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On Formality of Some Homogeneous Spaces

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Abstract: Let G/H be a homogeneous space of a compact simple classical Lie group G. Assume that the maximal torus T_H of H is conjugate to a torus T_β whose Lie algebra $\mathfrak{t}_\beta$ is the kernel of the maximal root β of the root system of the complexified Lie algebra $\mathfrak{g}^c$. We prove that such homogeneous space is formal. As an application, we give a short direct proof of the formality property of compact homogeneous 3-Sasakian spaces of classical type. This is a complement to the work of Fernández, Muñoz, and Sanchez which contains a full analysis of the formality property of $SO(3)$-bundles over the Wolf spaces and the proof of the formality property of homogeneous 3-Sasakian manifolds as a corollary.

Keywords: formality; 3-Sasakian manifold; homogeneous space

1. Introduction

Formality is an important homotopic property of topological spaces. It is often related to the existence of particular geometric structures on manifolds. For example, Kaehler manifolds are formal [1], and the same holds for compact Riemannian symmetric spaces [2,3]. In general, Sasakian manifolds do not possess this property. However, their higher order Massey products vanish [4], and this can be regarded as a "formality-like" property as well. An interesting issue is the formality of homogeneous spaces of compact Lie groups. For example, Amann [5] found several characterizations of non-formality of homogeneous spaces. Some homogeneous spaces determined by characters of maximal tori are not formal [6,7]. On the other hand, compact homogeneous spaces of positive Euler characteristics are known to be formal [3,7] and the same holds for G/H generated by a finite order automorphism of G [8]. It should be noted that there is a general method of studying the formality property of homogeneous spaces in terms of the Lie group-theoretic data [3,7]. However, such methods may work for a *given* pair (G, H) together with the *known* embedding of H into G. Hence, it is still interesting to find geometrically important classes of homogeneous spaces satisfying formality or non-formality property. In this article, we prove the following result.

Theorem 1. *Let G/H be a homogeneous space of a compact simple classical Lie group G. Assume that the maximal torus T_H of H is conjugate (in G) to the torus T_β whose Lie algebra is the kernel* Ker β *of the maximal root β of the root system $\Delta(\mathfrak{g}^c)$. Then G/H is formal.*

This class of homogeneous spaces has geometric significance. To show this we present the following geometric application. In [9] the formality property of $SO(3)$-bundles over the Wolf spaces was analyzed. Consequently, one obtains the formality property of any compact homogeneous 3-Sasakian manifold. In this note we show that if one restricts himself to this class of Riemannian manifolds, then the proof can be obtained entirely in terms of the data of the 3-Sasakian homogeneous space G/H (at least for classical Lie groups G). Thus, we give a direct proof the following result [9].

Theorem 2. *Let G be a classical compact simple Lie group. Then, any 3-Sasakian homogeneous space G/H is formal.*

Although [9] contains much stronger and more general result, the direct proof still may be of independent interest. This is motivated by the fact that homogeneous 3-Sasakian manifolds G/H admit a description in terms of the root systems of the complexified Lie algebra $\mathfrak{g}^c$, and in some cases, the formality property can be expressed via the same data [7] (see also [5,6]). It seems to make a remark that Theorem 1 probably holds for all simple Lie groups. However, the method of proof uses the generators of the ring of invariants of the Weyl group, which becomes computationally difficult (compare, for example the expressions of such polynomials for the exceptional Lie groups [10]).

2. Preliminaries

2.1. Presentation and Notation

We approach the problem of formality from the point of view of the classical cohomology theory of homogeneous spaces of compact Lie groups [7,11]. We use the basic notions and facts from the theory of Lie groups and Lie algebras without explanations. Instead, we refer to [12]. We denote Lie groups by capital letters $G, H, \dots$, and their Lie algebras by the corresponding Gothic letters $\mathfrak{g}, \mathfrak{h}, \dots$. Let G be a compact semisimple Lie group. The real cohomology algebra $H^*(G)$ is isomorphic to the exterior algebra over the space of primitive elements $P_G = \langle y_1, \dots, y_n \rangle$:

$$H^*(G) \cong \Lambda P_G = \Lambda(y_1, \dots, y_n),\, y_i \in P_G, i = 1, \dots, n = \text{rank } G.$$

