{\varphi_1}\right)^{\lambda}+(\xi-1)\left(1-\nu_{\varphi_1}\right)^{\lambda}} \end{array} \right) \end{array} \right\rangle.$$

4. Picture 2-Tuple Linguistic MSM Operators Based on ATT

Based on the new operations for picture 2-tuple linguistic sets and MSM operators, we developed a few novel *P2TLMSM* operators.

4.1. The ATT-P2TLMSM and ATT-P2TLGMSM Operators

Definition 17. *Let $\Phi_i = \left\langle (s_i, \alpha_i), \left(\mu_{\varphi i}, \eta_{\varphi i}, \nu_{\varphi i}\right)\right\rangle (i = 1, 2, \ldots, n)$ be a set of P2TLNs. Then, the ATT-P2TLMSM operator $\Lambda^n \to \Lambda$ is as follows.*

$$ATT - P2TLMSM^{(m)}(\Phi_1, \ldots, \Phi_n) = \left(\frac{\underset{1\le i_1<\ldots<i_m\le n}{\oplus}\left(\overset{m}{\underset{j=1}{\otimes}}\Phi_{i_j}\right)}{C_n^m}\right)^{\frac{1}{m}} \tag{9}$$

where Λ is a collection of P2TLNs and $m = 1, 2, \ldots, n$.

According to the operation rules of P2TLNs in Definition 16, the *ATT-P2TLMSM* operators are shown below.

Theorem 1. *Let $\Phi_i = \left\langle (s_i, \alpha_i), \left(\mu_{\varphi i}, \eta_{\varphi i}, \nu_{\varphi i}\right)\right\rangle$ $(i = 1, 2, \ldots, n)$ be a collection of P2TLNs with $m = 1, 2, \ldots, n$. The result of aggregating by Definition 17 is still a P2TLN.*

$$\begin{aligned} &ATT - P2TLMSM^{(m)}(\Phi_1, \ldots, \Phi_n) = \left(\frac{\underset{1\le i_1<\ldots<i_m\le n}{\oplus}\left(\overset{m}{\underset{j=1}{\otimes}}\Phi_{i_j}\right)}{C_n^m}\right)^{\frac{1}{m}} \\ &= \left\langle \begin{array}{l} \Delta\left((l-1)g^{-1}\left(\frac{1}{m}g\left(f^{-1}\left(\frac{1}{C_n^m}\left(\sum_{1\le i_1<\ldots<i_m\le n} f\left(g^{-1}\left(\sum_{j=1}^m g\left(\frac{\Delta^{-1}\left(s_{i_j},\alpha_{i_j}\right)}{(l-1)}\right)\right)\right)\right)\right)\right)\right)\right), \\ \left(\begin{array}{l} g^{-1}\left(\frac{1}{m}g\left(f^{-1}\left(\frac{1}{C_n^m}\left(\sum_{1\le i_1<\ldots<i_m\le n} f\left(g^{-1}\left(\sum_{j=1}^m g\left(\mu_{\varphi i_j}\right)\right)\right)\right)\right)\right)\right), \\ f^{-1}\left(\frac{1}{m}f\left(g^{-1}\left(\frac{1}{C_n^m}\left(\sum_{1\le i_1<\ldots<i_m\le n} g\left(f^{-1}\left(\sum_{j=1}^m f\left(\eta_{\varphi i_j}\right)\right)\right)\right)\right)\right)\right), \\ f^{-1}\left(\frac{1}{m}f\left(g^{-1}\left(\frac{1}{C_n^m}\left(\sum_{1\le i_1<\ldots<i_m\le n} g\left(f^{-1}\left(\sum_{j=1}^m f\left(\nu_{\varphi i_j}\right)\right)\right)\right)\right)\right)\right) \end{array}\right) \end{array} \right\rangle \end{aligned} \tag{10}$$

where $(i_1, i_2, \ldots, i_n)$ traverses all m-tuple combinations of $(1, 2, \ldots, n)$ and $C_n^m = \frac{n!}{m!(n-m)!}$ is the binomial coefficient.

Proof. The flowchart of proof is shown in Figure 1.

The specific proof process is as follows.

Because $\Phi_{i_j} = \left\langle \left(s_{i_j}, \alpha_{i_j}\right), \left(\mu_{\varphi i_j}, \eta_{\varphi i_j}, \nu_{\varphi i_j}\right)\right\rangle$ $(i = 1, 2, \ldots, nj = 1, 2, \ldots, m)$

$$\Rightarrow \overset{m}{\underset{j=1}{\otimes}}\Phi_{i_j} = \left\langle \begin{array}{l} \Delta\left((l-1)g^{-1}\left(\sum_{j=1}^m g\left(\frac{\Delta^{-1}\left(s_{i_j},\alpha_{i_j}\right)}{(l-1)}\right)\right)\right), \\ \left(g^{-1}\left(\sum_{j=1}^m g\left(\mu_{\varphi i_j}\right)\right), f^{-1}\left(\sum_{j=1}^m f\left(\eta_{\varphi i_j}\right)\right), f^{-1}\left(\sum_{j=1}^m f\left(\nu_{\varphi i_j}\right)\right)\right) \end{array} \right\rangle$$

$$\begin{aligned} &\Rightarrow \underset{1\le i_1<\ldots<i_m\le n}{\oplus}\left(\overset{m}{\underset{j=1}{\otimes}}\Phi_{i_j}\right) \\ &= \left\langle \begin{array}{l} \Delta\left((l-1)f^{-1}\left(\sum_{1\le i_1<\ldots<i_m\le n} f\left(g^{-1}\left(\sum_{j=1}^m g\left(\frac{\Delta^{-1}\left(s_{i_j},\alpha_{i_j}\right)}{(l-1)}\right)\right)\right)\right)\right), \\ \left(\begin{array}{l} f^{-1}\left(\sum_{1\le i_1<\ldots<i_m\le n} f\left(g^{-1}\left(\sum_{j=1}^m g\left(\mu_{\varphi i_j}\right)\right)\right)\right), g^{-1}\left(\sum_{1\le i_1<\ldots<i_m\le n} g\left(f^{-1}\left(\sum_{j=1}^m f\left(\eta_{\varphi i_j}\right)\right)\right)\right), \\ g^{-1}\left(\sum_{1\le i_1<\ldots<i_m\le n} g\left(f^{-1}\left(\sum_{j=1}^m f\left(\nu_{\varphi i_j}\right)\right)\right)\right) \end{array}\right) \end{array} \right\rangle \end{aligned}$$

$$\Rightarrow \frac{\underset{1\le i_1<\ldots<i_m\le n}{\oplus}\left(\overset{m}{\underset{j=1}{\otimes}}\Phi_{i_j}\right)}{C_n^m} = \left\langle \begin{array}{c} \Delta\left((l-1)f^{-1}\left(\frac{1}{C_n^m}f\left(f^{-1}\left(\sum_{1\le i_1<\ldots<i_m\le n} f\left(g^{-1}\left(\sum_{j=1}^m g\left(\frac{\Delta^{-1}\left(s_{i_j},\alpha_{i_j}\right)}{(l-1)}\right)\right)\right)\right)\right)\right)\right), \\ \left(\begin{array}{l} f^{-1}\left(\frac{1}{C_n^m}f\left(f^{-1}\left(\sum_{1\le i_1<\ldots<i_m\le n} f\left(g^{-1}\left(\sum_{j=1}^m g\left(\mu_{\varphi i_j}\right)\right)\right)\right)\right)\right), \\ g^{-1}\left(\frac{1}{C_n^m}g\left(g^{-1}\left(\sum_{1\le i_1<\ldots<i_m\le n} g\left(f^{-1}\left(\sum_{j=1}^m f\left(\eta_{\varphi i_j}\right)\right)\right)\right)\right)\right), \\ g^{-1}\left(\frac{1}{C_n^m}g\left(g^{-1}\left(\sum_{1\le i_1<\ldots<i_m\le n} g\left(f^{-1}\left(\sum_{j=1}^m f\left(\nu_{\varphi i_j}\right)\right)\right)\right)\right)\right) \end{array}\right) \end{array} \right\rangle$$

$$= \left\langle \begin{array}{c} \Delta\left((l-1)f^{-1}\left(\frac{1}{C_n^m}\left(\sum_{1\le i_1<\ldots<i_m\le n} f\left(g^{-1}\left(\sum_{j=1}^m g\left(\frac{\Delta^{-1}\left(s_{i_j},\alpha_{i_j}\right)}{(l-1)}\right)\right)\right)\right)\right)\right), \\ \left(\begin{array}{l} f^{-1}\left(\frac{1}{C_n^m}\left(\sum_{1\le i_1<\ldots<i_m\le n} f\left(g^{-1}\left(\sum_{j=1}^m g\left(\mu_{\varphi i_j}\right)\right)\right)\right)\right), \\ g^{-1}\left(\frac{1}{C_n^m}\left(\sum_{1\le i_1<\ldots<i_m\le n} g\left(f^{-1}\left(\sum_{j=1}^m f\left(\eta_{\varphi i_j}\right)\right)\right)\right)\right), \\ g^{-1}\left(\frac{1}{C_n^m}\left(\sum_{1\le i_1<\ldots<i_m\le n} g\left(f^{-1}\left(\sum_{j=1}^m f\left(\nu_{\varphi i_j}\right)\right)\right)\right)\right) \end{array}\right) \end{array} \right\rangle$$

$$\Rightarrow \left(\frac{\underset{1\le i_1<\ldots<i_m\le n}{\oplus}\left(\overset{m}{\underset{j=1}{\otimes}}\Phi_{i_j}\right)}{C_n^m}\right)^{\frac{1}{m}}$$

$$= \left\langle \begin{array}{c} \Delta\left((l-1)g^{-1}\left(\frac{1}{m}g\left(f^{-1}\left(\frac{1}{C_n^m}\left(\sum_{1\le i_1<\ldots<i_m\le n} f\left(g^{-1}\left(\sum_{j=1}^m g\left(\frac{\Delta^{-1}\left(s_{i_j},\alpha_{i_j}\right)}{(l-1)}\right)\right)\right)\right)\right)\right)\right)\right), \\ \left(\begin{array}{l} g^{-1}\left(\frac{1}{m}g\left(f^{-1}\left(\frac{1}{C_n^m}\left(\sum_{1\le i_1<\ldots<i_m\le n} f\left(g^{-1}\left(\sum_{j=1}^m g\left(\mu_{\varphi i_j}\right)\right)\right)\right)\right)\right)\right), \\ f^{-1}\left(\frac{1}{m}f\left(g^{-1}\left(\frac{1}{C_n^m}\left(\sum_{1\le i_1<\ldots<i_m\le n} g\left(f^{-1}\left(\sum_{j=1}^m f\left(\eta_{\varphi i_j}\right)\right)\right)\right)\right)\right)\right), \\ f^{-1}\left(\frac{1}{m}f\left(g^{-1}\left(\frac{1}{C_n^m}\left(\sum_{1\le i_1<\ldots<i_m\le n} g\left(f^{-1}\left(\sum_{j=1}^m f\left(\nu_{\varphi i_j}\right)\right)\right)\right)\right)\right)\right) \end{array}\right) \end{array} \right\rangle$$

Therefore, Theorem 1 proved to be correct. □

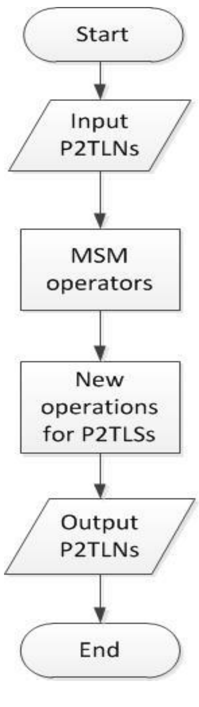

Figure 1. The flowchart of proof.

