*symmetry*

Article

# Tribonacci Numbers and Some Related Interesting Identities

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**Abstract:** The main purpose of this paper is, by using elementary methods and symmetry properties of the summation procedures, to study the computational problem of a certain power series related to the Tribonacci numbers, and to give some interesting identities for these numbers.

**Keywords:** the Tribonacci numbers; third-order linear recurrence sequence; convolution formula; power series; identity

**MSC:** 11B39; 11B83

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## 1. Introduction

For integers $n \geq 0$, the Fibonacci polynomials $F_n(x)$ are defined by $F_0(x) = 0$, $F_1(x) = 1$ and the second-order linear recurrence sequence:

$$F_{n+1}(x) = xF_n(x) + F_{n-1}(x), \text{ for all } n \geq 1.$$

If we take $x = 1$, then $\{F_n(1)\}$ becomes the famous Fibonacci sequence. Many experts and scholars have studied various elementary properties of $F_n(x)$, and obtained a series of valuable research results. For example, Ma Yuankui and Zhang Wenpeng [1] have studied the calculating problem of a certain sum of products of Fibonacci polynomials, and proved the equality in the formula below.

Let $h$ be a positive integer. Then, for any integer $n \geq 0$, one has the identity:

$$\sum_{a_1+a_2+\cdots+a_{h+1}=n} F_{a_1}(x)F_{a_2}(x)\cdots F_{a_{h+1}}(x) = \frac{1}{h!} \cdot \sum_{j=1}^{h} \frac{(-1)^{h-j} \cdot S(h,j)}{x^{2h-j}}$$
$$\times \left( \sum_{i=0}^{n} \frac{(n-i+j)!}{(n-i)!} \cdot \binom{2h+i-j-1}{i} \cdot \frac{(-1)^i \cdot 2^i \cdot F_{n-i+j}(x)}{x^i} \right),$$

where, as usual, the summation is taken over all $(h+1)$-dimension non-negative integer coordinates $(a_1, a_2, \cdots, a_{h+1})$ such that $a_1 + a_2 + \cdots + a_{h+1} = n$, and $S(h,i)$ is defined by $S(h,0) = 0$, $S(h,h) = 1$, and:

$$S(h+1, i+1) = 2 \cdot (2h-1-i) \cdot S(h, i+1) + S(h,i)$$

for all positive integers $1 \leq i \leq h-1$.

Taekyun Kim et al. [2] first introduced the convolved Fibonacci numbers $p_n(x)$, which are defined by the generating function:

$$\left( \frac{1}{1-t-t^2} \right)^x = \sum_{n=0}^{\infty} p_n(x) \cdot \frac{t^n}{n!}, \; x \in \mathbf{R}.$$

Then, they used the elementary and combinatorial methods to prove a series of important conclusions, one of them is the following identity:

$$p_n(x) = \sum_{l=0}^{n} \binom{n}{l} \cdot p_l(r) \cdot p_{n-l}(x-r) = \sum_{l=0}^{n} \binom{n}{l} \cdot p_{n-l}(r) \cdot p_l(x-r).$$

Chen Zhuoyu and Qi Lan [3] used a different method to prove the identity:

$$p_n(x) = \frac{1}{2} \sum_{i=0}^{n} (-1)^i \binom{n}{i} \langle x \rangle_i \cdot \langle x \rangle_{n-i} \cdot L_{n-2i},$$

where $L_n$ denote the $n$th Lucas numbers, $\langle x \rangle_0 = 1$, and:

$$\langle x \rangle_n = x(x+1)(x+2)\cdots(x+n-1)$$

for all integers $n \geq 1$.

As an interesting corollary of [3], Chen Zhuoyu and Qi Lan proved that, for any positive integer $k$, one has the identity:

$$\begin{aligned} & \sum_{a_1+a_2+a_3+\cdots+a_k=n} F_{a_1} \cdot F_{a_2} \cdot F_{a_3} \cdots F_{a_k} \\ = \; & \frac{1}{2((k-1)!)^2} \sum_{i=0}^{n} (-1)^i \cdot \frac{(k+i-1)! \cdot (k+n-i-1)!}{i! \cdot (n-i)!} \cdot L_{n-2i}. \end{aligned}$$

Papers related to linear recurrence sequences of numbers and polynomials include [4–17], there are too many to list all of them.

