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Ball Convergence for Combined Three-Step Methods Under Generalized Conditions in Banach Space

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Abstract: Problems from numerous disciplines such as applied sciences, scientific computing, applied mathematics, engineering to mention some can be converted to solving an equation. That is why, we suggest higher-order iterative method to solve equations with Banach space valued operators. Researchers used the suppositions involving seventh-order derivative by Chen, S.P. and Qian, Y.H. But, here, we only use suppositions on the first-order derivative and Lipschitz constrains. In addition, we do not only enlarge the applicability region of them but also suggest computable radii. Finally, we consider a good mixture of numerical examples in order to demonstrate the applicability of our results in cases not covered before.

Keywords: local convergence; convergence order; Banach space; iterative method

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1. Introduction

One of the most useful task in numerical analysis concerns finding a solution κ of

$$\Theta(x) = 0, \tag{1}$$

where $\Theta : \mathbb{D} \subset \mathbb{X} \to \mathbb{Y}$ is a Fréchet-differentiable operator, $\mathbb{X},\ \mathbb{Y}$ are Banach spaces and $\mathbb{D}$ is a convex subset of $\mathbb{X}$. The $L(\mathbb{X},\ \mathbb{Y})$ is the space of bounded linear operators from $\mathbb{X}$ to $\mathbb{Y}$.

Consider, a three step higher-order convergent method defined for each $l = 0, 1, 2, \dots$ by

$$\begin{aligned} y_l &= x_l - \Theta'(x_l)^{-1}\Theta(x_l),\\ z_l &= \phi\Big(x_l,\ \Theta(x_l),\ \Theta'(x_l),\ \Theta'(y_l)\Big),\\ x_{l+1} &= z_l - \beta A_l^{-1}\Theta(z_l), \end{aligned} \tag{2}$$

where $\alpha,\ \beta \in \mathbb{S}$, $A_l = (\beta-\alpha)\Theta'(x_l) + \alpha\Theta'(y_l)$, $(\mathbb{S} = \mathbb{R}$ or $\mathbb{S} = \mathbb{C})$ and the second sub step represents any iterative method, in which the order of convergence is at least $m = 1,\ 2,\ 3, \dots$. If $\mathbb{X} = \mathbb{Y} = \mathbb{R}$, then it was shown in [1]. The proof uses Taylor series expansions and the conditions on function Θ is up to the seventh differentiable. These suppositions of derivatives on the considered function Θ hamper the applicability of (2). Consider, a function μ on $\mathbb{X} = \mathbb{Y} = \mathbb{R}$, $\mathbb{D} = [-0.5, 1.5]$ by

$$\mu(t) = \begin{cases} 0, & t = 0\\ t^3 \ln t^2 + t^5 - t^4, & t \neq 0 \end{cases}.$$

Then, we have that

$$\mu'(t) = 3t^2 \ln t^2 + 5t^4 - 4t^3 + 2t^2,$$

$$\mu''(t) = 6t \ln t^2 + 20t^3 - 12t^2 + 10t$$

and

$$\mu'''(t) = 6 \ln t^2 + 60t^2 - 24t + 22.$$

Then, obviously the third-order derivative $\mu'''(t)$ is not bounded on $\mathbb{D}$. Method (2) studied in [1], for $\mathbb{X} = \mathbb{Y} = \mathbb{R}$ suffers from several following defects:

(i) Applicable only on the real line.
(ii) Range of initial guesses for granted convergence is not discussed.
(iii) Higher than first order derivatives and Taylor series expansions were used limiting the applicability.
(iv) No computable error bounds on $\|\Omega_l\|$ (where $\Omega_l = x_l - \kappa$) were given.
(v) No uniqueness result was addressed.
(vi) The convergence order claim by them is also not correct, e.g., see the following method 43 [1]

$$\begin{aligned} y_l &= x_l - \frac{\Theta(x_l)}{\Theta'(x_l)}, \\ z_l &= x_l - \frac{2\Theta(x_l)}{\Theta'(y_l) + \Theta'(x_l)}, \\ x_{l+1} &= z_l - \frac{\beta\Theta(z_l)}{\alpha\Theta'(y_l) + (\beta - \alpha)\Theta'(x_l)}. \end{aligned} \tag{3}$$

It has fifth-order of convergence for $\alpha = \beta$ but $\alpha \neq \beta \in \mathbb{R}$ provides fourth-order convergence. But, authors claimed sixth-order convergence for every $\alpha, \ \beta \in \mathbb{R}$ that is not correct. The new proof is given in Section 2.
(vii) They can't choose special cases like methods 41, 47 and 49 (numbering from their paper [1]) because Chen and Qian [1], consider $y_l = x_l - \frac{f(x_l)}{f'(x_l)}$ in the proof of theorem. Additionally, it is clearly mentioned in the expression of (21) (from their paper [1]).