Property 1. *Let* $\Phi_i = \left\langle (s_i, \alpha_i), \left(\mu_{\varphi i}, \eta_{\varphi i}, \nu_{\varphi i}\right)\right\rangle (i = 1, 2, \ldots, n)$ *and* $\Phi_i^* = \left\langle \left(s_i^*, \alpha_i^*\right), \left(\mu_{\varphi i}^*, \eta_{\varphi i}^*, \nu_{\varphi i}^*\right)\right\rangle$ $(i = 1, 2, \ldots, n)$ *be sets of P2TLNs.* $ATT - P2TLMSM^{(m)}$ *then has several properties.*

(1) *Idempotency: If the P2TLNs* $\Phi_i = \Phi = \left\langle \left(s_\varphi, \alpha_\varphi\right), \left(\mu_\varphi, \eta_\varphi, \nu_\varphi\right)\right\rangle$ *for all i, then* $ATT - P2TLMSM^{(m)} = \Phi = \left\langle \left(s_\varphi, \alpha_\varphi\right), \left(\mu_\varphi, \eta_\varphi, \nu_\varphi\right)\right\rangle$.

(2) *Commutativity: Assume* Φ_i *is a permutation of* Φ_i^* *for all i; then,* $ATT - P2TLMSM^{(m)}(\Phi_1, \Phi_2, \ldots, \Phi_n) = ATT - P2TLMSM^{(m)}\left(\Phi_1^*, \Phi_2^*, \ldots, \Phi_n^*\right)$.

(3) *Monotonicity: If* $(s_i, \alpha_i) \geq \left(s_i^*, \alpha_i^*\right)$, $\mu_{\varphi i} \geq \mu_{\varphi i}^*$, $\eta_{\varphi i} \leq \eta_{\varphi i}^*$ *and* $\nu_{\varphi i} \leq \nu_{\varphi i}^*$ *for each i* $(i = 1, 2, \ldots, n)$, *then* $\Phi_i \geq \Phi_i^*$ *and* $ATT - P2TLMSM^{(m)}(\Phi_1, \Phi_2, \ldots, \Phi_n) \geq ATT - P2TLMSM^{(m)}\left(\Phi_1^*, \Phi_2^*, \ldots, \Phi_n^*\right)$.

(4) *Boundedness: If* $\Phi^- = \min_i \Phi_i = \left\langle \min_i (s_i, \alpha_i), \left(\min_i\left(\mu_{\varphi i}\right), \min_i\left(\eta_{\varphi i}\right), \min_i\left(\nu_{\varphi i}\right)\right)\right\rangle$ *and* $\Phi^+ = \max_i \Phi_i = \left\langle \max_i (s_i, \alpha_i), \left(\max_i\left(\mu_{\varphi i}\right), \max_i\left(\eta_{\varphi i}\right), \max_i\left(\nu_{\varphi i}\right)\right)\right\rangle$, *then* $\Phi^- \leq ATT - P2TLMSM^{(m)}(\Phi_1, \Phi_2, \ldots, \Phi_n) \leq \Phi^+$.

Proof.

1. Since each $\Phi_i = \Phi = \left\langle \left(s_\varphi, \alpha_\varphi\right), \left(\mu_\varphi, \eta_\varphi, \nu_\varphi\right)\right\rangle$, that is,

$$
\begin{aligned}
& ATT - P2TLMSM^{(m)}(\Phi, \Phi, \ldots, \Phi) \\
&= \left\langle \begin{pmatrix} \Delta\left((l-1)g^{-1}\left(\frac{1}{m}g\left(f^{-1}\left(\frac{1}{C_n^m}\left(\sum_{1 \leq i_1 < \ldots < i_m \leq n} f\left(g^{-1}\left(\sum_{j=1}^m g\left(\frac{\Delta^{-1}\left(s_\varphi, \alpha_\varphi\right)}{(l-1)}\right)\right)\right)\right)\right)\right)\right)\right), \\ g^{-1}\left(\frac{1}{m}g\left(f^{-1}\left(\frac{1}{C_n^m}\left(\sum_{1 \leq i_1 < \ldots < i_m \leq n} f\left(g^{-1}\left(\sum_{j=1}^m g\left(\mu_\varphi\right)\right)\right)\right)\right)\right)\right), \\ f^{-1}\left(\frac{1}{m}f\left(g^{-1}\left(\frac{1}{C_n^m}\left(\sum_{1 \leq i_1 < \ldots < i_m \leq n} g\left(f^{-1}\left(\sum_{j=1}^m f\left(\eta_\varphi\right)\right)\right)\right)\right)\right)\right), \\ f^{-1}\left(\frac{1}{m}f\left(g^{-1}\left(\frac{1}{C_n^m}\left(\sum_{1 \leq i_1 < \ldots < i_m \leq n} g\left(f^{-1}\left(\sum_{j=1}^m f\left(\nu_\varphi\right)\right)\right)\right)\right)\right)\right) \end{pmatrix} \right\rangle \\
&= \left\langle \begin{matrix} \Delta\left((l-1)g^{-1}\left(\frac{1}{m}g\left(f^{-1}\left(\frac{C_n^m f\left(g^{-1}\left(\sum_{j=1}^m g\left(\frac{\Delta^{-1}(s_\varphi, \alpha_\varphi)}{(l-1)}\right)\right)\right)}{C_n^m}\right)\right)\right)\right), \\ \begin{pmatrix} g^{-1}\left(\frac{1}{m}g\left(f^{-1}\left(\frac{C_n^m f\left(g^{-1}\left(\sum_{j=1}^m g\left(\mu_\varphi\right)\right)\right)}{C_n^m}\right)\right)\right), f^{-1}\left(\frac{1}{m}f\left(g^{-1}\left(\frac{C_n^m g\left(f^{-1}\left(\sum_{j=1}^m f\left(\eta_\varphi\right)\right)\right)}{C_n^m}\right)\right)\right), \\ f^{-1}\left(\frac{1}{m}f\left(g^{-1}\left(\frac{C_n^m g\left(f^{-1}\left(\sum_{j=1}^m f\left(\nu_\varphi\right)\right)\right)}{C_n^m}\right)\right)\right) \end{pmatrix} \end{matrix} \right\rangle \\
&= \left\langle \left(s_\varphi, \alpha_\varphi\right), \left(\mu_\varphi, \eta_\varphi, \nu_\varphi\right)\right\rangle
\end{aligned}
$$

2. This property is obvious, and we do not prove it here.

3. If $(s_i, \alpha_i) \geq \left(s_i^*, \alpha_i^*\right)$, $\mu_{\varphi i} \geq \mu_{\varphi i}^*$, $\eta_{\varphi i} \leq \eta_{\varphi i}^*$, and $\nu_{\varphi i} \leq \nu_{\varphi i}^*$ for each i $(i = 1, 2, \ldots, n)$ and $\Phi_i = \Phi = \left\langle \left(s_\varphi, \alpha_\varphi\right), \left(\mu_\varphi, \eta_\varphi, \nu_\varphi\right)\right\rangle$, $\Phi_i^* = \Phi^* = \left\langle \left(s_\varphi^*, \alpha_\varphi^*\right), \left(\mu_\varphi^*, \eta_\varphi^*, \nu_\varphi^*\right)\right\rangle$, according to idempotency, $ATT - P2TLMSM^{(m)}(\Phi, \Phi, \ldots, \Phi) = \left\langle \left(s_\varphi, \alpha_\varphi\right), \left(\mu_\varphi, \eta_\varphi, \nu_\varphi\right)\right\rangle$ and $ATT - P2TLMSM^{(m)}(\Phi^*, \Phi^*, \ldots, \Phi^*) = \left\langle \left(s_\varphi^*, \alpha_\varphi^*\right), \left(\mu_\varphi^*, \eta_\varphi^*, \nu_\varphi^*\right)\right\rangle$. Therefore, we have

$$
P2TLMSM^{(m)}(\Phi_1, \Phi_2, \ldots, \Phi_n) \leq P2TLMSM^{(m)}\left(\Phi_1^*, \Phi_2^*, \ldots, \Phi_n^*\right)
$$

4. According to idempotency, let $\min\{\Phi_1, \Phi_2, \ldots, \Phi_n\} = \Phi = ATT - P2TLMSM^{(m)}(\Phi, \Phi, \ldots, \Phi)$ and $\max\{\Phi_1, \Phi_2, \ldots, \Phi_n\} = \Phi^* = ATT - P2TLMSM^{(m)}(\Phi^*, \Phi^*, \ldots, \Phi^*)$. Based on the monotonicity, if $\Phi \leq \Phi_i$ and $\Phi_i \leq \Phi^*$ for each i $(i = 1, 2, \ldots, n)$, then we have $\Phi = ATT - P2TLMSM^{(m)}(\Phi, \Phi, \ldots, \Phi) \leq ATT - P2TLMSM^{(m)}(\Phi_1, \Phi_2, \ldots, \Phi_n)$ and $ATT -$

$P2TLMSM^{(m)}(\Phi_1, \Phi_2, \ldots, \Phi_n) \le ATT - P2TLMSM^{(m)}(\Phi^*, \Phi^*, \ldots, \Phi^*) = \Phi^*$. Therefore, the following conclusion can be obtained.

$$\min\{\Phi_1, \Phi_2, \ldots, \Phi_n\} \le ATT - P2TLMSM^{(m)}\{\Phi_1, \Phi_2, \ldots, \Phi_n\} \le \max\{\Phi_1, \Phi_2, \ldots, \Phi_n\}$$