The degrees of y_i are equal to $2p_i - 1$, where p_i are the exponents of $\mathfrak{g}$. We denote by S_G the ring of G-invariant polynomials on the Lie algebra $\mathfrak{g}$. Let T be a maximal torus of G. Consider the Weyl group $W_G = N_G(T)/T$. It acts on $\mathfrak{t}$ and on the polynomial algebra $\mathbb{R}[\mathfrak{t}]$ of all polynomials over $\mathfrak{t}$. The subring S_{W_G} of W_G-invariants in $\mathbb{R}[\mathfrak{t}]$ is generated by $n = \text{rank } G$ polynomials $F_1, \dots, F_n$ of degrees $2p_i$. The following isomorphism is well known [7,11]:

$$S_G \cong S_{W_G} \cong \mathbb{R}[\mathfrak{t}]^{W_G} \cong \mathbb{R}[F_1, \dots, F_n].$$

We will use a map $\tau_G : \Lambda P_G \to S_G$ called the *transgression map* [7,11]. The transgression τ_G maps y_i, $i = 1, \dots, n$ onto some free generators of S_{W_G}. We follow [9] in the presentation of Sasakian and 3-Sasakian manifolds. One can also consult [13].

2.2. Formality

Here we recall some definitions and facts from the theory of minimal models and formality [14].

We consider *differential graded commutative algebras*, or DGAs, over the field $\mathbb{R}$ of real numbers. The degree of an element a of a DGA is denoted by $|a|$.

Definition 1. *A DGA $(\mathcal{A}, d)$ is minimal if:*

1. *$\mathcal{A}$ is the free algebra $\bigwedge V$ over a graded vector space $V = \bigoplus_i V^i$, and*
2. *there is a family of generators $\{a_\tau\}_{\tau \in I}$ indexed by some well-ordered set I, such that $|a_\mu| \leq |a_\tau|$ if $\mu < \tau$ and each da_τ is expressed in terms of preceding a_μ, $\mu < \tau$. Thus, da_τ does not have a linear part.*

An important example of DGA is the de Rham algebra $(\Omega^*(M), d)$ of a differentiable manifold M, where d is the exterior differential. This DGA will be used in this article.

Given a differential graded commutative algebra $(\mathcal{A}, d)$, we denote its cohomology by $H^*(\mathcal{A})$. The cohomology of a differential graded algebra $H^*(\mathcal{A})$ is also a DGA with the multiplication inherited from that on $\mathcal{A}$ and with zero differential. The DGA $(\mathcal{A}, d)$ is *connected* if $H^0(\mathcal{A}) = \mathbb{R}$, and $\mathcal{A}$ is

1-*connected* if, in addition, $H^1(\mathcal{A}) = 0$. Morphisms between DGAs are required to preserve the degree and to commute with the differential.

Definition 2. *A free graded differential algebra* $(\bigwedge V, d)$ *is called a minimal model of the differential graded commutative algebra* $(\mathcal{A}, d)$ *if* $(\bigwedge V, d)$ *is minimal and there exists a morphism of differential graded algebras*

$$\rho : (\bigwedge V, d) \longrightarrow (\mathcal{A}, d)$$

inducing an isomorphism $\rho^* : H^*(\bigwedge V) \xrightarrow{\sim} H^*(\mathcal{A})$ *of cohomologies.*

Definition 3. *Two DGAs* $(\mathcal{A}, d_A)$ *and* $(\mathcal{B}, d_B)$ *are quasi-isomorphic, if there is a sequence of DGA algebras* $(\mathcal{A}_i, d_i)$ *and a sequence of morphisms between* $(\mathcal{A}_i, d_i)$ *and* $(\mathcal{A}_{i+1}, d_{i+1})$ *with* $(\mathcal{A}_1, d_1) = (\mathcal{A}, d_A)$ *and* $(\mathcal{A}_n, d_n) = (\mathcal{B}, d_B)$ *such that these morphisms induce isomorphisms of the corresponding cohomology algebras (the morphisms may be directed arbitrarily).*

It is known [14] that any connected differential graded algebra $(\mathcal{A}, d)$ has a minimal model which is unique up to isomorphism.