To address all these problems, we first extend method (2) to Banach space valued operators. The order of convergence is computed by using *COC* or *ACOC* (see remark 2.2(d)). Our technique uses only the first derivative in the analysis of method (2), so we can solve classes of equations not possible before in [1].

The remaining material of the paper is ordered as proceeds: Section 2 suggest convergence study of scheme (2). The applicability of our technique appears in Section 3.

2. Convergence Analysis

We consider some scalars functions and constraints for convergence study. Therefore, we assume that functions $v, \ w_0, \ w, \ \bar{g}_2 : [0, \ +\infty) \to [0, \ +\infty)$ are continuous and nondecreasing with $w_0(0) = w(0) = 0$ and $\alpha, \ \beta \in \mathbb{S}$. Assume equation

$$w_0(t) = 1 \tag{4}$$

has a minimal positive solution r_0.

Functions $g_1,\ h_1,\ p$ and h_p defined on $[0,\ r_0)$ as follow:

$$g_1(t) = \frac{\int_0^1 w\big((1-\eta)t\big)d\eta}{1-w_0(t)},$$
$$h_1(t) = g_1(t) - 1,$$
$$p(t) = |\beta|^{-1}\Big[|\beta-\alpha|w_0(t) + |\alpha|w_0\big(g_1(t)t\big)\Big],\ \beta \neq 0,$$
and
$$h_p(t) = p(t) - 1.$$

Notice, that $h_1(0) = h_p(0) = -1 < 0$ and $h_1(t) \to +\infty$, $h_q(t) \to +\infty$ as $t \to r_0^-$. Then, by the intermediate value theorem (IVT), the functions h_1 and h_p have roots in $(0,\ r_0)$. Let r_1 and r_p, stand respectively the smallest such roots of the function h_1 and h_p. Additionally, we consider two functions g_2 and h_2 on $(0,\ r_0)$ by

$$g_2(t) = \bar{g}_2(t)t^{m-1},$$
and
$$h_2(t) = g_2(t) - 1.$$

Suppose that

$$\bar{g}_2(0) < 1,\ \text{if}\ m = 1 \tag{5}$$

and

$$g_2(t) \to a\ (\text{a number greater than one or } +\infty) \tag{6}$$

as $t \to \bar{r}_0^-$ for some $\bar{r}_0 \leq r_0$. Then, again by adopting IVT that function h_2 has some roots $(0,\ \bar{r}_0)$. Let r_2 be the smallest such root. Notice that, if $m > 1$ condition (5) is not needed to show $h_2(0) < 0$, since in this case $h_2(0) = g_2(0) - 1 = 0 - 1 = -1 < 0$.

Finally, functions g_3 and h_3 on $[0,\ \bar{r}_p)$ by

$$g_3(t) = \left(1 + \frac{\int_0^1 v(\eta g_2(t)t)d\eta}{1-p(t)}\right)g_2(t),$$
and
$$h_3(t) = g_3(t) - 1,$$

where $\bar{r}_p = \min\{r_p,\ r_2\}$. Suppose that

$$(1+v(0))\bar{g}_2(0) < 1,\ \text{if}\ m = 1, \tag{7}$$

we get by (7) that $h_3(0) = (1+v(0))\bar{g}_2(0) - 1 < 0$ and $h_3(t) \to +\infty$ or positive number as $t \to \bar{r}_p^-$. Let r_3 stand for the smallest root of function h_3 in $(0,\ r_p)$. Consider a radius of convergence r as

$$r = \min\{r_1,\ r_3\}. \tag{8}$$

Then, it holds

$$0 \leq g_i(t) < 1,\ i = 1,\ 2,\ 3 \text{ for each } t \in [0,\ r). \tag{9}$$

Let us assume that we have center $z \in \mathbb{X}$ and radius $\rho > 0$ of $U(z,\ \rho)$ and $\bar{U}(z,\ \rho)$ open and closed ball, respectively, in the Banach space $\mathbb{X}$.