□

In the following, we present a detailed formula as an example to introduce the *P2TLMSM* operator in the context MADM. When $g(x) = -\log x$, based on Formula (10), we can obtain:

$$\begin{aligned} &ATT - P2TLMSM^{(m)}(\Phi_1, \ldots, \Phi_n) = \left(\frac{\underset{1 \le i_1 < \ldots < i_m \le n}{\oplus} \left(\overset{m}{\underset{j=1}{\otimes}} \Phi_{i_j} \right)}{C_n^m} \right)^{\frac{1}{m}} \\ &= \left\langle \begin{array}{l} \Delta\left((l-1)\left(1 - \left(\prod_{1 \le i_1 < \ldots < i_m \le n} \left(1 - \prod_{j=1}^m \left(\frac{\Delta^{-1}\left(s_{i_j}, \alpha_{i_j}\right)}{(l-1)} \right) \right) \right)^{\frac{1}{C_n^m}} \right)^{\frac{1}{m}} \right), \\ \left(\begin{array}{l} \left(1 - \left(\prod_{1 \le i_1 < \ldots < i_m \le n} \left(1 - \prod_{j=1}^m \mu_{\varphi i_j} \right) \right)^{\frac{1}{C_n^m}} \right)^{\frac{1}{m}}, 1 - \left(1 - \left(\prod_{1 \le i_1 < \ldots < i_m \le n} \left(1 - \prod_{j=1}^m \left(1 - \eta_{\varphi i_j} \right) \right) \right)^{\frac{1}{C_n^m}} \right)^{\frac{1}{m}}, \\ 1 - \left(1 - \left(\prod_{1 \le i_1 < \ldots < i_m \le n} \left(1 - \prod_{j=1}^m \left(1 - \nu_{\varphi i_j} \right) \right) \right)^{\frac{1}{C_n^m}} \right)^{\frac{1}{m}} \end{array} \right) \end{array} \right\rangle \end{aligned} \tag{11}$$

Next, we study some special cases of the *P2TLMSM* operator with respect to the parameter m.

(1) When $m = 1$, Equation (10) degrades to the following formula.

$$\begin{aligned} &ATT - P2TLMSM^{(1)}(\Phi_1, \ldots, \Phi_n) = \left(\frac{\underset{1 \le i_1 \le n}{\oplus} \Phi_i}{C_n^1} \right) \\ &= \left\langle \Delta\left((l-1)\left(1 - \left(\prod_{1 \le i_1 \le n} \left(1 - \frac{\Delta^{-1}(s_i, \alpha_i)}{(l-1)} \right) \right)^{\frac{1}{n}} \right) \right), \left(1 - \left(\prod_{1 \le i_1 \le n} \left(1 - \mu_{\varphi i} \right) \right)^{\frac{1}{n}}, \left(\prod_{1 \le i_1 \le n} \eta_{\varphi i} \right)^{\frac{1}{n}}, \left(\prod_{1 \le i_1 \le n} \nu_{\varphi i} \right)^{\frac{1}{n}} \right) \right\rangle \end{aligned} \tag{12}$$

(2) When $m = 2$, Equation (10) degrades to the following formula.

$$\begin{aligned} &ATT - P2TLMSM^{(2)}(\Phi_1, \ldots, \Phi_n) = \left(\frac{\underset{1 \le i_1 < i_2 \le n}{\oplus} \left(\overset{2}{\underset{j=1}{\otimes}} \Phi_{i_j} \right)}{C_n^2} \right)^{\frac{1}{2}} \\ &= \left\langle \begin{array}{l} \Delta\left((l-1)\left(1 - \left(\prod_{1 \le i_1 < i_2 \le n} \left(1 - \frac{\left(\Delta^{-1}\left(s_{i_1}, \alpha_{i_1}\right)\right)}{(l-1)} \frac{\left(\Delta^{-1}\left(s_{i_2}, \alpha_{i_2}\right)\right)}{(l-1)} \right) \right)^{\frac{1}{C_n^m}} \right)^{\frac{1}{m}} \right), \\ \left(\begin{array}{l} \left(1 - \left(\prod_{1 \le i_1 < i_2 \le n} \left(1 - \mu_{\varphi i_1} \mu_{\varphi i_2} \right) \right)^{\frac{2}{n(n-1)}} \right)^{\frac{1}{2}}, 1 - \left(1 - \left(\prod_{1 \le i_1 < i_2 \le n} \left(1 - \left(1 - \eta_{\varphi i_1} \right)\left(1 - \eta_{\varphi i_2} \right) \right) \right)^{\frac{2}{n(n-1)}} \right)^{\frac{1}{2}}, \\ 1 - \left(1 - \left(\prod_{1 \le i_1 < i_2 \le n} \left(1 - \left(1 - \nu_{\varphi i_1} \right)\left(1 - \nu_{\varphi i_2} \right) \right) \right)^{\frac{2}{n(n-1)}} \right)^{\frac{1}{2}} \end{array} \right) \end{array} \right\rangle \end{aligned} \tag{13}$$

(3) When $m = n$, Equation (10) degrades to the following formula.

$$\begin{aligned} &ATT - P2TLMSM^{(n)}(\Phi_1, \ldots, \Phi_n) = \left(\frac{\underset{1 \le i_1 < \ldots < i_m \le n}{\oplus} \left(\overset{n}{\underset{j=1}{\otimes}} \Phi_{i_j} \right)}{C_n^n} \right)^{\frac{1}{n}} \\ &= \left\langle \Delta\left((l-1)\left(\prod_{j=1}^n \frac{\Delta^{-1}\left(s_{i_j}, \alpha_{i_j}\right)}{(l-1)} \right)^{\frac{1}{n}} \right), \left(\left(\prod_{j=1}^n \mu_{\varphi i_j} \right)^{\frac{1}{n}}, 1 - \left(\prod_{j=1}^n \left(1 - \eta_{\varphi i_j} \right) \right)^{\frac{1}{n}}, 1 - \left(\prod_{j=1}^n \left(1 - \nu_{\varphi i_j} \right) \right)^{\frac{1}{n}} \right) \right\rangle \end{aligned} \tag{14}$$

Definition 18. *Let* $\Phi_i = \left\langle (s_i, \alpha_i), \left(\mu_{\varphi i}, \eta_{\varphi i}, \nu_{\varphi i}\right)\right\rangle (i = 1, 2, \ldots, n)$ *be a set of P2TLNs. Then, the ATT-P2TLGMSM operator* $\Lambda^n \to \Lambda$ *is as follows.*

$$ATT - P2TLGMSM^{(m,p_1,p_2,\ldots,p_m)}(\Phi_1, \ldots, \Phi_n) = \left(\frac{\underset{1\le i_1<\ldots<i_m\le n}{\oplus}\left(\overset{m}{\underset{j=1}{\otimes}} \Phi_{i_j}^{p_j}\right)}{C_n^m} \right)^{\frac{1}{p_1+p_2+\ldots+p_m}}, \tag{15}$$

where Λ *is a collection of P2TLNs and* $m = 1, 2, \ldots, n$.

On the basis of the operation rules of P2TLNs in Definition 16, the *ATT-P2TLMSM* operators are presented below.

Theorem 2. *Let* $\Phi_i = \left\langle (s_i, \alpha_i), \left(\mu_{\varphi i}, \eta_{\varphi i}, \nu_{\varphi i}\right)\right\rangle$ $(i = 1, 2, \ldots, n)$ *be a set of P2TLNs and* $m = 1, 2, \ldots, n$. *Then, the aggregation result from Definition 17 is also a P2TLN.*

$$\begin{aligned} &ATT - P2TLGMSM^{(m,p_1,p_2,\ldots,p_m)}(\Phi_1, \ldots, \Phi_n) = \left(\frac{\underset{1\le i_1<\ldots<i_m\le n}{\oplus}\left(\overset{m}{\underset{j=1}{\otimes}} \Phi_{i_j}^{p_j}\right)}{C_n^m} \right)^{\frac{1}{p_1+p_2+\ldots+p_m}} \\ &\Delta\left((l-1)g^{-1}\left(\frac{1}{p_1+p_2+\ldots+p_m} g\left(f^{-1}\left(\frac{1}{C_n^m}\left(\textstyle\sum_{1\le i_1<\ldots<i_m\le n} f\left(g^{-1}\left(\sum_{j=1}^m g\left(g^{-1}\left(p_j g\left(\frac{\Delta^{-1}\left(s_{i_j},\alpha_{i_j}\right)}{(l-1)}\right)\right)\right)\right)\right)\right)\right)\right)\right)\right), \\ &= \left\langle \begin{pmatrix} g^{-1}\left(\frac{1}{p_1+p_2+\ldots+p_m} g\left(f^{-1}\left(\frac{1}{C_n^m}\left(\sum_{1\le i_1<\ldots<i_m\le n} f\left(g^{-1}\left(\sum_{j=1}^m g\left(g^{-1}\left(p_j g\left(\mu_{\varphi i_j}\right)\right)\right)\right)\right)\right)\right)\right)\right), \\ f^{-1}\left(\frac{1}{p_1+p_2+\ldots+p_m} f\left(g^{-1}\left(\frac{1}{C_n^m}\left(\sum_{1\le i_1<\ldots<i_m\le n} g\left(f^{-1}\left(\sum_{j=1}^m f\left(f^{-1}\left(p_j f\left(\eta_{\varphi i_j}\right)\right)\right)\right)\right)\right)\right)\right)\right), \\ f^{-1}\left(\frac{1}{p_1+p_2+\ldots+p_m} f\left(g^{-1}\left(\frac{1}{C_n^m}\left(\sum_{1\le i_1<\ldots<i_m\le n} g\left(f^{-1}\left(\sum_{j=1}^m f\left(f^{-1}\left(p_j f\left(\nu_{\varphi i_j}\right)\right)\right)\right)\right)\right)\right)\right)\right) \end{pmatrix} \right\rangle \end{aligned} \tag{16}$$

where $(i_1, i_2, \ldots, i_n)$ *traverses all m-tuple combinations of* $(1, 2, \ldots, n)$ *and* $C_n^m = \frac{n!}{m!(n-m)!}$ *is the binomial coefficient.*

Because the proof is analogous to Theorem 1, we do not repeat it here.

