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On Quasi-Homogeneous Production Functions

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Abstract: In this paper, we investigate the class of quasi-homogeneous production models, obtaining the classification of such models with constant elasticity with respect to an input as well as with respect to all inputs. Moreover, we prove that a quasi-homogeneous production function f satisfies the proportional marginal rate of substitution property if and only f reduces to some symmetric production functions.

Keywords: production function; quasi-homogeneous production model; marginal rate of substitution; elasticity of production

MSC: 91B02

1. Introduction

The concept of a production function is a basic one in economic analysis, representing the mathematical formalization of the relationship between production and the factors that actually contribute to that production. Such a function is a mapping $f : \mathbb{R}^n_+ \to \mathbb{R}_+$, $n \geq 2$, given by

$$f = f(x_1, \ldots, x_n),$$

where n denotes the number of factors of production (inputs), $x_1, \ldots, x_n$ are the inputs and f is the level of output (production).

A fundamental problem in both macroeconomics and microeconomics is to investigate the behavior of a production process (on the firm level and on the aggregate level, respectively) under the action of inputs. From this point of view, the main indicators that characterize this behavior are the elasticity of production with respect to an input, the marginal rate of technical substitution and the elasticity of substitution between two inputs [1,2]. The characterization of the production models with constant elasticity of production, with proportional marginal rate of substitution (PMRS) property and with constant elasticity of substitution (CES) property is a challenging problem [3–7] and several classification results were obtained in the last years for different production functions, such as homogeneous, homothetic, quasi-sum and quasi-product production functions [8–11]. Other notable results concerning the above production models were recently derived using a differential geometric approach [12–25]. This treatment is based on the fact that one can associate a graph hypersurface to any production function and it is remarkable that one can relate basic concepts from production theory with some differential geometric invariants (intrinsic and extrinsic) of the associated hypersurface [26,27].

The aim of this work is to investigate a production model recently studied in [28] under the name of quasi-homogeneous two-factor production function, as a natural generalization of the family of classical homogeneous production functions with two inputs (labor and capital). It is important to highlight that the quasi-homogeneity property of production functions was originally considered in economics by Eichhorn and Oettli [29], see also ([30], Section 6.2) and the relevance of these production models was outlined by many authors over the years, see e.g., ([31], Chapter 12, [32–34]). In [28], the authors investigated quasi-homogeneous two-factor production functions with CES property. In the present paper, we focus on the general case of quasi-homogeneous production functions with arbitrary number of inputs, giving a complete classification of such production models with constant elasticity of production with respect to a certain input and with PMRS property. In particular, we generalize the main results of [35].

2. Preliminaries

Let f be a differentiable production function with n inputs $x_1, x_2, ..., x_n$, $n \geq 2$, and non-vanishing first partial derivatives. Next, we denote the partial derivatives $\frac{\partial f}{\partial x_i}, \frac{\partial^2 f}{\partial x_i \partial x_j}, \ldots$, etc. by $f_{x_i}, f_{x_i x_j}, \ldots$, etc. Then, the elasticity of production with respect to an input x_i, also known as partial output elasticity, is an economic concept defined by

$$E_{x_i} = \frac{x_i}{f} f_{x_i} \tag{1}$$

that quantifies the responsiveness of the output (production) to changes in an input (production factor).

The marginal rate of technical substitution between two inputs x_j and x_i, $i \neq j$, is a basic indicator of production defined by

$$\mathrm{MRS}_{ij} = \frac{f_{x_j}}{f_{x_i}}, \tag{2}$$

that measures the rate at which one input can be replaced by another such that the same level of output is obtained. We say that a production function satisfies the proportional marginal rate of substitution (PMRS) property if

$$\mathrm{MRS}_{ij} = \frac{x_i}{x_j}, \ 1 \leq i \neq j \leq n. \tag{3}$$

We recall that the most known and widely used production function is the famous Cobb–Douglas (CD) production function introduced by Charles W. Cobb and Paul H. Douglas in [36]. In the case of n factors of production $(n \geq 2)$, the CD production function is given by