In this paper, we consider the Tribonacci numbers $T_n$ (see ([18], A000073)), which are defined by the third-order linear recurrence relation:

$$T_n = T_{n-1} + T_{n-2} + T_{n-3}, \; n \geq 3 \;\text{ with }\; T_0 = 0, T_1 = T_2 = 1.$$

For example, the first eleven terms of $T_n$ are $T_0 = 0$, $T_1 = 1$, $T_2 = 1$, $T_3 = 2$, $T_4 = 4$, $T_5 = 7$, $T_6 = 13$, $T_7 = 24$, $T_8 = 44$, $T_9 = 81$, $T_{10} = 149$, $T_{11} = 274$, $\cdots$.

The generating function $F(x)$ of the sequences $\{T_n\}$ is given by:

$$F(x) = \frac{1}{1-x-x^2-x^3} = \sum_{n=0}^{\infty} T_{n+1} \cdot x^n. \tag{1}$$

Let $\alpha$, $\beta$ and $\gamma$ be the three roots of the equation $x^3 - x^2 - x - 1 = 0$, then from references [19,20] we have:

$$\alpha = \frac{\sqrt[3]{19+3\sqrt{33}} + \sqrt[3]{19-3\sqrt{33}} + 1}{3},$$

$$\beta = \frac{2 - \left(1+\sqrt{-3}\right)\sqrt[3]{19-3\sqrt{33}} - \left(1-\sqrt{-3}\right)\sqrt[3]{19+3\sqrt{33}}}{6}$$

and:

$$\gamma = \frac{2 - \left(1-\sqrt{-3}\right)\sqrt[3]{19-3\sqrt{33}} - \left(1+\sqrt{-3}\right)\sqrt[3]{19+3\sqrt{33}}}{6}.$$

For any integer $n$, $T_n$ can be expressed as a Binet-type formula (see [21]):

$$T_n = c_1\alpha^n + c_2\beta^n + c_3\gamma^n. \tag{2}$$

Then note that $T_0 = 0$, $T_1 = T_2 = 1$, from Equation (2) we have:

$$\begin{cases} c_1 + c_2 + c_3 = 0, \\ c_1\alpha + c_2\beta + c_3\gamma = 1, \\ c_1\alpha^2 + c_2\beta^2 + c_3\gamma^2 = 1. \end{cases} \tag{3}$$

It is clear that Equation (3) implies:

$$\begin{aligned} c_1 &= \frac{\alpha}{(\alpha-\beta)(\alpha-\gamma)} = \frac{1}{-\alpha^2+4\alpha-1}, \\ c_2 &= \frac{\beta}{(\beta-\alpha)(\beta-\gamma)} = \frac{1}{-\beta^2+4\beta-1}, \\ c_3 &= \frac{\gamma}{(\gamma-\alpha)(\gamma-\beta)} = \frac{1}{-\gamma^2+4\gamma-1}. \end{aligned} \tag{4}$$

T. Komatsu et al. [19,20,22], E. Kilic [21] studied the arithmetical properties of Tribonacci numbers and obtained many meaningful convolution identities for $T_n$.

Inspired by the ideas in [2,3], it is natural to ask, for any real number $h$, what are the properties of the coefficients $T_n(h)$ of the power series of the function:

$$F(h,x) = \left(\frac{1}{1-x-x^2-x^3}\right)^h = \sum_{n=0}^{\infty} T_n(h)\cdot x^n \ ?$$

Moreover, is there any close relationship between $T_n(h)$ and $T_n$?

In view of these problems, in this paper we carry out a preliminary discussion and prove the following main result:

**Theorem 1.** *Let $h$ denote any fixed real number. Then for any integer $n \geq 0$, the following identity holds:*

$$\begin{aligned} T_n(h) = {} & \frac{1}{6}\sum_{u+v+w=n} \frac{\langle h\rangle_u}{u!}\frac{\langle h\rangle_v}{v!}\frac{\langle h\rangle_w}{w!}\left(3T_{w+1-u} - 2T_{w-u} - T_{w-u-1}\right) \\ & \times \left(3T_{w+1-v} - 2T_{w-v} - T_{w-v-1}\right) \\ & - \frac{1}{6}\sum_{u+v+w=n} \frac{\langle h\rangle_u}{u!}\frac{\langle h\rangle_v}{v!}\frac{\langle h\rangle_w}{w!}\left(3T_{3w+1-n} - 2T_{3w-n} - T_{3w-1-n}\right), \end{aligned}$$

*where $\sum\limits_{u+v+w=n}$ denotes the summation over all three-dimensional nonnegative integer coordinates $(u,v,w)$ such that $u+v+w=n$, and $\langle h\rangle_0 = 1$:*

$$\langle h\rangle_n = h(h+1)(h+2)\cdots(h+n-1)$$

*for all positive integers $n$.*

Note that $T_n(1) = T_{n+1}$ and $\frac{\langle 1\rangle_n}{n!} = 1$; from this theorem we may immediately deduce the following three corollaries:

**Corollary 1.** *For any positive integer $n$, the following identity is true:*

$$\begin{aligned} T_{n+1} = {} & \frac{1}{6}\sum_{u+v+w=n}\left(3T_{w+1-u} - 2T_{w-u} - T_{w-u-1}\right)\cdot\left(3T_{w+1-v} - 2T_{w-v} - T_{w-v-1}\right) \\ & - \frac{1}{6}\sum_{u+v+w=n}\left(3T_{3w+1-n} - 2T_{3w-n} - T_{3w-1-n}\right). \end{aligned}$$

**Corollary 2.** *For any positive integers h and n, the following identity holds:*

$$\begin{aligned} T_n(h) &= \sum_{a_1+a_2+\cdots+a_h=n} T_{a_1+1}\cdot T_{a_2+1}\cdots T_{a_h+1} \\ &= \frac{1}{6}\sum_{u+v+w=n}\frac{\langle h\rangle_u}{u!}\frac{\langle h\rangle_v}{v!}\frac{\langle h\rangle_w}{w!}\left(3T_{w+1-u}-2T_{w-u}-T_{w-u-1}\right) \\ &\quad\times\left(3T_{w+1-v}-2T_{w-v}-T_{w-v-1}\right) \\ &\quad-\frac{1}{6}\sum_{u+v+w=n}\frac{\langle h\rangle_u}{u!}\frac{\langle h\rangle_v}{v!}\frac{\langle h\rangle_w}{w!}\left(3T_{3w+1-n}-2T_{3w-n}-T_{3w-1-n}\right). \end{aligned}$$

**Corollary 3.** *For any positive integer n, the following identity holds:*

$$\begin{aligned} T_n\left(\frac{1}{2}\right) &= \frac{1}{6\cdot 4^n}\sum_{u+v+w=n}\frac{(2u)!}{(u!)^2}\frac{(2v)!}{(v!)^2}\frac{(2w)!}{(w!)^2}\left(3T_{w+1-u}-2T_{w-u}-T_{w-u-1}\right) \\ &\quad\times\left(3T_{w+1-v}-2T_{w-v}-T_{w-v-1}\right) \\ &\quad-\frac{1}{6\cdot 4^n}\sum_{u+v+w=n}\frac{(2u)!}{(u!)^2}\frac{(2v)!}{(v!)^2}\frac{(2w)!}{(w!)^2}\left(3T_{3w+1-n}-2T_{3w-n}-T_{3w-1-n}\right). \end{aligned}$$

## 2. A Simple Lemma

In this section, we present a simple identity, which is required in the proof of the theorem. Of course, simple number theories and knowledge of mathematical analysis is used in the proof of the following lemma. This topics can be found in [23], so there is no need it repeat here. The next lemma contains the relevant identities:

**Lemma 1.** *Let h be a fixed positive number. Then for any integer $n \geq 0$, we have the identity:*

$$\begin{aligned} \sum_{u+v+w=n}\frac{\langle h\rangle_u}{u!}\frac{\langle h\rangle_v}{v!}\frac{\langle h\rangle_w}{w!}\frac{1}{\alpha^u\beta^v\gamma^w} &= \frac{1}{6}\sum_{u+v+w=n}\frac{\langle h\rangle_u}{u!}\frac{\langle h\rangle_v}{v!}\frac{\langle h\rangle_w}{w!} \\ &\quad\times\left(3T_{w+1-u}-2T_{w-u}-T_{w-u-1}\right)\cdot\left(3T_{w+1-v}-2T_{w-v}-T_{w-v-1}\right) \\ &\quad-\frac{1}{6}\sum_{u+v+w=n}\frac{\langle h\rangle_u}{u!}\frac{\langle h\rangle_v}{v!}\frac{\langle h\rangle_w}{w!}\left(3T_{3w+1-n}-2T_{3w-n}-T_{3w-1-n}\right). \end{aligned}$$