Definition 4. *A minimal model of a connected differentiable manifold M is a minimal model* $(\bigwedge V, d)$ *for the de Rham complex* $(\Omega^*(M), d)$ *of differential forms on M.*

If M is a simply connected manifold, then the dual $(\pi_i(M) \otimes \mathbb{R})^*$ of the vector space $\pi_i(M) \otimes \mathbb{R}$ is isomorphic to V^i for any i. This duality shows the relation between minimal models and homotopy groups. The same result is valid when $i > 1$, the fundamental group $\pi_1(M)$ is nilpotent and its action on $\pi_j(M)$ is nilpotent for all $j > 1$.

Definition 5. *A minimal algebra* $(\bigwedge V, d)$ *is called formal if there exists a morphism of differential algebras* $\psi : (\bigwedge V, d) \longrightarrow (H^*(\bigwedge V), 0)$ *inducing the identity map on cohomology.*

A smooth manifold M is called *formal* if its minimal model is formal. Examples of formal manifolds are ubiquitous: spheres, projective spaces, compact Lie groups, some homogeneous spaces, flag manifolds, and all compact Kaehler manifolds [1,3,5,8,14].

It is important to note that quasi-isomorphic minimal algebras have isomorphic minimal models. Therefore, to study formality of manifolds, one can use other "algebraic models". This means that one may take any DGAs $(\mathcal{A}, d_A)$ which are quasi-isomorphic to the de Rham algebra. This will be used in our analysis of formality of homogeneous spaces.

2.3. Quaternionic-Kaehler and 3-Sasakian Manifolds

A Riemannian $4n$-dimensional manifold (X, h) is called *quaternionic-Kaehler*, if the holonomy group $\mathrm{Hol}(X, h)$ is contained in $Sp(n)Sp(1)$.

An odd dimensional Riemannian manifold (M, g) is *Sasakian* if its cone $(M \times \mathbb{R}^+, g^c = t^2 g + dt^2)$ is Kaehler. This means that there is a compatible integrable almost complex structure J so that $(M \times \mathbb{R}^+, g^c, J)$ is a Kaehler manifold. In this case, the vector field $\xi = J\frac{\partial}{\partial t}$ is a Killing vector field of unit length. The 1-form η defined by $\eta(X) = g(\xi, X)$ for any vector field X on M is a contact form, whose Reeb vector field is ξ. Let ∇ denote the Levi-Civita connection of g. The $(1,1)$-tensor $\phi(X) = \nabla_X \xi$ satisfies the identities

$$\phi^2 = -\,\mathrm{id} + \eta \otimes \xi,\; g(\phi(X), \phi(Y)) = g(X, Y) - \eta(X)\eta(Y),$$

$$d\eta(X, Y) = 2g(\phi(X), Y),$$

for any vector fields X, Y.

A Riemannian manifold (M, g) of dimension $4n+3$ is called 3-*Sasakian*, if the cone $(M \times \mathbb{R}^+, g^c)$ admits three compatible integrable almost complex structures J_1, J_2, J_3 such that

$$J_1 J_2 = -J_2 J_1 = J_3,$$

and such that $(M \times \mathbb{R}^+, g^c, J_1, J_2, J_3)$ is a hyperkaehler manifold. Thus, (M, g) admits three Sasakian structures with Reeb vector fields ξ_1, ξ_2, ξ_3 of the contact forms η_1, η_2, η_3, and three tensors ϕ_1, ϕ_2, ϕ_3. The following relations are satisfied:

$$\eta_i(\xi_j) = g(\xi_i, \xi_j) = \delta_{ij}, \phi_i(\xi_j) = -\phi_j(\xi_i) = \xi_k,$$

$$\eta_i \circ \phi_j = -\eta_j \circ \phi_i = \eta_k$$

$$\phi_i \circ \phi_j - \eta_j \otimes \xi_i = -\phi_j \circ \phi_i + \eta_i \otimes \xi_j = \phi_k,$$

$$[\xi_i, \xi_j] = 2\xi_k,$$

for any cyclic permutation of (i, j, k) of $(1, 2, 3)$.

Let (M, g) be a Riemannian manifold carrying a 3-Sasakian structure. Denote by $\mathrm{Aut}(M, g)$ the subgroup of the isometry group $\mathrm{Iso}(M, g)$ consisting of all isometries preserving the 3-Sasakian structure

$$(g, \xi_s, \eta_s, \phi_s, s = 1, 2, 3).$$

By definition, a 3-Sasakian manifold (M, g) is called *homogeneous*, if $\mathrm{Aut}(M, g)$ acts transitively on M.