$$f(x_1, ..., x_n) = A \cdot \prod_{i=1}^{n} x_i^{k_i}, \tag{4}$$

where $A > 0, k_1, \ldots, k_n \neq 0$. It is obvious that the CD production function is homogeneous of degree $p = \sum_{i=1}^{n} k_i$. It is also clear that f defined by Equation (4) is a symmetric function of n variables if and only if $k_1 = \ldots = k_n$; in this case we say that f is a symmetric CD production function.

A second production function of great interest was later considered by Arrow, Chenery, Minhas and Solow [37], Uzawa [38] and McFadden [39] as a natural generalization of the CD production function, by

$$f(x_1, ..., x_n) = A \left(\sum_{i=1}^{n} k_i x_i^{\rho} \right)^{\frac{\gamma}{\rho}}, \ (x_1, \ldots, x_n) \in D \subset \mathbb{R}^n_+, \tag{5}$$

where $A, k_1, \ldots, k_n, \gamma > 0, \rho < 1, \rho \neq 0$. It is also obvious that the above function, which is called the ACMS production function, but is also known as generalized CES production function because it has the CES property, is homogeneous of degree γ. For other classical homogeneous production models, the reader can refer to [40].

Recently, a more general class of production models was investigated in [28] under the name of quasi-homogeneous two-factor production functions. This class includes the family of homogeneous production functions with two inputs. We recall that a function f of n variables $x_1, x_2, ..., x_n$, $n \geq 2$, defined on a domain $D \subset \mathbb{R}^n$, is said to be a quasi-homogeneous (QH) function of degree q with weight vector $g = (g_1, \ldots, g_n) \in \mathbb{R}^n \setminus \{(0, \ldots, 0)\}$ if the following condition holds [41]

$$f(\lambda^{g_1} x_1, \ldots, \lambda^{g_n} x_n) = \lambda^q f(x_1, \ldots, x_n), \tag{6}$$

for all points $(x_1, \ldots, x_n) \in D$ and all $\lambda > 0$, provided that $(\lambda^{g_1} x_1, \ldots, \lambda^{g_n} x_n) \in D$.

3. Main Results

Definition 1. *Let $f : \mathbb{R}^n_+ \to \mathbb{R}_+$ be a production function with n inputs $x_1, x_2, ..., x_n$, where $n \geq 2$. If f satisfies the condition in Equation (6) for $(g_1, \ldots, g_n) \in \mathbb{R}^n \setminus \{(0, \ldots, 0)\}$ and $q \in \mathbb{R}$, then we say that f is a weight-homogeneous (WH) production function or a quasi-homogeneous (QH) production function of degree q with weight vector $g = (g_1, \ldots, g_n)$.*

Remark 1. *The quasi-homogeneity property in the above definition is a natural condition with a precise economic interpretation: if the production variables (inputs) are multiplied by different powers of the same factor, i.e., they grow in different number of times, then the output is multiplied by some power of this factor.*

Remark 2. *Obviously, the concept of QH production function generalizes the classical notion of homogeneous production function because a QH function of degree q with weight vector $g = (1, \ldots, 1)$ is nothing but a q-homogeneous function. On the other hand, it is clear that the concept of QH production function is more general because there are many examples of QH production functions which are not homogeneous. For instance, the production function $f : \mathbb{R}^3_+ \to \mathbb{R}_+$ with three inputs (K, L, N) (where K denotes the capital, L is the labor and N stands for the natural resources) defined by*

$$f(K, L, N) = 2KLN + \frac{3N^4}{K^6 + L^3 + N^2}$$

is a non-homogeneous QH production function of degree 6 with weight vector $g = (1, 2, 3)$.