Property 2. *Let* $\Phi_i = \left\langle (s_i, \alpha_i), \left(\mu_{\varphi i}, \eta_{\varphi i}, \nu_{\varphi i}\right)\right\rangle (i = 1, 2, \ldots, n)$ *and* $\Phi_i^* = \left\langle \left(s_i^*, \alpha_i^*\right), \left(\mu_{\varphi i}^*, \eta_{\varphi i}^*, \nu_{\varphi i}^*\right)\right\rangle$ $(i = 1, 2, \ldots, n)$ *be collections of P2TLNs.* $ATT - P2TLGMSM^{(m,p_1,p_2,\ldots,p_m)}$ *then has a number of properties.*

(1) *Idempotency: If the P2TLNs* $\Phi_i = \Phi = \left\langle \left(s_\varphi, \alpha_\varphi\right), \left(\mu_\varphi, \eta_\varphi, \nu_\varphi\right)\right\rangle$ *for all i, then* $ATT - P2TLGMSM^{(m,p_1,p_2,\ldots,p_m)} = \Phi = \left\langle \left(s_\varphi, \alpha_\varphi\right), \left(\mu_\varphi, \eta_\varphi, \nu_\varphi\right)\right\rangle$.

(2) *Commutativity: Assume* Φ_i *is a permutation of* Φ_i^* *for all i; then,* $ATT - P2TLGMSM^{(m,p_1,p_2,\ldots,p_m)}(\Phi_1, \Phi_2, \ldots, \Phi_n) = ATT - P2TLGMSM^{(m,p_1,p_2,\ldots,p_m)}\left(\Phi_1^*, \Phi_2^*, \ldots, \Phi_n^*\right)$.

(3) *Monotonicity: If* $(s_i, \alpha_i) \ge \left(s_i^*, \alpha_i^*\right)$, $\mu_{\varphi i} \ge \mu_{\varphi i}^*$, $\eta_{\varphi i} \le \eta_{\varphi i}^*$ *and* $\nu_{\varphi i} \le \nu_{\varphi i}^*$ *for each i* $(i = 1, 2, \ldots, n)$, *then* $\Phi_i \ge \Phi_i^*$ *and* $ATT - P2TLGMSM^{(m,p_1,p_2,\ldots,p_m)}(\Phi_1, \Phi_2, \ldots, \Phi_n) \ge ATT - P2TLGMSM^{(m,p_1,p_2,\ldots,p_m)}\left(\Phi_1^*, \Phi_2^*, \ldots, \Phi_n^*\right)$.

(4) *Boundedness: If* $\Phi^- = \min_i \Phi_i = \left\langle \min_i(s_i, \alpha_i), \left(\min_i\left(\mu_{\varphi i}\right), \min_i\left(\eta_{\varphi i}\right), \min_i\left(\nu_{\varphi i}\right)\right)\right\rangle$ *and* $\Phi^+ = \max_i \Phi_i = \left\langle \max_i(s_i, \alpha_i), \left(\max_i\left(\mu_{\varphi i}\right), \max_i\left(\eta_{\varphi i}\right), \max_i\left(\nu_{\varphi i}\right)\right)\right\rangle$, *then* $\Phi^- \le ATT - P2TLGMSM^{(m,p_1,p_2,\ldots,p_m)}(\Phi_1, \Phi_2, \ldots, \Phi_n) \le \Phi^+$.

Because the proof is analogous to Property 1, we do not repeat it here.

In the following, we present a detailed formula as an example to introduce the *P2TLMSM* operator in the context of MADM. When $g(x) = -\log x$, based on Formula (16), we obtain:

$$
\begin{aligned}
&ATT-P2TLGMSM^{(m,p_1,p_2,\ldots,p_m)}(\Phi_1,\ldots,\Phi_n) = \left(\frac{\underset{1\le i_1<\ldots<i_m\le n}{\oplus} \left(\overset{m}{\underset{j=1}{\otimes}} \Phi_{i_j}^{p_j} \right)}{C_n^m} \right)^{\frac{1}{p_1+p_2+\ldots+p_m}} \\
&= \left\langle \begin{array}{l} \Delta\left((l-1)\left(1-\left(\prod_{1\le i_1<\ldots<i_m\le n}\left(1-\prod_{j=1}^{m}\left(\frac{\Delta^{-1}\left(s_{i_j},\alpha_{i_j}\right)}{(l-1)}\right)^{p_j}\right)\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{p_1+p_2+\ldots+p_m}} \right), \\ \left(\begin{array}{l} \left(1-\left(\prod_{1\le i_1<\ldots<i_m\le n}\left(1-\prod_{j=1}^{m}\mu_{\varphi i_j}^{p_j}\right)\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{p_1+p_2+\ldots+p_m}}, 1-\left(1-\left(\prod_{1\le i_1<\ldots<i_m\le n}\left(1-\prod_{j=1}^{m}\left(1-\eta_{\varphi i_j}^{p_j}\right)\right)\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{p_1+p_2+\ldots+p_m}}, \\ 1-\left(1-\left(\prod_{1\le i_1<\ldots<i_m\le n}\left(1-\prod_{j=1}^{m}\left(1-\nu_{\varphi i_j}^{p_j}\right)\right)\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{p_1+p_2+\ldots+p_m}} \end{array} \right) \end{array} \right\rangle
\end{aligned} \tag{17}
$$

Next, we study several specific cases of the *P2TLGMSM* operator with respect to the argument m.

(1) When $m = 1$, Equation (17) degrades to the following formula.

$$
\begin{aligned}
&ATT-P2TLGMSM^{(1)}(\Phi_1,\ldots,\Phi_n) = \left(\frac{\underset{1\le i_1\le n}{\oplus} \Phi_i^{p_1}}{C_n^1} \right)^{\frac{1}{p_1}} \\
&= \left\langle \begin{array}{l} \Delta\left((l-1)\left(1-\left(\prod_{1\le i_1\le n}\left(1-\left(\frac{\Delta^{-1}(s_i,\alpha_i)}{(l-1)}\right)^{p_1}\right)\right)^{\frac{1}{n}}\right)^{\frac{1}{p_1}} \right), \\ \left(\left(1-\left(\prod_{1\le i_1\le n}\left(1-\mu_{\varphi i}^{p_1}\right)\right)^{\frac{1}{n}}\right)^{\frac{1}{p_1}}, 1-\left(1-\left(\eta_{\varphi i_j}^{p_j}\right)^{\frac{1}{n}}\right)^{\frac{1}{p_1+p_2+\ldots+p_m}}, 1-\left(1-\left(\nu_{\varphi i_j}^{p_j}\right)^{\frac{1}{n}}\right)^{\frac{1}{p_1+p_2+\ldots+p_m}} \right) \end{array} \right\rangle
\end{aligned} \tag{18}
$$

(2) When $m = 2$, Equation (17) degrades to the following formula.

$$
\begin{aligned}
&ATT-P2TLGMSM^{(2,p_1,p_2)}(\Phi_1,\ldots,\Phi_n) = \left(\frac{\underset{1\le i_1<i_2\le n}{\oplus} \left(\overset{2}{\underset{j=1}{\otimes}} \Phi_{i_j}^{p_j} \right)}{C_n^2} \right)^{\frac{1}{p_1+p_2}} \\
&= \left\langle \begin{array}{l} \Delta\left((l-1)\left(1-\left(\prod_{1\le i_1<i_2\le n}\left(1-\prod_{j=1}^{2}\left(\frac{\Delta^{-1}\left(s_{i_j},\alpha_{i_j}\right)}{(l-1)}\right)^{p_j}\right)\right)^{\frac{2}{n(n-1)}}\right)^{\frac{1}{p_1+p_2}} \right), \\ \left(\begin{array}{l} \left(1-\left(\prod_{1\le i_1<i_2\le n}\left(1-\prod_{j=1}^{2}\mu_{\varphi i_j}^{p_j}\right)\right)^{\frac{2}{n(n-1)}}\right)^{\frac{1}{p_1+p_2}}, 1-\left(1-\left(\prod_{1\le i_1<i_2\le n}\left(1-\prod_{j=1}^{2}\left(1-\eta_{\varphi i_j}^{p_j}\right)\right)\right)^{\frac{2}{n(n-1)}}\right)^{\frac{1}{p_1+p_2}}, \\ 1-\left(1-\left(\prod_{1\le i_1<i_2\le n}\left(1-\prod_{j=1}^{2}\left(1-\nu_{\varphi i_j}^{p_j}\right)\right)\right)^{\frac{2}{n(n-1)}}\right)^{\frac{1}{p_1+p_2}} \end{array} \right) \end{array} \right\rangle
\end{aligned} \tag{19}
$$

(3) When $m = n$, Equation (17) degrades to the following formula.

$$
\begin{aligned}
&ATT-P2TLGMSM^{(m,p_1,p_2,\ldots,p_m)}(\Phi_1,\ldots,\Phi_n) = \left(\frac{\underset{1\le i_1<\ldots<i_m\le n}{\oplus} \left(\overset{m}{\underset{j=1}{\otimes}} \Phi_{i_j}^{p_j} \right)}{C_n^m} \right)^{\frac{1}{p_1+p_2+\ldots+p_m}} \\
&= \left\langle \begin{array}{l} \Delta\left((l-1)\left(\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{i_j},\alpha_{i_j}\right)}{(l-1)}\right)^{p_j}\right)^{\frac{1}{p_1+p_2+\ldots+p_m}} \right), \\ \left(\left(\prod_{j=1}^{n}\mu_{\varphi i_j}^{p_j}\right)^{\frac{1}{p_1+p_2+\ldots+p_m}}, 1-\left(\prod_{j=1}^{n}\left(1-\eta_{\varphi i_j}^{p_j}\right)\right)^{\frac{1}{p_1+p_2+\ldots+p_m}}, 1-\left(\prod_{j=1}^{m}\left(1-\nu_{\varphi i_j}^{p_j}\right)\right)^{\frac{1}{p_1+p_2+\ldots+p_m}} \right) \end{array} \right\rangle
\end{aligned} \tag{20}
$$

4.2. The ATT-P2TLWMSM and ATT-P2TLWGMSM Operators

We now introduce the weighted *ATT-P2TLMSM* and *ATT-P2TLGMSM* operators to improve the decision making accuracy.

Definition 19. *Let* $\Phi_i = \left\langle (s_i, \alpha_i), \left(\mu_{\varphi i}, \eta_{\varphi i}, \nu_{\varphi i}\right)\right\rangle (i = 1, 2, \ldots, n)$ *be a set of P2TLNs and* $W = (w_1, w_2, \ldots, w_n)^T$ *be the weight vector, where* w_i *represents the importance degree of* Φ_i*, satisfying* $\sum_{i=1}^{n} w_i = 1$ *with* $w_i > 0$*. Then, the ATT-P2TLWMSM operator* $\Lambda^n \to \Lambda$ *is as follows.*

$$ATT - P2TLWMSM^{(m)}(\Phi_1, \ldots, \Phi_n) = \left(\frac{\underset{1 \le i_1 < \ldots < i_m \le n}{\oplus} \left(\overset{m}{\underset{j=1}{\otimes}} w_{i_j} \Phi_{i_j} \right)}{C_n^m} \right)^{\frac{1}{m}} \tag{21}$$

where Λ *is a set of P2TLNs and* $m = 1, 2, \ldots, n$.