By definition, a *Wolf space* is a homogeneous quaternionic-Kaehler manifold of positive scalar curvature. The classification of the Wolf spaces is known [15,16] and can be reproduced as follows:

$$\mathbb{HP}^n = Sp(n+1)/(Sp(n) \times Sp(1)), \ \mathbb{G}r_2(\mathbb{C}^{n+2}), \ \tilde{\mathbb{G}}r_4(\mathbb{R}^{n+4}),$$

$$GI = G_2/SO(4), \ FI = F_4/Sp(3) \cdot Sp(1), \ EII = E_6/SU(6) \cdot Sp(1),$$

$$EVI = E_7/Spin(12) \cdot Spin(1), \ EIX = E_8/E_7 \cdot Sp(1).$$

Here $\tilde{\mathbb{G}}r_4(\mathbb{R}^{n+4})$ denotes the Grassmannian of oriented real 4-planes. It follows that the classification of homogeneous 3-Sasakian manifolds is given by the following result (see [9], Section 2).

Theorem 3. *Let (M, g) be a 3-Sasakian homogeneous space. Then M is the total space of the fiber bundle*

$$F \to M \to W$$

over a Wolf space W. The fiber F is $Sp(1)$ for $M = S^{4n+3}$ and it equals $SO(3)$ in all other cases. Moreover, M is the one of the following homogeneous spaces:

$$Sp(n+1)/Sp(n) \cong S^{4n+3}, \ Sp(n+1)/(Sp(n) \times \mathbb{Z}_2),$$

$$SU(n+2)/S(U(n) \times U(1)), SO(m+4)/SO(m) \times Sp(1),$$

$$G_2/Sp(1), F_4/Sp(3), E_6/SU(6), E_7/Spin(12), E_8/E_7,$$

where $k \geq 0, n \geq 1, m \geq 3$. For the first two cases $Sp(0)$ means the trivial group.

3. Proof of Theorem 1

3.1. A Theorem on Formality of Homogeneous Spaces

Theorem 4 ([5]). *Let G/H be a homogeneous space of a compact semisimple Lie group G and let T_H be a maximal torus in H. Then G/H is formal if and only G/T_H is formal.*

3.2. Cartan Algebras

The material of this subsection is presented following [7]. It is well known that a homogeneous space G/H of a compact semisimple Lie group G has an algebraic model (which is called the Cartan algebra) of the form

$$(C(\mathfrak{g},\mathfrak{h}),d) = (S_H \otimes \Lambda P_G, d)$$

where

$$d(q \otimes 1) = 0,\ \forall q \in S_H$$

$$d(1 \otimes p) = j^*(\tau_G(p)),\ \forall p \in \Lambda P_G.$$

Here $\tau_G : \Lambda P_G \to S_G$ is the transgression, $j^* : S_G \to S_H$ is a restriction map, and S_G, S_H are the algebras of invariant polynomials on $\mathfrak{g}$ and $\mathfrak{h}$, respectively. In particular, if $H = T$ for some torus in G, then j^* is a restriction of any invariant polynomial in S_G onto the Lie algebra $\mathfrak{t}$. Please note that T need not be maximal.

More generally, consider the DGA algebra of the form

$$(C,d) = (\mathbb{R}[x_1,\dots,x_m] \otimes \Lambda(y_1,\dots,y_n), d)$$

with the differential d vanishing on x_i, $i = 1,\dots,m$ and

$$d(y_j) = F_j(x_1,\dots,x_m).$$

We assume that y_j have some odd degrees $2l_j - 1$. Let $H^*(C)$ be the cohomology algebra of (C,d). We will also use the notation

$$H^*(C) = H(F_1,\dots,F_n)$$

to stress the role of the ideal $I = (F_1,\dots,F_n)$ (in the polynomial ring $\mathbb{R}[x_1,\dots,x_m]$).

Recall the following definition. Let A be any commutative ring. A sequence $a_1,\dots,a_k$ of elements in A is called *regular*, if a_i is not a zero divisor in $A/(a_1,\dots,a_{i-1})$.

The following characterization of formality of a general Cartan algebra (C,d) is well known [7].