Remark 3. *Notice that a necessary and sufficient condition for a differentiable function f of n variables $x_1, x_2, ..., x_n$, $n \geq 2$, to be QH of degree q with weight vector $g = (g_1, \ldots, g_n)$, is to satisfy the generalized Euler identity [41,42]:*

$$\sum_{i=1}^{n} g_i x_i f_{x_i} = qf. \tag{7}$$

Lemma 1. *Let f be a differentiable function of n variables $x_1, x_2, ..., x_n$, $n \geq 2$. If f is a QH function of degree q with weight vector $g = (g_1, \ldots, g_n) \in \mathbb{R}^n \setminus \{(0, \ldots, 0)\}$, then there exists at least one index $i \in \{1, \ldots, n\}$ such that f can be expressed in the form*

$$f(x_1, \ldots, x_n) = x_i^{\frac{q}{g_i}} h\left(\frac{x_1^{g_i}}{x_i^{g_1}}, \ldots, \frac{x_{i-1}^{g_i}}{x_i^{g_{i-1}}}, \frac{x_{i+1}^{g_i}}{x_i^{g_{i+1}}}, \ldots, \frac{x_n^{g_i}}{x_i^{g_n}}\right), \tag{8}$$

where h is a differentiable function of $n - 1$ variables.

Proof of Lemma 1. Because f is a QH function of degree q with weight vector g, it follows that the generalized Euler identity in Equation (7) is satisfied. In the following, we use the method of characteristics to solve the quasi-linear PDE in Equation (7). The characteristic equations in the nonparametric form are

$$\frac{dx_1}{g_1 x_1} = \ldots = \frac{dx_n}{g_n x_n} = \frac{df}{qf}.$$

However, since g has at least one nonzero entry, it is clear that there exists at least one index $i \in \{1, \ldots, n\}$ such that $g_i \neq 0$. Next, we set this index i. Consequently, the above characteristic equations can be written as

$$\frac{dx_i}{g_i x_i} = \frac{dx_j}{g_j x_j},$$

for $j = 1, \ldots, i-1, i+1, \ldots, n$, and

$$\frac{dx_i}{g_i x_i} = \frac{df}{qf}.$$

Integrating the above equations, we derive

$$\frac{x_j^{g_i}}{x_i^{g_j}} = C_j,$$

for $j = 1, \ldots, i-1, i+1, \ldots, n$, where $C_1, \ldots, C_{i-1}, C_{i+1}, \ldots, C_n$ are arbitrary real constants, and

$$\frac{f}{x_i^{\frac{q}{g_i}}} = C,$$

where C is an arbitrary real constant.

Therefore, the general solution of Equation (7) can be written in the implicit form as

$$\Phi\left(\frac{x_1^{g_i}}{x_i^{g_1}}, \ldots, \frac{x_{i-1}^{g_i}}{x_i^{g_{i-1}}}, \frac{x_{i+1}^{g_i}}{x_i^{g_{i+1}}}, \ldots, \frac{x_n^{g_i}}{x_i^{g_n}}, \frac{f}{x_i^{\frac{q}{g_i}}}\right) = 0,$$

where Φ is an arbitrary function, and the conclusion follows immediately. □

Theorem 1. *Let $f : \mathbb{R}_+^n \to \mathbb{R}_+$ be a twice differentiable production function of n inputs $(x_1, x_2, \ldots, x_n)$, $n \geq 2$. If f is a quasi-homogeneous production function of degree q with weight vector $(g_1, \ldots, g_n)$, then:*

i. The production has constant elasticity k_i with respect to an input x_i if and only if f reduces to the one of the following:

(a) a function of the form

$$f(x_1, x_2, \ldots, x_n) = x_i^{\frac{q}{g_i}} F(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n), \tag{9}$$

where F is a twice differentiable real valued function of $n-1$ variables, provided that the weights are

$$g_1 = 0, \ldots, g_{i-1} = 0, g_i = \frac{q}{k_i} \neq 0, g_{i+1} = 0, \ldots, g_n = 0;$$

or

(b) a function of the form

$$f(x_1, x_2, \ldots, x_n) = x_i^{k_i} x_j^{\frac{q - k_i g_i}{g_j}} F(u_1, \ldots, u_{n-2}), \tag{10}$$

where j is any element settled from the set $\{1, \ldots, n\} \setminus \{i\}$ for which $g_j \neq 0$ and F is a twice differentiable real valued function of $n-2$ variables

$$\{u_1, \ldots, u_{n-2}\} = \left\{\frac{x_k^{g_j}}{x_j^{g_k}} | k \in \{1, \ldots, n\} \setminus \{i, j\}\right\}. \tag{11}$$

ii. *The production has constant elasticity* k_i *with respect to all inputs* x_i, $i \in \{1, 2, \ldots, n\}$, *if and only if*

$$g_1 k_1 + g_2 k_2 + \ldots + g_n k_n = q \tag{12}$$

and f *reduces to the following CD production function:*

$$f(x_1, x_2, \ldots, x_n) = C x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n}, \tag{13}$$

where C *is a positive real constant.*

iii. f *satisfies the PMRS property if and only* f *reduces to the one of the following:*

(a) *a quasi-homogeneous function of degree* 0 *defined by the following homothetic symmetric CD production function*

$$f(x_1, x_2, \ldots, x_n) = F\left(x_1^{g_j} x_2^{g_j} \ldots x_n^{g_j}\right), \tag{14}$$

where F *is a twice differentiable real valued function of one variable and* j *is any element settled from the set* $\{1, \ldots, n\}$ *for which* $g_j > 0$, *provided that* $\sum_{i=1}^{n} g_i = 0$; *or*

(b) *a quasi-homogeneous function of degree* $q \neq 0$ *defined by the following symmetric CD production function*

$$f(x_1, x_2, \ldots, x_n) = C x_1^{\frac{q}{\sum_{i=1}^{n} g_i}} x_2^{\frac{q}{\sum_{i=1}^{n} g_i}} \ldots x_n^{\frac{q}{\sum_{i=1}^{n} g_i}}, \tag{15}$$

where C *is a positive real constant, provided that* $\sum_{i=1}^{n} g_i \neq 0$.

Proof of Theorem 1. i. The "if" part of the assertion follows by simple direct computation. Next, we prove the "only if" part of the assertion. Suppose that the production has constant elasticity k_i with respect to an input x_i. Then, we have from Equation (1):

$$f_{x_i} = k_i \frac{f}{x_i}. \tag{16}$$

We can distinguish now two cases.

Case 1. If there exists $j \neq i$ such that $g_j \neq 0$, then it follows from Lemma 1 that the function f can be expressed as

$$f(x_1, \ldots, x_n) = x_j^{\frac{q}{g_j}} h(u_1, \ldots, u_{n-1}), \tag{17}$$

where

$$u_k = \begin{cases} \dfrac{x_k^{g_j}}{x_j^{g_k}}, & 1 \leq k \leq j-1; \\ \dfrac{x_{k+1}^{g_j}}{x_j^{g_{k+1}}}, & j \leq k \leq n-1. \end{cases} \tag{18}$$

Taking into account that $i \neq j$, from Equation (17), we get

$$f_{x_i} = \begin{cases} g_j x_j^{\frac{q}{g_j} - g_i} x_i^{g_j - 1} h_{u_i}, & \text{if } i < j; \\ g_j x_j^{\frac{q}{g_j} - g_i} x_i^{g_j - 1} h_{u_{i-1}}, & \text{if } i > j. \end{cases} \tag{19}$$