**Proof.** Since $\alpha$, $\beta$ and $\gamma$ are the three roots of the equation $x^3-x^2-x-1=0$, so by the relationship between the roots and the coefficients of the equation we have $\alpha\cdot\beta\cdot\gamma=1$. Thus, for any non-negative integers $u$, $v$ and $w$:

$$\begin{aligned} &\left(\alpha^{w-u}+\beta^{w-u}+\gamma^{w-u}\right)\left(\alpha^{w-v}+\beta^{w-v}+\gamma^{w-v}\right) \\ =\ &\alpha^{2w-u-v}+\beta^{2w-u-v}+\gamma^{2w-u-v}+\alpha^{w-u}\beta^{w-v}+\alpha^{w-u}\gamma^{w-v} \\ &+\beta^{w-u}\alpha^{w-v}+\beta^{w-u}\gamma^{w-v}+\gamma^{w-u}\alpha^{w-v}+\gamma^{w-u}\beta^{w-v} \\ =\ &\alpha^{2w-u-v}+\beta^{2w-u-v}+\gamma^{2w-u-v}+\frac{1}{\alpha^u\beta^v\gamma^w}+\frac{1}{\alpha^u\beta^w\gamma^v} \\ &+\frac{1}{\alpha^v\beta^u\gamma^w}+\frac{1}{\alpha^v\beta^w\gamma^u}+\frac{1}{\alpha^w\beta^u\gamma^v}+\frac{1}{\alpha^w\beta^v\gamma^u}. \end{aligned} \tag{5}$$

On the other hand, from Equation (4) we also have:

$$c_1\left(-\alpha^2+4\alpha-1\right)=-c_1\alpha^2+4c_1\alpha-c_1=1,$$

$$c_2\left(-\beta^2+4\beta-1\right)=-c_2\beta^2+4c_2\beta-c_2=1$$

and:

$$c_3\left(-\gamma^2+4\gamma-1\right)=-c_3\gamma^2+4c_3\gamma-c_3=1.$$

So for any integer $r$, we have:

$$\alpha^r=c_1\left(-\alpha^2+4\alpha-1\right)\alpha^r=-c_1\alpha^{2+r}+4c_1\alpha^{1+r}-c_1\alpha^r,$$

$$\beta^r=c_2\left(-\beta^2+4\beta-1\right)\beta^r=-c_2\beta^{2+r}+4c_2\beta^{1+r}-c_2\beta^r$$

and:

$$\gamma^r=c_3\left(-\gamma^2+4\gamma-1\right)\beta^r=-c_3\gamma^{2+r}+4c_3\gamma^{1+r}-c_3\gamma^r.$$

From these identities and in combination with Equation (2) we may immediately deduce:

$$\begin{aligned}
& \alpha^r+\beta^r+\gamma^r=-\left(c_1\alpha^{2+r}+c_2\beta^{2+r}+c_3\gamma^{2+r}\right) \\
& +4\left(c_1\alpha^{1+r}+c_2\beta^{1+r}+c_3\gamma^{1+r}\right)-\left(c_1\alpha^r+c_2\beta^r+c_3\gamma^r\right) \\
= \; & -T_{2+r}+4T_{1+r}-T_r=3T_{1+r}-2T_r-T_{r-1}.
\end{aligned} \tag{6}$$

Combining Equations (5) and (6) and noting that the non-negative integer coordinates $(u,v,w)$ with $w+v+w=n$ are symmetrical, we have:

$$\begin{aligned}
& \sum_{u+v+w=n}\frac{\langle h\rangle_u}{u!}\frac{\langle h\rangle_v}{v!}\frac{\langle h\rangle_w}{w!}\left(\alpha^{w-u}+\beta^{w-u}+\gamma^{w-u}\right)\left(\alpha^{w-v}+\beta^{w-v}+\gamma^{w-v}\right) \\
= & \sum_{u+v+w=n}\frac{\langle h\rangle_u}{u!}\frac{\langle h\rangle_v}{v!}\frac{\langle h\rangle_w}{w!}\left(3T_{w+1-u}-2T_{w-u}-T_{w-u-1}\right) \\
& \times\left(3T_{w+1-v}-2T_{w-v}-T_{w-v-1}\right)
\end{aligned} \tag{7}$$

and:

$$\begin{aligned}
& \sum_{u+v+w=n}\frac{\langle h\rangle_u}{u!}\frac{\langle h\rangle_v}{v!}\frac{\langle h\rangle_w}{w!}\left(\alpha^{w-u}+\beta^{w-u}+\gamma^{w-u}\right)\left(\alpha^{w-v}+\beta^{w-v}+\gamma^{w-v}\right) \\
= & \sum_{u+v+w=n}\frac{\langle h\rangle_u}{u!}\frac{\langle h\rangle_v}{v!}\frac{\langle h\rangle_w}{w!}\left(3T_{3w+1-n}-2T_{3w-n}-T_{3w-n-1}\right) \\
& +\sum_{u+v+w=n}\frac{\langle h\rangle_u}{u!}\frac{\langle h\rangle_v}{v!}\frac{\langle h\rangle_w}{w!}\left(\frac{1}{\alpha^u\beta^v\gamma^w}+\frac{1}{\alpha^u\beta^w\gamma^v}+\frac{1}{\alpha^v\beta^u\gamma^w}\right) \\
& +\sum_{u+v+w=n}\frac{\langle h\rangle_u}{u!}\frac{\langle h\rangle_v}{v!}\frac{\langle h\rangle_w}{w!}\left(\frac{1}{\alpha^v\beta^w\gamma^u}+\frac{1}{\alpha^w\beta^u\gamma^v}+\frac{1}{\alpha^w\beta^v\gamma^u}\right) \\
= & \sum_{u+v+w=n}\frac{\langle h\rangle_u}{u!}\frac{\langle h\rangle_v}{v!}\frac{\langle h\rangle_w}{w!}\left(3T_{3w+1-n}-2T_{3w-n}-T_{3w-n-1}\right) \\
& +6\sum_{u+v+w=n}\frac{\langle h\rangle_u}{u!}\frac{\langle h\rangle_v}{v!}\frac{\langle h\rangle_w}{w!}\frac{1}{\alpha^u\beta^v\gamma^w}.
\end{aligned} \tag{8}$$

Now the lemma follows from Equations (7) and (8). □

## 3. Proof of the Theorem

Now we can easily prove our theorem. In fact, for any real number $h$, from the lemma and noting that the power series expansion of $(1-x)^{-h}$, which reads as follows:

$$\frac{1}{(1-x)^h} = \sum_{n=0}^{\infty} \frac{\langle h \rangle_n}{n!} \cdot x^n, \ \ |x| < 1$$

we have:

$$\begin{aligned} F(h,x) &= \frac{1}{(1-x-x^2-x^3)^h} = \frac{1}{\left(1-\frac{x}{\alpha}\right)^h \left(1-\frac{x}{\beta}\right)^h \left(1-\frac{x}{\gamma}\right)^h} \\ &= \left(\sum_{n=0}^{\infty} \frac{\langle h \rangle_n}{n!} \cdot \frac{x^n}{\alpha^n}\right) \left(\sum_{n=0}^{\infty} \frac{\langle h \rangle_n}{n!} \cdot \frac{x^n}{\beta^n}\right) \left(\sum_{n=0}^{\infty} \frac{\langle h \rangle_n}{n!} \cdot \frac{x^n}{\gamma^n}\right) \\ &= \sum_{n=0}^{\infty} \left( \sum_{u+v+w=n} \frac{\langle h \rangle_u}{u!} \frac{\langle h \rangle_v}{v!} \frac{\langle h \rangle_w}{w!} \frac{1}{\alpha^u \beta^v \gamma^w} \right) \cdot x^n. \end{aligned} \tag{9}$$

On the other hand, we also have:

$$F(h,x) = \sum_{n=0}^{\infty} T_n(h) \cdot x^n. \tag{10}$$

Applying Equations (9) and (10), the lemma and the uniqueness of power series expansion, we deduce:

$$\begin{aligned} T_n(h) &= \sum_{u+v+w=n} \frac{\langle h \rangle_u}{u!} \frac{\langle h \rangle_v}{v!} \frac{\langle h \rangle_w}{w!} \frac{1}{\alpha^u \beta^v \gamma^w} \\ &= \frac{1}{6} \sum_{u+v+w=n} \frac{\langle h \rangle_u}{u!} \frac{\langle h \rangle_v}{v!} \frac{\langle h \rangle_w}{w!} \left(3T_{w+1-u} - 2T_{w-u} - T_{w-u-1}\right) \\ &\quad \times \left(3T_{w+1-v} - 2T_{w-v} - T_{w-v-1}\right) \\ &\quad - \frac{1}{6} \sum_{u+v+w=n} \frac{\langle h \rangle_u}{u!} \frac{\langle h \rangle_v}{v!} \frac{\langle h \rangle_w}{w!} \left(3T_{3w+1-n} - 2T_{3w-n} - T_{3w-1-n}\right). \end{aligned}$$

This completes the proof of our theorem.

## 4. Conclusions

The main results of this paper are a theorem and three corollaries. The theorem establishes a close relationship between $T_n(h)$ and $T_n$. In other words, $T_n(h)$ can be expressed as a combination of $T_n$. Three corollaries are actually simplified versions of the particular values of $h$ in the theorem. It is clear that the research method in our paper can also be used as a reference for a further study of the properties of higher-order linear recursive sequence line Tribonacci polynomials.

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