On the basis of the operation rules of P2TLNs in Definition 16, the *ATT-P2TLWMSM* operators are shown below.

Theorem 3. *Let* $\Phi_i = \left\langle (s_i, \alpha_i), \left(\mu_{\varphi i}, \eta_{\varphi i}, \nu_{\varphi i}\right)\right\rangle$ $(i = 1, 2, \ldots, n)$ *be a set of P2TLNs and* $m = 1, 2, \ldots, n$*. Then, the result of aggregating via Definition 19 is also a P2TLN.*

$$\begin{aligned} &ATT - P2TLWMSM^{(m)}(\Phi_1, \ldots, \Phi_n) = \left(\frac{\underset{1 \le i_1 < \ldots < i_m \le n}{\oplus} \left(\overset{m}{\underset{j=1}{\otimes}} w_{i_j} \Phi_{i_j} \right)}{C_n^m} \right)^{\frac{1}{m}} \\ &\Delta\left((l-1) g^{-1}\left(\frac{1}{m} g\left(f^{-1}\left(\frac{1}{C_n^m}\left(\sum_{1 \le i_1 < \ldots < i_m \le n} f\left(g^{-1}\left(\sum_{j=1}^{m} g\left(f^{-1}\left(w_{i_j} f\left(\frac{\Delta^{-1}\left(s_{i_j}, \alpha_{i_j}\right)}{(l-1)} \right)\right)\right)\right)\right)\right)\right)\right)\right)\right)\right), \\ &= \left\langle \begin{pmatrix} g^{-1}\left(\frac{1}{m} g\left(f^{-1}\left(\frac{1}{C_n^m}\left(\sum_{1 \le i_1 < \ldots < i_m \le n} f\left(g^{-1}\left(\sum_{j=1}^{m} g\left(f^{-1}\left(w_{i_j} f\left(\mu_{\varphi i_j} \right)\right)\right)\right)\right)\right)\right)\right)\right)\right), \\ f^{-1}\left(\frac{1}{m} f\left(g^{-1}\left(\frac{1}{C_n^m}\left(\sum_{1 \le i_1 < \ldots < i_m \le n} g\left(f^{-1}\left(\sum_{j=1}^{m} f\left(g^{-1}\left(w_{i_j} g\left(\eta_{\varphi i_j} \right)\right)\right)\right)\right)\right)\right)\right)\right)\right), \\ f^{-1}\left(\frac{1}{m} f\left(g^{-1}\left(\frac{1}{C_n^m}\left(\sum_{1 \le i_1 < \ldots < i_m \le n} g\left(f^{-1}\left(\sum_{j=1}^{m} f\left(g^{-1}\left(w_{i_j} g\left(\nu_{\varphi i_j} \right)\right)\right)\right)\right)\right)\right)\right)\right)\right) \end{pmatrix} \right\rangle \end{aligned} \tag{22}$$

where $(i_1, i_2, \ldots, i_n)$ *traverses all m-tuple combinations of* $(1, 2, \ldots, n)$ *and* $C_n^m = \frac{n!}{m!(n-m)!}$ *is the binomial coefficient. The proof of Theorem 3 is analogous to that of Theorem 1, and we do not repeat the proof here.*

Property 3. *Let* $\Phi_i = \left\langle (s_i, \alpha_i), \left(\mu_{\varphi i}, \eta_{\varphi i}, \nu_{\varphi i}\right)\right\rangle (i = 1, 2, \ldots, n)$ *and* $\Phi_i^* = \left\langle \left(s_i^*, \alpha_i^*\right), \left(\mu_{\varphi i}^*, \eta_{\varphi i}^*, \nu_{\varphi i}^*\right)\right\rangle$ $(i = 1, 2, \ldots, n)$ *be sets of P2TLNs.* $ATT - P2TLWMSM^{(m, p_1, p_2, \ldots, p_m)}$ *then has a number of properties.*

(1) Monotonicity: If $(s_i, \alpha_i) \ge \left(s_i^*, \alpha_i^*\right)$, $\mu_{\varphi i} \ge \mu_{\varphi i}^*$, $\eta_{\varphi i} \le \eta_{\varphi i}^*$ *and* $\nu_{\varphi i} \le \nu_{\varphi i}^*$ *for each* i $(i = 1, 2, \ldots, n)$*, then* $\Phi_i \ge \Phi_i^*$ *and* $ATT - P2TLWMSM^{(m, p_1, p_2, \ldots, p_m)}(\Phi_1, \Phi_2, \ldots, \Phi_n) \ge ATT - P2TLWMSM^{(m, p_1, p_2, \ldots, p_m)}\left(\Phi_1^*, \Phi_2^*, \ldots, \Phi_n^*\right)$.

(2) Boundedness: If $\Phi^- = \min_i \Phi_i = \left\langle \min_i (s_i, \alpha_i), \left(\min_i\left(\mu_{\varphi i}\right), \min_i\left(\eta_{\varphi i}\right), \min_i\left(\nu_{\varphi i}\right)\right)\right\rangle$ *and* $\Phi^+ = \max_i \Phi_i = \left\langle \max_i (s_i, \alpha_i), \left(\max_i\left(\mu_{\varphi i}\right), \max_i\left(\eta_{\varphi i}\right), \max_i\left(\nu_{\varphi i}\right)\right)\right\rangle$*, then* $\Phi^- \le ATT - P2TLWMSM^{(m, p_1, p_2, \ldots, p_m)}(\Phi_1, \Phi_2, \ldots, \Phi_n) \le \Phi^+$.

The proof is analogous to that of Property 1 and is therefore omitted.

In the following, we present a detailed formula as an example to introduce the *P2TLMSM* operator in the context of MADM. When $g(x) = -\log x$, based on Formula (22), we obtain:

$$ATT-P2TLWMSM^{(m)}(\Phi_1,\ldots,\Phi_n)=\left(\frac{\underset{1\le i_1<\ldots<i_m\le n}{\oplus}\left(\overset{m}{\underset{j=1}{\otimes}} w_{i_j}\Phi_{i_j}\right)}{C_n^m}\right)^{\frac{1}{m}}$$
$$=\left\langle \begin{array}{l} \Delta\left((l-1)\left(1-\left(\prod_{1\le i_1<\ldots<i_m\le n}\left(1-\prod_{j=1}^{m}\left(1-\left(1-\frac{\Delta^{-1}\left(s_{i_j},\alpha_{i_j}\right)}{(l-1)}\right)^{w_{ij}}\right)\right)\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{m}}\right), \\ \left(\begin{array}{l}\left(1-\left(\prod_{1\le i_1<\ldots<i_m\le n}\left(1-\prod_{j=1}^{m}\left(1-\left(1-\mu_{\varphi i_j}\right)^{w_{i_j}}\right)\right)\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{m}}, 1-\left(1-\left(\prod_{1\le i_1<\ldots<i_m\le n}\left(1-\prod_{j=1}^{m}\left(1-\eta_{\varphi i_j}^{w_{i_j}}\right)\right)\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{m}}, \\ 1-\left(1-\left(\prod_{1\le i_1<\ldots<i_m\le n}\left(1-\prod_{j=1}^{m}\left(1-\nu_{\varphi i_j}^{w_{i_j}}\right)\right)\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{m}}\end{array}\right)\end{array}\right\rangle \tag{23}$$

If we consider some specific values of m, the following formulas can be obtained.

(1) When $m = 1$, Equation (23) degrades to the following formula.

$$ATT-P2TLWMSM^{(1)}(\Phi_1,\ldots,\Phi_n)=\frac{\underset{1\le i_1<\ldots<i_m\le n}{\oplus}\left(\overset{1}{\underset{j=1}{\otimes}}\Phi_{i_j}\right)}{C_n^1}$$
$$=\left\langle\Delta\left((l-1)\left(1-\left(\prod_{1\le i\le n}\left(1-\frac{\Delta^{-1}(s_i,\alpha_i)}{(l-1)}\right)^{w_i}\right)^{\frac{1}{n}}\right)\right),\left(1-\left(\prod_{1\le i\le n}\left(1-\mu_{\varphi i_j}\right)^{w_{i_j}}\right)^{\frac{1}{n}},\left(\prod_{1\le i_1<\ldots<i_m\le n}\eta_{\varphi i_j}^{w_{i_j}}\right)^{\frac{1}{n}},\left(\prod_{1\le i_1<\ldots<i_m\le n}\nu_{\varphi i_j}^{w_{i_j}}\right)^{\frac{1}{n}}\right)\right\rangle \tag{24}$$

(2) When $m = 2$, Equation (23) degrades to the following formula.

$$ATT-P2TLWMSM^{(2)}(\Phi_1,\ldots,\Phi_n)=\left(\frac{\underset{1\le i_1<i_2\le n}{\oplus}\left(\overset{2}{\underset{j=1}{\otimes}}\Phi_{i_j}\right)}{C_n^2}\right)^{\frac{1}{2}}$$
$$=\left\langle\begin{array}{l}\Delta\left((l-1)\left(1-\left(\prod_{1\le i_1<i_2\le n}\left(1-\prod_{j=1}^{2}\left(1-\left(1-\frac{\Delta^{-1}\left(s_{i_j},\alpha_{i_j}\right)}{(l-1)}\right)^{w_{ij}}\right)\right)\right)^{\frac{2}{n(n-1)}}\right)^{\frac{1}{2}}\right),\\ \left(\begin{array}{l}\left(1-\left(\prod_{1\le i_1<i_2\le n}\left(1-\prod_{j=1}^{2}\left(1-\left(1-\mu_{\varphi i_j}\right)^{w_{i_j}}\right)\right)\right)^{\frac{2}{n(n-1)}}\right)^{\frac{1}{2}},\\ 1-\left(1-\left(\prod_{1\le i_1<i_2\le n}\left(1-\prod_{j=1}^{2}\left(1-\eta_{\varphi i_j}^{w_{i_j}}\right)\right)\right)^{\frac{2}{n(n-1)}}\right)^{\frac{1}{2}},\\ 1-\left(1-\left(\prod_{1\le i_1<i_2\le n}\left(1-\prod_{j=1}^{2}\left(1-\nu_{\varphi i_j}^{w_{i_j}}\right)\right)\right)^{\frac{2}{n(n-1)}}\right)^{\frac{1}{2}}\end{array}\right)\end{array}\right\rangle \tag{25}$$

(3) When $m = n$, Equation (23) degrades to the following formula.