Theorem 1. *Let $\Theta : \mathbb{D} \subseteq \mathbb{X} \to \mathbb{Y}$ be a differentiable operator. Let $v,\ w_0,\ w,\ \bar{g}_2 : [0,\ \infty) \to [0,\ \infty)$ be nondecreasing continuous functions with $w_0(0) = w(0) = 0$. Additionally, we consider that $r_0 \in [0,\ \infty)$, $\alpha \in S$, $\beta \in S - \{0\}$ and $m \geq 1$. Assume that there exists $\kappa \in \mathbb{D}$ such that for every $\lambda_1 \in \mathbb{D}$*

$$\Theta(\kappa) = 0, \quad \Theta'(\kappa)^{-1} \in L(\mathbb{Y},\ \mathbb{X}), \tag{10}$$

$$\|\Theta'(\kappa)^{-1}\big(\Theta'(\lambda_1) - \Theta'(\kappa)\big)\| \leq w_0(\|\lambda_1 - \kappa\|). \tag{11}$$

and Equation (4) has a minimal solution r_0 and (5) holds.

Moreover, assume that for each $\lambda_1,\ \lambda_2 \in \mathbb{D}_0 := \mathbb{D} \cap U(\kappa,\ r_0)$

$$\|\Theta'(\kappa)^{-1}\big(\Theta'(\lambda_1) - \Theta'(\lambda_2)\big)\| \leq w(\|\lambda_1 - \lambda_2\|), \tag{12}$$

$$\|\Theta'(\kappa)^{-1}\Theta'(\lambda_1)\| \leq v(\|\lambda_1 - \kappa\|), \tag{13}$$

$$\|\phi(\lambda_1,\ \Theta(\lambda_1),\ \Theta'(\lambda_1),\ \Theta'(\lambda_2))\| \leq \bar{g}_2(\|\lambda_1 - \kappa\|)\|\lambda_1 - \kappa\|^m \tag{14}$$

and

$$\bar{U}(\kappa, r) \subseteq \mathbb{D}. \tag{15}$$

Then, for $x_0 \in U(\kappa,\ r) - \{\kappa\}$, we have $\lim\limits_{l\to\infty} x_l = \kappa$, where $\{x_l\} \subset U(\kappa,\ r)$ and the following assertions hold

$$\|y_l - \kappa\| \leq g_1(\|\Omega_l\|)\|\Omega_l\| \leq \|\Omega_l\| < r, \tag{16}$$

$$\|z_l - \kappa\| \leq g_2(\|\Omega_l\|)\|\Omega_l\| \leq \|\Omega_l\| \tag{17}$$

and

$$\|x_{l+1} - \kappa\| \leq g_3(\|\Omega_l\|)\|\Omega_l\| \leq \|\Omega_l\|, \tag{18}$$

where $x_l - \kappa = \Omega_l$ and functions $g_i,\ i = 1,\ 2,\ 3$ are given previously. Moreover, if $R \geq r$

$$\int_0^1 w_0(\eta R)d\eta < 1, \tag{19}$$

then κ is unique in $\mathbb{D}_1 := \mathbb{D} \cap \bar{U}(\kappa,\ R)$.

Proof. We demonstrate that the sequence $\{x_l\}$ is well-defined in $U(\kappa,\ r)$ and converges to κ by adopting mathematical induction. By the hypothesis $x_0 \in U(\kappa, r) - \{\kappa\}$, (4), (6) and (13), we yield

$$\|\Theta'(\kappa)^{-1}(\Theta'(x_0) - \Theta'(\kappa))\| \leq w_0(\|\Omega_0\|) < w_0(r) < 1, \tag{20}$$

where $\Omega_0 = x_0 - \kappa$ and $\Theta'(x_0)^{-1} \in L(\mathbb{Y},\ \mathbb{X})$, y_0 exists by the first two sub steps of method (2) and

$$\big\|\Theta'(x_0)^{-1}\Theta'(\kappa)\big\| \leq \frac{1}{1 - w_0(\|\Omega_0\|)}. \tag{21}$$

From (4), (8), (9) (for $i = 1$), (10), (12), (21) and the first substep of (2), we have

$$\begin{aligned}
\|y_0 - \kappa\| &= \|\Omega_0 - \Theta'(x_0)^{-1}\Theta(x_0) - \kappa\| \\
&= \left\|\Theta'(x_0)^{-1}\left[\Theta'(x_0)(\Omega_0 - \kappa) - \big(\Theta(x_0) - \Theta(\kappa)\big)\right]\right\| \\
&= \left\|\left[\Theta'(x_0)^{-1}\Theta'(\kappa)\right]\left[\Theta'(\kappa)^{-1}\Big(\Theta'(x_0)(\Omega_0 - \kappa) - \big(\Theta(x_0) - \Theta(\kappa)\big)\Big)\right]\right\| \\
&\leq \|\Theta'(x_0)^{-1}\Theta(\kappa)\|\left\|\int_0^1 \Big(\Theta'(\kappa)^{-1}(\Theta'(\kappa + \eta(\Omega_0 - \kappa)) - \Theta'(x_0))(\Omega_0)\Big)d\eta\right\| \\
&\leq \frac{\int_0^1 w((1-\eta)\|\Omega_0\|)d\eta\|\Omega_0\|}{1 - w_0(\|\Omega_0\|)} \\
&\leq g_1(\|\Omega_0\|)\|\Omega_0\| \leq \|\Omega_0\| < r,
\end{aligned} \tag{22}$$