Using now Equations (16) and (19), we deduce

$$k_i h = \begin{cases} g_j u_i h_{u_i}, & \text{if } i < j; \\ g_j u_{i-1} h_{u_{i-1}}, & \text{if } i > j \end{cases} \tag{20}$$

and, solving the partial differential equation (Equation (20)), we obtain the solution

$$h(u_1, \ldots, u_{n-1}) = \begin{cases} u_i^{\frac{k_i}{g_j}} F(u_1, \ldots, \widehat{u_i}, \ldots, u_{n-1}), & \text{if } i < j \\ (u_{i-1})^{\frac{k_i}{g_j}} F(u_1, \ldots, \widehat{u_{i-1}}, \ldots, u_{n-1}), & \text{if } i > j \end{cases}, \tag{21}$$

where F is a twice differentiable real valued function of $n-2$ variables and "$\widehat{\ }$" over u_k indicates that u_k is omitted.

Replacing now Equation (21) into Equation (17), and taking account of Equation (18), we derive that f takes the form of Equation (10). Hence, we have Case (b) of the statement.

Case 2. If $g_j = 0$, for all $j \neq i$, then it follows that $g_i \neq 0$. Therefore, Equation (7) reduces to

$$g_i x_i f_{x_i} = qf. \tag{22}$$

From Equations (16) and (22), it follows immediately that $q \neq 0$ and $g_i = \frac{q}{k_i}$, and the function f is given by Equation (9). Hence, we deduce Case (a) of the statement.

ii. If f is given by Equation (13) and the weights satisfy the condition in Equation (12), then we can easily check by direct computation that the production has constant elasticity k_i with respect to all inputs x_i, $i \in \{1, 2, \ldots, n\}$. Conversely, suppose that the partial output elasticity E_{x_i} with respect to the input x_i is a constant k_i, for all $i \in \{1, 2, \ldots, n\}$. Then, according to the number of non-zero values of weights, we distinguish the following two situations.

Case 1. Only one weight is non-zero. Then, we suppose that $g_i \neq 0$ and $g_j = 0$, for all $j \neq i$. Now, from the condition $E_{x_i} = k_i$, we deduce from i. that $k_i = \frac{q}{g_i}$ and f reduces to Equation (9). Imposing now the condition

$$E_{x_j} = k_j, \ \forall j \neq i,$$

and taking account of Equations (1) and (9), we get the following system of PDEs

$$\frac{F_{x_j}}{F} = \frac{k_j}{x_j}, \ \forall j \neq i$$

with the solution

$$F(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) = C \prod_{j \neq i} x_j^{k_j}, \tag{23}$$

where C is a positive real constant. Replacing now Equation (23) into Equation (9), we obtain that f reduces to the CD production function given by Equation (13). Moreover, we remark that Equation (12) is valid in this case.

Case 2. At least two weights are non-zero. Then, we suppose that $g_i \neq 0$ and $g_j \neq 0$, where $i, j \in \{1, \ldots, n\}$, $i \neq j$. Next, from the condition $E_{x_i} = k_i$, we deduce from i. that f reduces to Equation (10). Imposing now the conditions

$$E_{x_r} = k_r, \forall r \in \{1, \ldots, n\} - \{i, j\}$$

and taking account of Equations (1), (10) and (11), we get the following system of PDEs

$$\frac{F_{u_r}}{F} = \frac{k_r}{g_j} \frac{1}{u_r}, \ r = 1, \ldots, n-2,$$

having the solution

$$F(u_1, \ldots, u_{n-2}) = C \prod_{r=1}^{n-2} u_r^{\frac{k_r}{g_j}}, \tag{24}$$

where C is a positive real constant. Replacing now Equation (24) into Equation (10) and taking account of Equation (11), we obtain that f takes the form

$$f(x_1, \ldots, x_n) = C x_j^{\frac{q - \sum_{r \neq j} g_r k_r}{g_j}} \prod_{i \neq j} x_i^{k_i}, \tag{25}$$

where C is a positive real constant.