$$ATT-P2TLWMSM^{(n)}(\Phi_1,\ldots,\Phi_n)=\left(\frac{\underset{1\le i_1<\ldots<i_n\le n}{\oplus}\left(\overset{n}{\underset{j=1}{\otimes}}\Phi_{i_j}\right)}{C_n^n}\right)^{\frac{1}{n}}$$
$$=\left\langle\Delta\left((l-1)\left(\prod_{j=1}^{n}\left(1-\left(1-\frac{\Delta^{-1}\left(s_{i_j},\alpha_{i_j}\right)}{(l-1)}\right)^{w_{ij}}\right)\right)^{\frac{1}{n}}\right),\left(\left(\prod_{j=1}^{n}\left(1-\left(1-\mu_{\varphi i_j}\right)^{w_{ij}}\right)\right)^{\frac{1}{n}},1-\left(\prod_{j=1}^{n}\left(1-\eta_{\varphi i_j}^{w_{ij}}\right)\right)^{\frac{1}{n}},1-\left(\prod_{j=1}^{n}\left(1-\nu_{\varphi i_j}^{w_{ij}}\right)\right)^{\frac{1}{n}}\right)\right\rangle \tag{26}$$

Definition 20. *Let* $\Phi_i = \left\langle (s_i, \alpha_i), \left(\mu_{\varphi i}, \eta_{\varphi i}, \nu_{\varphi i}\right) \right\rangle (i = 1, 2, \ldots, n)$ *be a set of P2TLNs and* $W = (w_1, w_2, \ldots, w_n)^T$ *be the weight vector, where* w_i *represents the importance degree of* Φ_i*, satisfying* $\sum_{i=1}^{n} w_i = 1$ *with* $w_i > 0$*. Then, the ATT-P2TLGWMSM operator* $\Lambda^n \to \Lambda$ *is as follows.*

$$ATT - P2TLGWMSM^{(m, p_1, p_2, \ldots, p_m)}(\Phi_1, \ldots, \Phi_n) = \left(\frac{\underset{1 \le i_1 < \ldots < i_m \le n}{\oplus} \left(\overset{m}{\underset{j=1}{\otimes}} \left(w_{i_j} \Phi_{i_j} \right)^{p_j} \right)}{C_n^m} \right)^{\frac{1}{p_1 + p_2 + \ldots + p_m}} \tag{27}$$

where Λ *is a collection of P2TLNs and* $m = 1, 2, \ldots, n$.

On the basis of the operation rules of P2TLNs in Definition 16, the *ATT-P2TLGWMSM* operator is presented below.

Theorem 4. *Let* $\Phi_i = \left\langle (s_i, \alpha_i), \left(\mu_{\varphi i}, \eta_{\varphi i}, \nu_{\varphi i}\right) \right\rangle$ $(i = 1, 2, \ldots, n)$ *be a collection of P2TLNs and* $m = 1, 2, \ldots, n$. *Then, the result of aggregating with Definition 20 is also a P2TLN.*

$$\begin{aligned} & ATT - P2TLGWMSM^{(m)}(\Phi_1, \ldots, \Phi_n) = \left(\frac{\underset{1 \le i_1 < \ldots < i_m \le n}{\oplus} \left(\overset{m}{\underset{j=1}{\otimes}} \left(w_{i_j} \Phi_{i_j} \right)^{p_j} \right)}{C_n^m} \right)^{\frac{1}{p_1 + p_2 + \ldots + p_m}} \\ & = \left\langle \begin{array}{l} \Delta\left((l-1) g^{-1}\left(\frac{1}{p_1 + p_2 + \ldots + p_m} g\left(f^{-1}\left(\frac{1}{C_n^m} \left(\sum_{1 \le i_1 < \ldots < i_m \le n} f\left(g^{-1}\left(\sum_{j=1}^{m} g\left(g^{-1}\left(p_j g\left(f^{-1}\left(w_{i_j} f\left(\frac{\Delta^{-1}\left(s_{i_j}, \alpha_{i_j}\right)}{(l-1)} \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right), \\ \left(\begin{array}{l} g^{-1}\left(\frac{1}{p_1 + p_2 + \ldots + p_m} g\left(f^{-1}\left(\frac{1}{C_n^m} \left(\sum_{1 \le i_1 < \ldots < i_m \le n} f\left(g^{-1}\left(\sum_{j=1}^{m} g\left(g^{-1}\left(p_j g\left(f^{-1}\left(w_{i_j} f\left(\mu_{\varphi i_j} \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right), \\ f^{-1}\left(\frac{1}{p_1 + p_2 + \ldots + p_m} f\left(g^{-1}\left(\frac{1}{C_n^m} \left(\sum_{1 \le i_1 < \ldots < i_m \le n} g\left(f^{-1}\left(\sum_{j=1}^{m} f\left(f^{-1}\left(p_j f\left(g^{-1}\left(w_{i_j} g\left(\eta_{\varphi i_j} \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right), \\ f^{-1}\left(\frac{1}{p_1 + p_2 + \ldots + p_m} f\left(g^{-1}\left(\frac{1}{C_n^m} \left(\sum_{1 \le i_1 < \ldots < i_m \le n} g\left(f^{-1}\left(\sum_{j=1}^{m} f\left(f^{-1}\left(p_j f\left(g^{-1}\left(w_{i_j} g\left(\nu_{\varphi i_j} \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \end{array} \right) \end{array} \right\rangle \end{aligned} \tag{28}$$

where $(i_1, i_2, \ldots, i_n)$ *traverses all m-tuple combinations of* $(1, 2, \ldots, n)$ *and* $C_n^m = \frac{n!}{m!(n-m)!}$ *is the binomial coefficient. The proof of Theorem 4 is similar to that of Theorem 1 and is therefore omitted.*

Property 4. *Let* $\Phi_i = \left\langle (s_i, \alpha_i), \left(\mu_{\varphi i}, \eta_{\varphi i}, \nu_{\varphi i}\right) \right\rangle (i = 1, 2, \ldots, n)$ *and* $\Phi_i^* = \left\langle \left(s_i^*, \alpha_i^*\right), \left(\mu_{\varphi i}^*, \eta_{\varphi i}^*, \nu_{\varphi i}^*\right) \right\rangle$ $(i = 1, 2, \ldots, n)$ *be sets of P2TLNs.* $ATT - P2TLGWMSM^{(m, p_1, p_2, \ldots, p_m)}$ *has the following important properties.*

(1) *Monotonicity: If* $(s_i, \alpha_i) \ge \left(s_i^*, \alpha_i^*\right)$, $\mu_{\varphi i} \ge \mu_{\varphi i}^*$, $\eta_{\varphi i} \le \eta_{\varphi i}^*$ *and* $\nu_{\varphi i} \le \nu_{\varphi i}^*$ *for each* i $(i = 1, 2, \ldots, n)$*, then* $\Phi_i \ge \Phi_i^*$ *and* $ATT - P2TLGWMSM^{(m, p_1, p_2, \ldots, p_m)}(\Phi_1, \Phi_2, \ldots, \Phi_n) \ge ATT - P2TLGWMSM^{(m, p_1, p_2, \ldots, p_m)}\left(\Phi_1^*, \Phi_2^*, \ldots, \Phi_n^*\right)$.

(2) *Boundedness: If* $\Phi^- = \min_i \Phi_i = \left\langle \min_i (s_i, \alpha_i), \left(\min_i\left(\mu_{\varphi i}\right), \min_i\left(\eta_{\varphi i}\right), \min_i\left(\nu_{\varphi i}\right) \right) \right\rangle$ *and* $\Phi^+ = \max_i \Phi_i = \left\langle \max_i (s_i, \alpha_i), \left(\max_i\left(\mu_{\varphi i}\right), \max_i\left(\eta_{\varphi i}\right), \max_i\left(\nu_{\varphi i}\right) \right) \right\rangle$*, then* $\Phi^- \le ATT - P2TLGWMSM^{(m, p_1, p_2, \ldots, p_m)}(\Phi_1, \Phi_2, \ldots, \Phi_n) \le \Phi^+$.

The proofs are analogous to those of Property 1 and are therefore omitted.

In the following, we present a detailed formula as an example to introduce the *P2TLGWMSM* operator in the context of MADM. When $g(x) = -\log x$, based on Formula (28), we obtain:

$$ATT-P2TLGWMSM^{(m,p_1,\ldots,p_m)}(\Phi_1,\ldots,\Phi_n) = \left(\frac{\underset{1\le i_1<\ldots<i_m\le n}{\oplus}\left(\overset{m}{\underset{j=1}{\otimes}}\left(w_{i_j}\Phi_{i_j}\right)^{p_j}\right)}{C_n^m}\right)^{\frac{1}{p_1+p_2+\ldots+p_m}}$$
$$\Delta\left((l-1)\left(1-\left(\prod_{1\le i_1<\ldots<i_m\le n}\left(1-\prod_{j=1}^m\left(1-\left(1-\frac{\Delta^{-1}\left(s_{i_j},\alpha_{i_j}\right)}{(l-1)}\right)^{w_{ij}}\right)^{p_j}\right)\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{p_1+p_2+\ldots+p_m}}\right),$$
$$= \left\langle \begin{pmatrix} \left(1-\left(\prod_{1\le i_1<\ldots<i_m\le n}\left(1-\prod_{j=1}^m\left(1-\left(1-\mu_{\varphi i_j}\right)^{w_{i_j}}\right)^{p_j}\right)\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{p_1+p_2+\ldots+p_m}}, \\ 1-\left(1-\left(\prod_{1\le i_1<\ldots<i_m\le n}\left(1-\prod_{j=1}^m\left(1-\eta_{\varphi i_j}^{w_{i_j}}\right)^{p_j}\right)\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{p_1+p_2+\ldots+p_m}}, \\ 1-\left(1-\left(\prod_{1\le i_1<\ldots<i_m\le n}\left(1-\prod_{j=1}^m\left(1-\nu_{\varphi i_j}^{w_{i_j}}\right)^{p_j}\right)\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{p_1+p_2+\ldots+p_m}} \end{pmatrix} \right\rangle \quad (29)$$

If we consider some special values of m, the following formulas are obtained.