which implies (16) for $l = 0$ and $y_0 \in U(\kappa, r)$.

By (8), (9) (for $i = 2$) and (14), we get

$$\begin{aligned}
\|z_0 - \kappa\| &= \|\phi(x_0, \Theta(x_0), \Theta'(x_0), \Theta'(y_0))\| \\
&\leq \bar{g}_2(\|\Omega_0\|)\|\Omega_0\|^m \\
&= g_2(\|\Omega_0\|)\|\Omega_0\| \leq \|\Omega_0\| < r,
\end{aligned} \tag{23}$$

so (17) holds $l = 0$ and $z_0 \in U(\kappa, r)$.

Using expressions (4), (8) and (11), we obtain

$$\begin{aligned}
&\left\|(\beta\Theta'(\kappa))^{-1}\left[(\beta - \alpha)(\Theta'(x_0) - \Theta'(\kappa)) + \alpha(\Theta'(y_0) - \Theta'(\kappa))\right]\right\| \\
&\qquad \leq |\beta|^{-1}\left[|\beta - \alpha|w_0(\|\Omega_0\|) + |\alpha|w_0(\|y_0 - \kappa\|)\right] \\
&\qquad \leq |\beta|^{-1}\left[|\beta - \alpha|w_0(\|\Omega_0\|) + |\alpha|w_0(g_1(\|\Omega_0\|)\|\Omega_0\|)\right] \\
&\qquad = p(\|\Omega_0\|) \leq p(r) < 1,
\end{aligned} \tag{24}$$

so

$$\|((\beta - \alpha)\Theta'(x_0) + \alpha\Theta'(y_0))^{-1}\Theta'(\kappa)\| \leq \frac{1}{1 - p(\|\Omega_0\|)}. \tag{25}$$

and x_1 is well-defined.

In view of (4), (8), (9) (for $i = 3$), (13), (22), (23) and (24), we get in turn that

$$\begin{aligned}
\|x_1 - \kappa\| &= \|z_0 - \kappa\| + |\beta|\int_0^1 v(\eta\|z_0 - \kappa\|)d\eta\|\Omega_0\| \\
&\leq \left(1 + \frac{|\beta|\int_0^1 v(\eta g_2(\|\Omega_0\|))d\eta}{|\beta|(1 - p(\|\Omega_0\|))}\right)g_2(\|\Omega_0\|)\|\Omega_0\| \\
&= g_3(\|\Omega_0\|)\|\Omega_0\| \leq \|\Omega_0\| < r,
\end{aligned} \tag{26}$$

that demonstrates (18) and $x_1 \in U(\kappa, r)$. If we substitute x_0, y_0, x_1 by x_l, y_l, x_{l+1}, we arrive at (18) and (19). By adopting the estimates

$$\|x_{l+1} - \kappa\| \leq c\|\Omega_l\| < r, \; c = g_2(\|\Omega_0\|) \in [0, 1), \tag{27}$$

so $\lim_{l\to\infty} x_l = \kappa$ and $x_{l+1} \in U(\kappa, r)$.

Now, only the uniqueness part is missing, so we assume that $\kappa^* \in \mathbb{D}_1$ with $\Theta(\kappa^*) = 0$. Consider, $Q = \int_0^1 \Theta'(\kappa + \eta(\kappa - \kappa^*))d\eta$. From (8) and (15), we obtain

$$\begin{aligned} \|\Theta'(\kappa)^{-1}(Q - \Theta'(\kappa))\| &\leq \| \int_0^1 w_0(\eta\|\kappa^* - \kappa\|)d\eta \\ &\leq \int_0^1 w_0(\eta R)d\eta < 1, \end{aligned} \tag{28}$$

and by

$$0 = \Theta(\kappa) - \Theta(\kappa^*) = Q(\kappa - \kappa^*), \tag{29}$$

we derive $\kappa = \kappa^*$. □

Remark 1.