Finally, we impose the condition

$$E_{x_j} = k_j$$

and taking account of Equations (1) and (25), we derive that the weights must satisfy the relation

$$\frac{q - \sum_{r \neq j} g_r k_r}{g_j} = k_j, \tag{26}$$

which is equivalent to the condition in Equation (12). Consequently, replacing Equation (26) into Equation (25), we deduce that f reduces to the CD production function given by Equation (13).

iii. If either of Situation (a) or (b) occurs, then one can check easily by a direct calculation that the production function f satisfies the PMRS property. Next, we prove the "only if" part of the assertion. Assume that a quasi-homogeneous production function f of degree q with weight vector $(g_1, \ldots, g_n)$ satisfies the PMRS property. Then, we deduce from Equations (2) and (3) that

$$x_1 f_{x_1} = x_2 f_{x_2} = \ldots = x_n f_{x_n}. \tag{27}$$

Using the generalized Euler identity in Equation (7) and the above relation in Equation (27), we derive

$$x_i f_{x_i} \sum_{j=1}^{n} g_j = q f, \ i = 1, \ldots, n. \tag{28}$$

We can distinguish now the following two situations.

Case 1. If the weights satisfy $\sum_{i=1}^{n} g_i \neq 0$, then it follows from Equation (28) that $q \neq 0$ and we get the following system of PDEs

$$\frac{f_{x_i}}{f} = \frac{q}{\sum_{j=1}^{n} g_j} \frac{1}{x_i}, \ i = 1, \ldots, n,$$

having the solution

$$f(x_1, x_2, \ldots, x_n) = C x_1^{\frac{q}{\sum_{j=1}^{n} g_j}} \ldots x_n^{\frac{q}{\sum_{j=1}^{n} g_j}},$$

where C is a positive real constant. Hence, we have Case (b) of the statement.

Case 2. If the weights satisfy $\sum_{i=1}^{n} g_i = 0$, then it follows from Equation (28) that $q = 0$. Next, we set an index j such that $g_j > 0$, and we deduce from Lemma 1 that f can be written as

$$f(x_1, \ldots, x_n) = h(u_1, \ldots, u_{n-1}), \tag{29}$$

where h is a differentiable function of $n - 1$ variables $(u_1, \ldots, u_{n-1})$ expressed by Equation (18).

Replacing now Equation (29) into Equation (27), we derive

$$g_j u_i h_{u_i} = -\sum_{r=1}^{j-1} g_r u_r h_{u_r} - \sum_{r=j}^{n-1} g_{r+1} u_r h_{u_r}, \tag{30}$$

for $1 \leq i \leq n-1$, with $i \neq j$. Taking into account now that $g_j \neq 0$ and the weights satisfy $\sum_{i=1}^{n} g_i = 0$, we find that Equation (30) reduces to

$$u_1 h_{u_1} = \ldots = u_{n-1} h_{u_{n-1}}. \tag{31}$$

Solving the above system of PDEs, we obtain the solution

$$h(u_1, \ldots, u_{n-1}) = F\left(\prod_{i=1}^{n-1} u_i\right), \tag{32}$$

where F is a twice differentiable function of one variable. Finally, replacing Equations (32) and (18) into Equation (29), we derive that f is given by

$$f(x_1, \ldots, x_n) = F\left(x_j^{-\sum_{i \neq j} g_i} \prod_{i \neq j} x_i^{g_j}\right)$$

and taking into account that $\sum_{i=1}^{n} g_i = 0$, we conclude that f takes the form of Equation (14). Hence, we have Case (a) of the statement. $\square$

In the particular case of two inputs, Theorem 1 reduces to the following.