Theorem 5. *A general Cartan algebra (C,d) is formal if and only if the ideal $(F_1,\dots,F_n)$ has the following property: the minimal system of generators is regular. The number of such generators cannot exceed m.*

Finally, recall the following isomorphism

$$S_G \cong S_{W_G} \cong \mathbb{R}[\mathfrak{t}]^{W_G},$$

where S_{W_G} denotes the ring of polynomials on $\mathfrak{t}$ which are invariant with respect to the action of the Weyl group W_G of G. Also, there is a commutative diagram

$$\begin{array}{ccc} S_G & \longrightarrow & S_{W_G} \\ j^* \downarrow & & j^* \downarrow \\ S_H & \longrightarrow & S_{W_H} \end{array}$$

which shows that the Cartan algebra $(C(\mathfrak{g},\mathfrak{h})$ is isomorphic to the general Cartan algebra of the form

$$(C,d) = \mathbb{R}[\mathfrak{t}_H]^{W_H} \otimes \Lambda(y_1,\ldots,y_n)$$

$$d(y_k) = j^*(F_k), k = 1,\ldots,n, F_k \in \mathbb{R}[\mathfrak{t}]^{W_G}.$$

Here F_k are free generators of the ring of invariants $\mathbb{R}[\mathfrak{t}]^{W_G}$ determined by the transgression.

Please note that in the sequel we will use the particular choices of free invariant generators of polynomial algebras $\mathbb{R}[\mathfrak{t}]^{W_G}$ for each simple compact Lie group. These can be found in many sources, we use [7], Example 1 on page 186.

3.3. Formality of G/T_β

Proposition 1. *Let G/T_β be a homogeneous space of a compact classical Lie group G and a torus T_β whose Lie algebra is the kernel of the maximal root. Then G/T_β is formal.*

Proof. The proof is based on the checking of the conditions of Theorem 5 for G/T_β in each case A_n, B_n, C_n, D_n separately (although the calculations are very similar). Also, due to the final remark in the previous section, we can consider the algebraic model of G/T_β in the form

$$(\mathbb{R}[\mathfrak{t}_\beta] \otimes \Lambda(y_1,\ldots,y_n), d)$$

with

$$d(y_i) = F_i|_{\mathfrak{t}_\beta}, i = 1,\ldots,n.$$

In the proof we use the description of the maximal roots of the root systems of classical type [15].

Case 1 (C_n)**.** In this case, in the coordinates $x_1,\ldots,x_n$ in $\mathfrak{t}$, the maximal root β has the form $\beta = 2x_1$. Thus, $\mathfrak{t}_\beta$ is determined by the equation $x_1 = 0$, and the restrictions of F_i on $\mathfrak{t}_\beta$ have the form $F_i|_{\mathfrak{t}_\beta} = F_i(0,x_2,\ldots,x_n)$. Please note that the ring of invariants $\mathbb{R}[\mathfrak{t}]^{W_G}$ may have different sets of generators, and in general we cannot take them arbitrarily, because they are determined by the transgression. However, by Theorem 5, *the formality property is determined not by the particular polynomials, but by the whole ideal* $(F_1,\ldots,F_n)$. It follows that one can work with any set of generators. In case of C_n we can take

$$F_i(x_1,\ldots,x_n) = x_1^{2i} + \cdots + x_n^{2i}, i = 1,\ldots,n.$$

The restrictions onto $\mathfrak{t}_\beta$ have the form

$$F_i(0,x_2,\ldots,x_n),$$

this sequence is obviously regular for $i = 1,\ldots,n-1$. Since the number of variables is also $n-1$, the result follows.

Case 2 (B_n)**.** Here $\beta = x_1 + x_2$. We make the same argument to the previous case. Again, one may choose the invariant generators in the form $F_i = \sum_{k=1}^n x_k^{2i}, i = 1,\ldots,n$. This time the restrictions will take the form

$$F_i|_{\mathfrak{t}_\beta} = F_i(-x_2,x_2,x_3 \ldots,x_n) = 2x_2^{2i} + x_3^{2i} + \cdots + x_n^{2i}.$$

Again, this sequence is obviously regular for $i = 1,\ldots,n-1$ and the result follows from Theorem 5.