(a) *By expression* (13) *hypothesis* (15) *can be omitted, if we set*

$$v(t) = 1 + w_0(t) \text{ or } v(t) = 1 + w_0(r_0), \tag{30}$$

since,

$$\begin{aligned} \|\Theta'(\kappa)^{-1}\left[(\Theta'(x) - \Theta'(\kappa)) + \Theta'(\kappa)\right]\| &= 1 + \|\Theta'(\kappa)^{-1}(\Theta'(x) - \Theta'(\kappa))\| \\ &\leq 1 + w_0(\|x - \kappa\|) \\ &= 1 + w_0(t) \quad \text{for} \quad \|x - \kappa\| \leq r_0. \end{aligned} \tag{31}$$

(b) *Consider w_0 to be strictly increasing, so we have*

$$r_0 = w_0^{-1}(1) \tag{32}$$

for (4).

(c) *If w_0 and w are constants, then*

$$r_1 = \frac{2}{2w_0 + w} \tag{33}$$

and

$$r \leq r_1, \tag{34}$$

where r_1 is the convergence radius for well-known Newton's method

$$x_{l+1} = x_l - \Theta'(x_l)^{-1}\Theta(x_l), \tag{35}$$

given in [2].

On the other hand, Rheindoldt [3] *and Traub* [4] *suggested*

$$r_{TR} = \frac{2}{3w_1}, \tag{36}$$

where as Argyros [2,5]

$$r_A = \frac{2}{2w_0 + w_1}, \tag{37}$$

where w_1 is the Lipschitz constant for (9) *on $\mathbb{D}$. Then,*

$$w \leq w_1, \ w_0 \leq w_1, \tag{38}$$

so

$$r_{TR} \leq r_A \leq r_1 \tag{39}$$

and

$$\frac{r_{TR}}{r_A} \to \frac{1}{3} \quad as \quad \frac{w_0}{w} \to 0. \tag{40}$$

(d) *We use the following rule for COC*

$$\xi = \frac{\ln \frac{\|x_{l+2}-\kappa\|}{\|x_{l+1}-\kappa\|}}{\ln \frac{\|x_{l+1}-\kappa\|}{\|x_l-\kappa\|}}, \qquad \text{for each } l = 0, 1, 2, \ldots \tag{41}$$

or ACOC [6], defined as

$$\xi^* = \frac{\ln \frac{\|x_{l+2}-x_{l+1}\|}{\|x_{l+1}-x_l\|}}{\ln \frac{\|x_{l+1}-x_n\|}{\|x_n-x_{l-1}\|}}, \qquad \text{for each } l = 1, 2, \ldots \tag{42}$$

not requiring derivatives and ξ^* does not depend on κ.

(e) *Our results can be adopted for operators Θ that satisfy [2,5]*

$$\Theta'(x) = P(\Theta(x)), \tag{43}$$

for a continuous operator P. The beauty of our study is that we can use the results without prior knowledge of solution κ, since $\Theta'(\kappa) = P(\Theta(\kappa)) = P(0)$. As an example $\Theta(x) = e^x - 1$, so we assume $P(x) = x + 1$.

(f) *Let us show how to consider functions ϕ, $\bar{g}_2$, g_2 and m. Define function ϕ by*

$$\phi(x_l,\ \Theta(x_l),\ \Theta'(x_l),\ \Theta'(y_l)) = y_l - \Theta'(y_l)^{-1}\Theta(y_l). \tag{44}$$

Then, we can choose

$$g_2(t) = \frac{\int_0^1 w((1-\eta)g_1(t)t)d\eta}{1 - w_0(g_1(t)t)} g_1(t). \tag{45}$$

If $w_0,\ w,\ v$ are given in particular by $w_0(t) = L_0 t,\ w(t) = Lt$ and $v(t) = M$ for some $L_0, L > 0$, and $M \geq 1$, then we have that

$$\bar{g}_2(t) = \frac{\frac{L^2}{8(1-L_0t)^2}}{1 - \frac{L_0Lt^2}{2(1-L_0t)}}, \qquad g_2(t) = \bar{g}_2(t)t^3 \text{ and } m = 4. \tag{46}$$

(g) *If $\beta = 0$, we can obtain the results for the two-step method*

$$\begin{aligned} y_l &= x_l - \Theta'(x_l)^{-1}\Theta(x_l), \\ x_{l+1} &= \phi(x_l,\ \Theta(x_l),\ \Theta'(x_l),\ \Theta'(y_l)) \end{aligned} \tag{47}$$

by setting $z_l = x_{l+1}$ in Theorem 1.