(1) When $m = 1$, Equation (29) degrades to the following formula.

$$ATT-P2TLGWMSM^{(1,p_1)}(\Phi_1,\ldots,\Phi_n) = \left(\frac{\underset{1\le i_1\le n}{\oplus}\left(\overset{1}{\underset{j=1}{\otimes}}\left(w_{i_j}\Phi_{i_j}\right)^{p_1}\right)}{C_n^1}\right)^{\frac{1}{p_1}}$$
$$\Delta\left((l-1)\left(1-\left(\prod_{1\le i_1\le n}\left(1-\left(1-\left(1-\frac{\Delta^{-1}\left(s_{i_1},\alpha_{i_1}\right)}{(l-1)}\right)^{w_{i1}}\right)^{p_1}\right)\right)^{\frac{1}{n}}\right)^{\frac{1}{p_1}}\right),$$
$$= \left\langle \begin{pmatrix} \left(1-\left(\prod_{1\le i_1\le n}\left(1-\left(1-\left(1-\mu_{\varphi i_1}\right)^{w_{i_1}}\right)^{p_1}\right)\right)^{\frac{1}{n}}\right)^{\frac{1}{p_1}}, 1-\left(1-\left(\prod_{1\le i_1\le n}\left(1-\left(1-\eta_{\varphi i_1}^{w_{i_1}}\right)^{p_1}\right)\right)^{\frac{1}{n}}\right)^{\frac{1}{p_1}}, \\ 1-\left(1-\left(\prod_{1\le i_1\le n}\left(1-\left(1-\nu_{\varphi i_1}^{w_{i_1}}\right)^{p_1}\right)\right)^{\frac{1}{n}}\right)^{\frac{1}{p_1}} \end{pmatrix} \right\rangle \quad (30)$$

(2) When $m = 2$, Equation (23) degrades to the following formula.

$$ATT-P2TLGWMSM^{(2,p_1,p_2)}(\Phi_1,\ldots,\Phi_n) = \left(\frac{\underset{1\le i_1<i_2\le n}{\oplus}\left(\overset{2}{\underset{j=1}{\otimes}}\left(w_{i_j}\Phi_{i_j}\right)^{p_j}\right)}{C_n^2}\right)^{\frac{1}{p_1+p_2}}$$
$$\Delta\left((l-1)\left(1-\left(\prod_{1\le i_1<i_2\le n}\left(1-\prod_{j=1}^2\left(1-\left(1-\frac{\Delta^{-1}\left(s_{i_j},\alpha_{i_j}\right)}{(l-1)}\right)^{w_{ij}}\right)^{p_j}\right)\right)^{\frac{2}{n(n-1)}}\right)^{\frac{1}{p_1+p_2}}\right),$$
$$= \left\langle \begin{pmatrix} \left(1-\left(\prod_{1\le i_1<i_2\le n}\left(1-\prod_{j=1}^2\left(1-\left(1-\mu_{\varphi i_j}\right)^{w_{i_j}}\right)^{p_j}\right)\right)^{\frac{2}{n(n-1)}}\right)^{\frac{1}{p_1+p_2}}, \\ 1-\left(1-\left(\prod_{1\le i_1<i_2\le n}\left(1-\prod_{j=1}^2\left(1-\eta_{\varphi i_j}^{w_{i_j}}\right)^{p_j}\right)\right)^{\frac{2}{n(n-1)}}\right)^{\frac{1}{p_1+p_2}}, \\ 1-\left(1-\left(\prod_{1\le i_1<i_2\le n}\left(1-\prod_{j=1}^2\left(1-\nu_{\varphi i_j}^{w_{i_j}}\right)^{p_j}\right)\right)^{\frac{2}{n(n-1)}}\right)^{\frac{1}{p_1+p_2}} \end{pmatrix} \right\rangle \quad (31)$$

(3) When $m = n$, Equation (23) degrades to the following formula.

$$\begin{aligned} & ATT-P2TLGWMSM^{(n,p_1,\ldots,p_n)}(\Phi_1,\ldots,\Phi_n) = \left(\frac{\underset{1\le i_1<\ldots<i_n\le n}{\oplus}\left(\overset{n}{\underset{j=1}{\otimes}}\left(w_{i_j}\Phi_{i_j}\right)^{p_j}\right)}{C_n^n} \right)^{\frac{1}{p_1+p_2+\ldots+p_n}} \\ & = \left\langle \begin{array}{l} \Delta\left((l-1)\left(\prod_{j=1}^{n}\left(1-\left(1-\frac{\Delta^{-1}\left(s_{i_j},\alpha_{i_j}\right)}{(l-1)}\right)^{w_{ij}}\right)^{p_j}\right)^{\frac{1}{p_1+p_2+\ldots+p_n}} \right), \\ \left(\left(\prod_{j=1}^{m}\left(1-\left(1-\mu_{\varphi i_j}\right)^{w_{i_j}}\right)^{p_j}\right)^{\frac{1}{p_1+p_2+\ldots+p_n}}, 1-\left(\prod_{j=1}^{m}\left(1-\eta_{\varphi i_j}^{w_{i_j}}\right)^{p_j}\right)^{\frac{1}{p_1+p_2+\ldots+p_n}}, 1-\left(\prod_{j=1}^{m}\left(1-\nu_{\varphi i_j}^{w_{i_j}}\right)^{p_j}\right)^{\frac{1}{p_1+p_2+\ldots+p_n}} \right) \end{array} \right\rangle \end{aligned} \tag{32}$$

5. MADM Based on the *ATT-P2TLMSM* Operator

Based on the *ATT-P2TLWMSM* and *ATT-P2TLGWMSM* operators, in this section, we address the MADM problems in which the attribute preference values take the form of picture 2-tuple linguistic variables.

Let $A = \{A_1, A_2 \ldots, A_m\}$ be a discrete set of alternatives, $C = \{C_1, C_2 \ldots, C_n\}$ be the set of attributes and $\omega = \{\omega_1, \omega_2 \ldots, \omega_n\}$ be the weighting vector of the attributes $C_j (j = 1, 2, \ldots, n)$, where $\omega_j \in [0, 1]$, $\sum_{j=1}^{n} \omega_j = 1$. For the alternative $A_i \in A$ of the attribute $C_j \in C$, the decision maker provides an attribute value φ_{i_j}, which is a picture 2-tuple linguistic variable. Each attribute value φ_{i_j} constitutes the decision matrix $R = \left(r_{ij}\right)_{m\times n} = \left\langle \left(s_{ij}, \alpha_{ij}\right), \left(\mu_{ij}, \eta_{ij}, \nu_{ij}\right)\right\rangle_{m\times n}$.

Next, we apply the *ATT-P2TLWMSM* and *ATT-P2TLGWMSM* operators to solve MADM problems in which the attribute values take the form of P2TLNs. The flowchart of the method is shown in Figure 2.

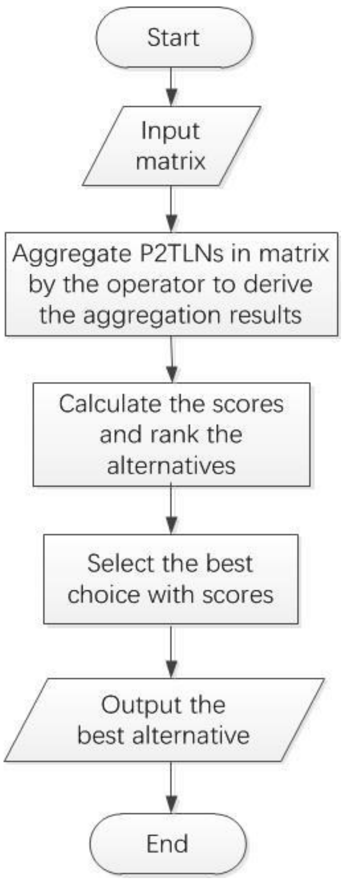

Figure 2. The flowchart of the method.

Step 1. Aggregate all P2TLNs via the *ATT-P2TLWMSM* or *ATT-P2TLGWMSM* operator to derive the aggregation results $\varphi_i(i = 1, 2, \ldots, m)$ of the alternatives $A = \{A_1, A_2 \ldots, A_m\}$.
Step 2. Calculate the scores $S(\varphi_i)(i = 1, 2, \ldots, m)$ of the P2TLNs $\varphi_i(i = 1, 2, \ldots, m)$ and rank the alternatives $A = \{A_1, A_2 \ldots, A_m\}$. If $S(\varphi_i)$ is equal to $S\left(\varphi_j\right)$, the accuracy degrees $H(\varphi_i)$ and $H\left(\varphi_j\right)$ must be calculated. Then, rank the alternatives $A = \{A_1, A_2 \ldots, A_m\}$ according to $H(\varphi_i)$ and $H\left(\varphi_j\right)$.
Step 3. Sort the alternatives $A = \{A_1, A_2 \ldots, A_m\}$ and select the best choice with $S(\varphi_i)(i = 1, 2, \ldots, m)$.
Step 4. End.

6. Illustrative Example

6.1. Data and Backdrop

In this section, we adapt a practical example from Wei et al. [36] to illustrate the methods proposed in this paper. A company plans to invest in an enterprise resource planning (ERP) system. Five optional ERP systems $A_i(i = 1, 2, \ldots, 5)$ are available, and the company considers the following four attributes when evaluating the alternatives: (1) C_1 represents functionality and technology; (2) C_2 represents strategic fitness; (3) C_3 represents vendor's ability; and (4) C_4 represents vendor's reputation. Furthermore, the weight vector of the four attributes is $\omega = (0.2, 0.1, 0.3, 0.4)$, and the decision matrix $\Phi = \left(\varphi_{ij}\right)_{5\times4}$ is given in Table 1, where $\varphi_{ij} \in S, S = (s_0, s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8)$ = (extremely bad, very poor, poor, slightly poor, fair, slightly good, good, very good, extremely good).

Table 1. Picture 2-tuple linguistic matrix.

Options\Attributes	C_1	C_2	C_3	C_4
A_1	$\langle(s_4, 0), (0.53, 0.33, 0.09)\rangle$	$\langle(s_2, 0), (0.89, 0.08, 0.03)\rangle$	$\langle(s_1, 0), (0.42, 0.35, 0.18)\rangle$	$\langle(s_3, 0), (0.08, 0.89, 0.02)\rangle$
A_2	$\langle(s_1, 0), (0.73, 0.12, 0.08)\rangle$	$\langle(s_4, 0), (0.13, 0.64, 0.21)\rangle$	$\langle(s_2, 0), (0.03, 0.82, 0.13)\rangle$	$\langle(s_4, 0), (0.73, 0.15, 0.08)\rangle$
A_3	$\langle(s_5, 0), (0.91, 0.03, 0.02)\rangle$	$\langle(s_1, 0), (0.07, 0.09, 0.05)\rangle$	$\langle(s_4, 0), (0.04, 0.85, 0.10)\rangle$	$\langle(s_2, 0), (0.68, 0.26, 0.06)\rangle$
A_4	$\langle(s_5, 0), (0.85, 0.09, 0.05)\rangle$	$\langle(s_6, 0), (0.74, 0.16, 0.10)\rangle$	$\langle(s_7, 0), (0.02, 0.89, 0.05)\rangle$	$\langle(s_1, 0), (0.08, 0.84, 0.06)\rangle$
A_5	$\langle(s_3, 0), (0.90, 0.05, 0.02)\rangle$	$\langle(s_1, 0), (0.68, 0.08, 0.21)\rangle$	$\langle(s_3, 0), (0.05, 0.87, 0.06)\rangle$	$\langle(s_1, 0), (0.13, 0.75, 0.09)\rangle$

6.2. Method Based on the ATT-P2TLWMSM and ATT-P2TLGWMSM Operators

In general, we set m = 2, according to Section 5. The procedures to address the MADM problem are as follows.