Case 3 (D_n)**.** In this case, again, $\beta = x_1 + x_2$. However, the invariant generators are different. One of the possible choices is

$$F_i(x_1,\ldots,x_n) = \sum_{k=1}^n x_k^{2i}, i = 1,\ldots,n-1, F_n = x_1 \cdots x_n.$$

Thus,

$$F_i|_{\mathfrak{t}_\beta} = F_i(-x_2, x_2, \ldots, x_n), i < n, F_n|_{\mathfrak{t}_\beta} = x_2^2 x_3 \cdots x_n.$$

Since $F_i(-x_2, x_2, \ldots, x_n)$ for $i < n$ obviously constitute a regular sequence, and the number of variables is $n-1$, necessarily $F_n|_{\mathfrak{t}_\beta} \in (F_1|_{\mathfrak{t}_\beta}, \ldots, F_{n-1}|_{\mathfrak{t}_\beta})$. The formality property follows.

Case 4 (A_n)**.** Here the standard coordinates in $\mathfrak{t}$ satisfy the equality

$$x_1 + \cdots + x_{n+1} = 0.$$

In these coordinates $\beta = x_1 - x_{l+1}$. One can choose the generating invariant polynomials in the form

$$F_i(x_1, \ldots, x_{n+1}) = \sum_{k=1}^{n+1} x_k^i, i = 2, \ldots, n+1.$$

The restrictions have the form

$$F_i(x_1, \ldots, x_n, x_1), i = 2, \ldots, n+1.$$

These polynomials form a regular sequence for $i = 2, \ldots, n$, as required. The proof is complete. □

3.4. *Completion of Proof of Theorem 1*

The proof of Theorem 1 follows from Theorem 4 and Proposition 1.

4. Application: Formality of 3-Sasakian Homogeneous Manifolds of Classical Type

4.1. *Quaternionic-Kaehler Symmetric Spaces (Wolf Spaces)*

In this subsection we present a version of Theorem 3 in terms of the root systems (see Theorems 6 and 7). Let $\mathfrak{g}$ be a compact simple Lie algebra and $\mathfrak{t}$ be its maximal abelian subalgebra. Consider the complexifications $\mathfrak{g}^c$ and $\mathfrak{t}^c$. Thus, $\mathfrak{t}^c$ is a Cartan subalgebra of $\mathfrak{g}^c$. Let $\Delta = \Delta(\mathfrak{g}^c, \mathfrak{t}^c)$ denote the root system determined by $\mathfrak{t}^c$. Choose the maximal root $\beta \in \Delta$ with respect to some fixed ordering of Δ. As usual, $\mathfrak{g}_\alpha$ denotes the root space of $\alpha \in \Delta$. Define

$$\mathfrak{l}_1 = \{H \in \mathfrak{t} \,|\, \beta(H) = 0\} + \sum_{\alpha>0, \langle\alpha,\beta\rangle=0} \mathfrak{g} \cap (\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}). \tag{1}$$

Put

$$\mathfrak{a}_1 = \mathfrak{g} \cap (\{H_\beta\} + \mathfrak{g}_\beta + \mathfrak{g}_{-\beta}), \tag{2}$$

and

$$\mathfrak{k} = \mathfrak{l}_1 + \mathfrak{a}_1. \tag{3}$$

Theorem 6 (Wolf, [16])**.** *If G/K is a quaternionic-Kaehler symmetric space, then $K = L_1 \cdot A_1$, where the Lie algebras $\mathfrak{l}_1$ and $\mathfrak{a}_1$ are determined by Equations (1)–(3).*

Theorem 7 ([9], Section 2)**.** *Let $G/K = G/L_1 \cdot A_1$ be the quaternionic symmetric space. Then the homogeneous space G/L_1 is 3-Sasakian. All compact homogeneous Sasakian manifolds are obtained in this way.*

Remark 1. Theorem 7 follows from the description of 3-Sasakian manifolds in [9] together with Theorem 6.

4.2. *Proof of Theorem 2*

By Theorem 7, any compact homogeneous 3-Sasakian manifold G/H has the form G/L_1 with L_1 given by Theorem 6. One can easily notice that the maximal torus T_{L_1} in L_1 has the Lie algebra of the form $\mathfrak{t}_\beta = \ker \beta$ for the maximal root β. By Theorem 1 the formality property of G/L_1 follows.

5. Conclusions

We have proved that if G is a classical compact Lie group, then the quotient of G by a torus determined by a maximal root, is formal. This result may have important applications in geometry of homogeneous spaces. As an example of such application we present a direct short proof of a result of Fernández, Muñoz and Sanchez about the formality property of some homogeneous 3-Sasakian manifolds.

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