Convergence Order of Expression (3) from [1]

Theorem 2. *Let $\Theta : \mathbb{R} \to \mathbb{R}$ has a simple zero ξ being a sufficiently many times differentiable function in an interval containing ξ. Further, we consider that initial guess $x = x_0$ is sufficiently close to ξ. Then, the iterative scheme defined by (3) from [1] has minimum fourth-order convergence and satisfy the following error equation*

$$\begin{aligned} e_{l+1} = &-\frac{c_2\left(2c_2^2 + c_3\right)(\alpha-\beta)}{\beta}e_l^4 + \frac{1}{2\beta^2}\Big[4\beta c_4 c_2(\beta-\alpha) - 4c_2^4(2\alpha^2 - 8\alpha\beta + 5\beta^2) \\ &- 2c_3c_2^2(2\alpha^2 + \alpha\beta - 4\beta^2) + 3\beta c_3^2(\beta-\alpha)\Big]e_l^5 + O(e_l^6), \end{aligned} \tag{48}$$

where $\alpha, \beta \in \mathbb{R}$, $e_l = x_l - \xi$ *and* $c_j = \frac{\Theta^{(j)}(\xi)}{j!\Theta'(\xi)}$ *for* $j = 1,\ 2,\ \ldots 6$.

Proof. The Taylor's series expansion of function $\Theta(x_l)$ and its first order derivative $\Theta'(x_l)$ around $x = \xi$ with the assumption $\Theta'(\xi) \neq 0$ leads us to:

$$\Theta(x_l) = \Theta'(\xi)\left[\sum_{j=1}^{6} c_j e_l^j + O(e_l^7)\right], \tag{49}$$

and

$$\Theta'(x_l) = \Theta'(\xi)\left[\sum_{j=1}^{6} j c_j e_l^j + O(e_l^7)\right], \tag{50}$$

respectively.

By using the Equations (49) and (50), we get

$$\begin{aligned} y_l - \xi =& c_2 e_l^2 - 2(c_2^2 - c_3)e_l^3 + (4c_2^3 - 7c_3c_2 + 3c_4)e_l^4 + (-8c_2^4 + 20c_3c_2^2 - 10c_4c_2 - 6c_3^2 + 4c_5)e_l^5 \\ &+ (16c_2^5 - 52c_3c_2^3 + 28c_4c_2^2 + (33c_3^2 - 13c_5)c_2 - 17c_3c_4 + 5c_6)e_l^6 + O(e_l^7). \end{aligned} \tag{51}$$

The following expansion of $\Theta(y_l)$ about ξ

$$\begin{aligned} \Theta'(y_l) =& \Theta'(\xi)\Big[1 + 2c_2^2e_l^2 + (4c_2c_3 - 4c_2^3)e_l^3 + c_2(8c_2^3 - 11c_3c_2 + 6c_4)e_l^4 - 4c_2(4c_2^4 - 7c_3c_2^2 + 5c_4c_2 - 2c_5)e_l^5 \\ &+ 2\big(16c_2^6 - 34c_3c_2^4 + 30c_4c_2^3 - 13c_5c_2^2 + (5c_6 - 8c_3c_4)c_2 + 6c_3^3\big)e_l^6\Big]. \end{aligned} \tag{52}$$

From Equations (50)–(52) in the second substep of (3), we have

$$\begin{aligned} z_l - \xi =& \left(c_2^2 + \frac{c_3}{2}\right)e_l^3 + \left(-3c_2^3 + \frac{3c_3c_2}{2} + c_4\right)e_l^4 + \left(6c_2^4 - 9c_3c_2^2 + 2c_4c_2 - \frac{3}{4}(c_3^2 - 2c_5)\right)e_l^5 \\ &+ \frac{1}{2}\left(-18c_2^5 + 50c_3c_2^3 - 30c_4c_2^2 - 5\left(c_3^2 - c_5\right)c_2 - 5c_3c_4 + 4c_6\right)e_l^6 + O(e_l^7). \end{aligned} \tag{53}$$

Similarly, we can expand function $f(z_l)$ about ξ with the help of Taylor series expansion, which is defined as follows:

$$\begin{aligned} \Theta(z_l) =& \Theta'(\xi)\Big[\left(c_2^2 + \frac{c_3}{2}\right)e_l^3 + \left(-3c_2^3 + \frac{3c_3c_2}{2} + c_4\right)e_l^4 + \left(6c_2^4 - 9c_3c_2^2 + 2c_4c_2 - \frac{3}{4}(c_3^2 - 2c_5)\right)e_l^5 \\ &+ \left\{c_2\left(c_2^2 + \frac{c_3}{2}\right)^2 + \frac{1}{2}\left(-18c_2^5 + 50c_3c_2^3 - 30c_4c_2^2 - 5(c_3^2 - c_5)c_2 - 5c_3c_4 + 4c_6\right)\right\}e_l^6 + O(e_l^7)\Big]. \end{aligned} \tag{54}$$