Step 1. Aggregate all P2TLNs by the *ATT-P2TLWMSM* or *ATT-P2TLGWMSM* operator to derive the aggregation results $\varphi_i(i = 1, 2, \ldots, m)$ of the alternatives $A = \{A_1, A_2 \ldots, A_m\}$. The aggregation results are listed in Table 2.

Table 2. The aggregation results of the ATT-P2TLWMSM and ATT-P2TLGWMSM operators.

Operators\Attributes	*ATT-P2TLWMSM*	*ATT-P2TLGWMSM* (p = 1, q = 2)
A_1	$\langle(s_1, -0.2945), (0.1259, 0.8214, 0.5405)\rangle$	$\langle(s_1, -0.1703), (0.1318, 0.8083, 0.4791)\rangle$
A_2	$\langle(s_1, -0.2271), (0.1368, 0.7807, 0.5978)\rangle$	$\langle(s_1, 0.0173), (0.1925, 0.7196, 0.5520)\rangle$
A_3	$\langle(s_1, -0.0282), (0.1649, 0.7178, 0.5077)\rangle$	$\langle(s_1, 0.0342), (0.2093, 0.7010, 0.4786)\rangle$
A_4	$\langle(s_2, 0.1086), (0.0981, 0.8565, 0.5212)\rangle$	$\langle(s_3, -0.0697), (0.0973, 0.8672, 0.4797)\rangle$
A_5	$\langle(s_1, 0.4509), (0.1113, 0.8132, 0.5332)\rangle$	$\langle(s_1, -0.3932), (0.0991, 0.8249, 0.5025)\rangle$

Step 2. Based on the aggregation results displayed in Table 2, the score functions of the ERP systems are given in Table 3.

Table 3. The score functions of the enterprise resource planning (ERP) systems.

Operators\Attributes	*ATT-P2TLWMSM*	*ATT-P2TLGWMSM* (p = 1, q = 2)
A_1	$(s_0, 0.2065)$	$(s_0, 0.2707)$
A_2	$(s_0, 0.2083)$	$(s_0, 0.3258)$
A_3	$(s_0, 0.3193)$	$(s_0, 0.3778)$
A_4	$(s_1, -0.3917)$	$(s_1, -0.0951)$
A_5	$(s_0, 0.1587)$	$(s_0, 0.1810)$

Step 3. Rank all the alternatives $A = \{A_1, A_2 \ldots, A_5\}$ based on the score functions in Table 3. The sorting results are given in Table 4.

Table 4. The final sorting results.

Operator	Parameter			Ranking
	m	p_1	p_2	
$ATT-P2TLWMSM^{(m)}$	2	-	-	$A_4 > A_3 > A_2 > A_1 > A_5$
$ATT-P2TLGWMSM^{(m,p_1,p_2)}$	2	1	2	$A_4 > A_3 > A_2 > A_1 > A_5$

The best choice is A_4.

Step 4. End

6.3. Comparative Analysis and Discussion

First, we compare the methods proposed in this paper with the methods developed by Wei [36,41]; the comparison results are presented in Table 5. As shown in Tables 4 and 5, the rank results of the methods proposed in this paper are the same as those of the methods in [36,41]; therefore, the approach developed in this paper is accurate and effective.

The merits of the methods proposed in this paper for addressing MADM problems are as follows:

(1) The calculation object of the operators proposed in this paper is P2TLN, which not only includes 2-tuple linguistic information but also expresses the degree of positive membership, the degree of neutral membership, the degree of negative membership and the degree of refusal membership of an element in linguistic terms. These functions make the representation of linguistic information more precise.

(2) The same ranking results as those in references [36,41] show that the methods proposed in this paper are valid and effective for solving MADM problems in which the attribute values take the form of picture 2-tuple linguistic information. Compared with the methods based on *P2TLWA* and *P2TLWGBM* proposed by Wei [36,41], the operators developed in this paper can capture the interrelationships among multiple input parameters. Therefore, the methods proposed in this paper are valid and correct and can solve MADM problems better.

Table 5. Comparison of different methods.

Method	Operator	Ranking
Method in [36]	$P2TLWA^{(m)}$ $(m = 2)$	$A_4 > A_3 > A_2 > A_1 > A_5$
Method in [41]	$P2TLWGBM^{(m,p_1,p_2)}$ $(p_1 = p_2 = 1)$	$A_4 > A_3 > A_2 > A_1 > A_5$
Proposed methods in this paper	$ATT-P2TLWMSM^{(m)}$ $(m = 2)$	$A_4 > A_3 > A_2 > A_1 > A_5$
Proposed methods in this paper	$ATT-P2TLGWMSM^{(m,p_1,p_2)}$ $(p_1 = p_2 = 1)$	$A_4 > A_3 > A_2 > A_1 > A_5$

In the following, we use Table 6 to illustrate the influence of parameters on the sorting results.

Table 6. Comparison results for different parameter values.

Operator	m	p_1	p_2	p_3	Ranking
$ATT-P2TLWMSM^{(m)}$	1	-	-	-	$A_4 > A_3 > A_2 > A_1 > A_5$
	2	-	-	-	$A_4 > A_3 > A_2 > A_1 > A_5$
	3	-	-	-	$A_4 > A_3 > A_1 > A_2 > A_5$
	4	-	-	-	$A_4 > A_3 > A_1 > A_2 > A_5$
$ATT-P2TLGWMSM^{(m,p_1,p_2)}$	2	0	1	-	$A_4 > A_2 > A_3 > A_1 > A_5$
		1	0	-	$A_4 > A_3 > A_5 > A_1 > A_2$
		1	1	-	$A_4 > A_3 > A_2 > A_1 > A_5$
		1	2	-	$A_4 > A_3 > A_2 > A_1 > A_5$
		1	3	-	$A_4 > A_3 > A_2 > A_1 > A_5$
		2	1	-	$A_4 > A_3 > A_5 > A_1 > A_2$
		2	2	-	$A_4 > A_3 > A_2 > A_1 > A_5$
		2	3	-	$A_4 > A_3 > A_2 > A_1 > A_5$
		3	1	-	$A_4 > A_3 > A_5 > A_1 > A_2$
		3	2	-	$A_4 > A_3 > A_1 > A_2 > A_5$
		3	3	-	$A_4 > A_3 > A_2 > A_1 > A_5$
$ATT-P2TLGWMSM^{(m,p_1,p_2,p_3)}$	3	1	0	0	$A_4 > A_3 > A_1 > A_5 > A_2$
		0	1	0	$A_4 > A_3 > A_5 > A_2 > A_1$
		0	0	1	$A_4 > A_2 > A_3 > A_1 > A_5$
		1	1	1	$A_4 > A_3 > A_1 > A_5 > A_2$
		1	1	2	$A_4 > A_3 > A_2 > A_1 > A_5$
		1	1	3	$A_4 > A_2 > A_3 > A_1 > A_5$
		1	1	4	$A_4 > A_2 > A_3 > A_1 > A_5$
		1	2	1	$A_4 > A_3 > A_5 > A_2 > A_1$
		1	3	1	$A_4 > A_3 > A_5 > A_2 > A_1$
		1	4	1	$A_4 > A_3 > A_5 > A_2 > A_1$
		2	1	1	$A_4 > A_3 > A_1 > A_5 > A_2$
		3	1	1	$A_4 > A_3 > A_1 > A_5 > A_2$
		4	1	1	$A_4 > A_3 > A_1 > A_5 > A_2$

As shown in Table 6, the best choice is A_4, and the worst choices are A_5 or A_2. When we select the *ATT-P2TLWMSM* operator to solve the MADM problem and m is given different values, although the optimal and worst choices remain the same, the ranking changes. The MSM operator captures the interrelationship among input parameters, and the value of m determines the relationships between how many input parameters must be considered. When the value of m is too large to be close to the number of attributes, ranking results have some small errors.

When we select the *ATT-P2TLGWMSM* operator to solve the MADM problem and $m = 2$, as shown in Table 6, the best option is still A_4. Furthermore, if p_1 or p_2 is equal to 0, the final sorting result has no practical meaning. Therefore, the arguments must be real numbers. In addition, when p_1 is less than or equal to p_2, the same ranking results are obtained.

Furthermore, there is a comparison for *ATT-P2TLGWMSM* with different values of p_1, p_2 and p_3 when $m = 3$. The ranking results are the same when two of the three parameters have the same value and the other is different. For example, when p_1 is equal to 2, 3, or 4 and p_2 is equal to p_3 (both equal 1), the ranking results are all $A_4 > A_3 > A_1 > A_5 > A_2$. The data in Table 6, indicate that if one of the arguments far exceeds the others, the sorting results may be disordered. Moreover, the content in Table 6 indicates that when $m = 3$, alternatives A_1, A_2 and A_5 are easily impacted.

In conclusion, the parameter values directly affect the final sorting results. Therefore, companies must choose suitable parameters to address MADM problems. In addition, the methods proposed in this paper are shown to be flexible and valid through the comparison and analysis of the above results.

7. Conclusions

In this paper, we solve MADM problems with picture 2-tuple linguistic information via novel aggregation operators—The ATT-P2TLMSM operator and the ATT-P2TLGMSM operator. Moreover, we discuss a few desired properties and series of specific cases of the proposed operators in detail. Considering that the input parameters have varying importance, the ATT-P2TLMSM operator and the ATT-P2TLGMSM operator are introduced. Finally, a method for MADM with picture 2-tuple linguistic information based on the proposed operators is developed, and an illustrative example is given to confirm the proposed operators. The method can not only capture the relationship of input parameters, but also take the ATT operational laws whose operations are closed into account. In addition, P2TLNs are suitable to describe decision makers' confidence that the final decision results would be effective and flexible.

In the future, we will study a series of novel operators based on P2TLNs that can accurately express uncertain information. In view of the fact that the ATT operation rules are not easy to understand, we will search for more easy-to-understand and simplified calculation rules without compromising accuracy. In addition, we will address more MADM problems, such as environmental evaluation, bank investment, and stock forecasting. Moreover, we will extend the MSM operators to handle MADM problems.

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