Corollary 1. *Let $f : \mathbb{R}_+^2 \to \mathbb{R}_+$ be a twice differentiable production function with two inputs (K, L), where K denotes the capital and L stands for labor. If f is a quasi-homogeneous production function of degree q with weight vector (g_K, g_L), then:*

i. *The output elasticity with respect to capital is a constant k if and only if f reduces to the one of the following:*
   (a) *a function having the form:*
   $$f(K, L) = K^{\frac{q}{g_K}} F(L), \tag{33}$$
   where F is a twice differentiable real valued function of one real variable, provided that the weights are
   $$g_K = \frac{q}{k} \neq 0, g_L = 0;$$
   or
   (b) *a CD production function given by:*
   $$f(K, L) = CK^k L^{\frac{q - k g_K}{g_L}}, \tag{34}$$
   where C is a positive real constant, provided that $g_L \neq 0$.

ii. *The output elasticity with respect to labor is a constant k if and only if f reduces to the one of the following:*
   (a) *a function having the form:*
   $$f(K, L) = L^{\frac{q}{g_L}} F(K), \tag{35}$$
   where F is a twice differentiable real valued function of one real variable, provided that the weights are
   $$g_K = 0, g_L = \frac{q}{k} \neq 0;$$
   or
   (b) *a CD production function given by:*
   $$f(K, L) = CK^{\frac{q - k g_L}{g_K}} L^k, \tag{36}$$
   where C is a positive real constant, provided that $g_K \neq 0$.

iii. The output elasticities of labor and capital are both constant, k and l, respectively, if and only if

$$g_K k + g_L l = q \tag{37}$$

and f reduces to the following CD production function:

$$f(K, L) = CK^k L^l, \tag{38}$$

where C is a positive real constant.

iv. f satisfies the PMRS property if and only if f reduces to the one of the following:

(a) a quasi-homogeneous function of degree 0 defined by the following homothetic CD production function

$$f(K, L) = F\left(K^{g_K} L^{-g_L}\right), \tag{39}$$

where F is a twice differentiable real valued function of one real variable, provided that $g_K + g_L = 0$; or

(b) a quasi-homogeneous function of degree $q \neq 0$ defined by the following CD production function

$$f(K, L) = CK^{\frac{q}{g_K + g_L}} L^{\frac{q}{g_K + g_L}}, \tag{40}$$

where C is a positive real constant, provided that $g_K + g_L \neq 0$.

Remark 4. *Notice that Theorem 1 and Corollary 1 generalize the main results of [35] concerning the classification of homogeneous production models with PMRS property and with constant elasticity of production with respect to inputs. Indeed, if the quasi-homogeneous production function f of degree q reduces to a homogeneous model, then the weight vector is $g = (1, \ldots, 1)$ and consequently some cases cannot occur in the classification theorems established in this article. Therefore, in the particular setting of homogeneous models, it is easy to see that Theorem 1 and Corollary 1 reduce to ([35], Theorem 5) and ([35], Theorem 4), respectively.*

4. Conclusions and Future Works

It is well known that one of the key concepts in economic theory is the production function. In this paper, we focus on the production functions satisfying the quasi-homogeneity property, a property originally considered in economics in [29] and investigated by many authors over the years. We are primarily interested in studying such production models that either exhibit a constant partial elasticity with respect to its inputs, or satisfy the PMRS property. The main theorem of this article characterizes the analytical form of such functions. Therefore, from the point of view of the specific contribution of the present work to the economic theory, this study completes the understanding of the relations between constant partial elasticity and proportional marginal rate of substitution, and the structure of the production function. Notice that the present work not only extends some classification results from homogeneous to non-homogeneous production models, but also enriches the classification by identifying some new production models that are not included in the original classification, such as the homothetic symmetric Cobb–Douglas production functions—a class of non-homogeneous production models satisfying the PMRS property. The results obtained in this article motivate further studies to obtain classification results for other non-homogeneous production models, such as homothetic [1], ray-homothetic [32], semi-homogenous [43] and almost ray-homothetic [44]. It would be of particular interest to investigate the feature that the MRS is constant for proportional change of all inputs, since the constant MRS along the same ray from the origin in the input space is an important property in economic analysis [45,46].

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