Adopting expressions (49)–(54), in the last sub-step of method (3), we have

$$\begin{aligned} e_{l+1} =& -\frac{c_2\left(2c_2^2 + c_3\right)(\alpha - \beta)}{\beta}e_l^4 + \frac{1}{2\beta^2}\Big[4\beta c_4c_2(\beta - \alpha) - 4c_2^4(2\alpha^2 - 8\alpha\beta + 5\beta^2) \\ &- 2c_3c_2^2(2\alpha^2 + \alpha\beta - 4\beta^2) + 3\beta c_3^2(\beta - \alpha)\Big]e_l^5 + O(e_l^6). \end{aligned} \tag{55}$$

For choosing $\alpha = \beta$ in (55), we obtain

$$e_{l+1} = \left(2c_2^4 + c_3c_2^2\right)e_l^5 + O(e_l^6). \tag{56}$$

The expression (55) confirms that the scheme (3) have maximum fifth-order convergence for $\alpha = \beta$ (that can be seen in (56)). This completes the proof and also contradict the claim of authors [1]. □

This type of proof and theme are close to work on generalization of the fixed point theorem [2,5,7,8]. We recall a standard definition.

Definition 2. *Let $\{x_l\}$ be a sequence in $\mathbb{X}$ which converges to κ. Then, the convergence is of order $\lambda \geq 1$ if there exist $\lambda > 0$, abd $l_0 \in \mathbb{N}$ such that*

$$\|x_{l+1} - \kappa\| \leq \lambda \|x_l - \kappa\|^{\lambda} \text{ for each } l \geq l_0.$$

3. Examples with Applications

Here, we test theoretical results on four numerical examples. In the whole section, we consider $\phi(x_l, \Theta(x_l), \Theta'(x_l), \Theta'(y_l)) = x_l - \frac{2f(x_l)}{f'(y_l)+f'(x_l)}$, that means $m = 2$ for the computational point of view, called by $(M1)$.

Example 1. *Set $\mathbb{X} = \mathbb{Y} = C[0, 1]$. Consider an integral equation [9], defined by*

$$x(\beta) = 1 + \int_0^1 T(\beta, \alpha) \left(x(\alpha)^{\frac{3}{2}} + \frac{x(\alpha)^2}{2} \right) d\alpha \tag{57}$$

where

$$T(\beta, \alpha) = \begin{cases} (1-\beta)\alpha, & \alpha \leq s, \\ \beta(1-\alpha), & s \leq \alpha. \end{cases} \tag{58}$$

Consider corresponding operator $\Theta : C[0, 1] \to C[0, 1]$ as

$$\Theta(x)(\beta) = x(\beta) - \int_0^{\alpha} T(\beta, \alpha) \left(x(\alpha)^{\frac{3}{2}} + \frac{x(\alpha)^2}{2} \right) d\alpha. \tag{59}$$

But

$$\left\| \int_0^{\alpha} T(\beta, \alpha) d\alpha \right\| \leq \frac{1}{8}, \tag{60}$$

and

$$\Theta'(x)y(\beta) = y(\beta) - \int_0^{\alpha} T(\beta, \alpha) \left(\frac{3}{2} x(\alpha)^{\frac{1}{2}} + x(\alpha) \right) d\alpha.$$

Using $\kappa(s) = 0$, we obtain

$$\left\| \Theta'(\kappa)^{-1} (\Theta'(x) - \Theta'(y)) \right\| \leq \frac{1}{8} \left(\frac{3}{2} \|x - y\|^{\frac{1}{2}} + \|x - y\| \right), \tag{61}$$

So, we can set

$$w_0(\alpha) = w(\alpha) = \frac{1}{8} \left(\frac{3}{2} \alpha^{\frac{1}{2}} + \alpha \right).$$

Hence, by adopting Remark 2.2(a), we have

$$v(\alpha) = 1 + w_0(\alpha) \text{ or } v_0(\alpha) = M,$$

The results in [1] are not applicable, since Θ' is not Lipschitz. But, our results can be used. The radii of convergence of method (2) for example (1) are described in Table 1.

Table 1. Radii of convergence for problem (1).

α	β	m	r_1	r_p	r_2	r_3	r	Methods
1	1	2	2.6303	3.13475	2.6303	2.1546	2.1546	M1
1	2	2	2.6303	3.35124	2.6303	2.0157	2.0157	M1

Example 2. *Consider a system of differential equations*

$$\begin{aligned} &\theta_1'(x) - \theta_1(x) - 1 = 0 \\ &\theta_2'(y) - (e-1)y - 1 = 0 \\ &\theta_3'(z) - 1 = 0 \end{aligned} \tag{62}$$

that model for the motion of an object for $\theta_1(0) = \theta_2(0) = \theta_3(0) = 0$. *Then, for* $v = (x,\ y,\ z)^T$ *consider* $\Theta := (\theta_1,\ \theta_2,\ \theta_3) : \mathbb{D} \to \mathbb{R}^3$ *defined by*

$$\Theta(v) = \left(e^x - 1,\ \frac{e-1}{2}y^2 + y,\ z\right)^T. \tag{63}$$

We have

$$\Theta'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then, we get $w_0(t) = L_0 t,\ w(t) = Lt,\ w_1(t) = L_1 t$ *and* $v(t) = M$, *where* $L_0 = e - 1 < L = e^{\frac{1}{L_0}} = 1.789572397,\ L_1 = e$ *and* $M = e^{\frac{1}{L_0}} = 1.7896$. *The convergence radii of scheme* (2) *for example* (2) *are depicted in Table* 2.

Table 2. Radii of convergence for problem (2).

α	β	r_1	r_p	r_2	r_3	r	Methods	x_0	n	ρ
1	1	0.382692	0.422359	0.321733	0.218933	0.218933	M1	0.15	3	4.9963
1	2	0.382692	0.441487	0.321733	0.218933	0.218933	M1	0.11	4	4.0000

We follow the stopping criteria for computer programming (i) $\|F(X_l)\|$ *and (ii)* $\|X_{l+1} - X_l\| < 10^{-100}$ *in all the examples.*

Example 3. *Set* $\mathbb{X} = \mathbb{Y} = C[0,\ 1]$ *and* $\mathbb{D} = \bar{U}(0,\ 1)$. *Consider* Θ *on* $\mathbb{D}$ *as*

$$\Theta(\varphi)(x) = \phi(x) - 5\int_0^1 x\eta\varphi(\eta)^3 d\eta. \tag{64}$$

We have that

$$\Theta'(\varphi(\xi))(x) = \xi(x) - 15\int_0^1 x\eta\varphi(\eta)^2\xi(\eta)d\eta,\ \textit{for each}\ \xi \in \mathbb{D}. \tag{65}$$

Then, we get $\kappa = 0,\ L_0 = 7.5,\ L_1 = L = 15$ *and* $M = 2$. *leading to* $w_0(t) = L_0 t,\ v(t) = 2 = M,\ w(t) = Lt,\ w_1(t) = L_1 t$. *The radii of convergence of scheme* (2) *for problem* (3) *are described in the Table* 3.

Table 3. Radii of convergence for problem (3).

α	β	m	r_1	r_p	r_2	r_3	r	Methods
1	1	2	0.0666667	0.0824045	0.0233123	0.00819825	0.00819825	M1
1	2	2	0.0666667	0.0888889	0.0233123	0.00819825	0.00819825	M1

Example 4. *We get* $L = L_0 = 96.662907$ *and* $M = 2$ *for example at introduction. Then, we can set* $w_0(t) = L_0 t,\ v(t) = M = 2,\ w(t) = Lt,\ w_1(t) = Lt$. *The convergence radii of the iterative method* (2) *for example* (4) *are mentioned in the Table* 4.

Table 4. Radii of convergence for problem (4).

α	β	m	r_1	r_p	r_2	r_3	r	**Methods**	x_0	n	ρ
1	1	2	0.0102914	0.0102917	0.00995072	0.00958025	0.00958025	M1	1.008	3	5.0000
1	2	2	0.0102914	0.010292	0.00995072	0.00958025	0.00958025	M1	1.007	4	3.0000

4. Conclusions

A major problem in the development of iterative methods is the convergence conditions. In the case of especially high order methods, such as (2), the operator involved must be seventh times differentiable according to the earlier study [1] which do not appear in the methods, limiting the applicability. Moreover, no error bounds or uniqueness of the solution that can be computed are given. That is why we address these problems based only on the first order derivative which actually appears in the method. The convergence order is determined using *COC* or *ACOC* that do not require higher than first order derivatives. Our technique can be used to expand the applicability of other iterative methods [1–13] along the same lines.

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