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Integral Inequalities of Chebyshev Type for Continuous Fields of Hermitian Operators Involving Tracy–Singh Products and Weighted Pythagorean Means

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Abstract: In this paper, we establish several integral inequalities of Chebyshev type for bounded continuous fields of Hermitian operators concerning Tracy-Singh products and weighted Pythagorean means. The weighted Pythagorean means considered here are parametrization versions of three symmetric means: the arithmetic mean, the geometric mean, and the harmonic mean. Every continuous field considered here is parametrized by a locally compact Hausdorff space equipped with a finite Radon measure. Tracy-Singh product versions of the Chebyshev-Grüss inequality via oscillations are also obtained. Such integral inequalities reduce to discrete inequalities when the space is a finite space equipped with the counting measure. Moreover, our results include Chebyshev-type inequalities for tensor product of operators and Tracy-Singh/Kronecker products of matrices.

Keywords: Chebyshev inequality; Tracy-Singh product; continuous field of operators; Bochner integral; weighted Pythagorean mean

1. Introduction

One of the fundamental inequalities in mathematics is the Chebyshev inequality, named after P.L. Chebyshev, which states that

$$\frac{1}{n}\sum_{i=1}^{n} a_i b_i \geqslant \left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} b_i\right) \tag{1}$$

for all real numbers a_i, b_i $(1 \leqslant i \leqslant n)$ such that $a_1 \leqslant \ldots \leqslant a_n$ and $b_1 \leqslant \ldots \leqslant b_n$, or $a_1 \geqslant \ldots \geqslant a_n$ and $b_1 \geqslant \ldots \geqslant b_n$. This inequality can be generalized to

$$\sum_{i=1}^{n} w_i a_i b_i \geqslant \left(\sum_{i=1}^{n} w_i a_i\right)\left(\sum_{i=1}^{n} w_i b_i\right) \tag{2}$$

where $w_i \geqslant 0$ for all $1 = 1, \ldots, n$. A matrix version of (2) involving the Hadamard product was obtained in [1].

A continuous version of the Chebyshev inequality [2] says that if $f, g : [a,b] \to \mathbb{R}$ are monotone functions in the same sense and $p : [a,b] \to [0,\infty)$ is an integrable function, then

$$\int_a^b p(x)dx \int_a^b p(x)f(x)g(x)dx \geqslant \int_a^b p(x)f(x)dx \cdot \int_a^b p(x)g(x)dx. \tag{3}$$

If f and g are monotone in the opposite sense, the reverse inequality holds. In [3], Moslehian and Bakherad extended this inequality to Hilbert space operators related with the Hadamard product by using the notion of synchronous Hadamard property. They also presented integral Chebyshev inequalities respecting operator means.

The Grüss inequality, first introduced by G. Grüss in 1935 [4], is a complement of the Chebyshev inequality. This inequality gives a bound of the difference between the product of the integrals and the integral of the product for two integrable functions. For each integral function $f : [a,b] \to \mathbb{R}$, let us denote

$$\mathcal{I}(f) = \frac{1}{b-a}\int_a^b f(x)dx.$$

The Grüss inequality states that if $f, g : [a,b] \to \mathbb{R}$ are integrable functions and there exist real constants k, K, l, L such that $k \leqslant f(x) \leqslant K$ and $l \leqslant g(x) \leqslant L$ for all $x \in [a,b]$, then

$$|\mathcal{I}(fg) - \mathcal{I}(f)\mathcal{I}(g)| \ \leqslant \ \frac{1}{4}(K-k)(L-l). \tag{4}$$

This inequality has been studied and generalized by several authors; see [5–7]. In [7], the term Chebyshev-Grüss inequalities is used mentioning to Grüss inequalities for Chebyshev functions $T_{\mathcal{I}}$ which defined as

$$T_{\mathcal{I}}(f,g) \ = \ \mathcal{I}(f \cdot g) - \mathcal{I}(f) \cdot \mathcal{I}(g).$$

A general form of Chebyshev-Grüss inequalities is given by

$$|T_{\mathcal{I}}(f,g)| \ \leqslant \ E(\mathcal{I}, f, g)$$

where E is an expression depending on the arithmetic integral mean $\mathcal{I}$ and oscillations of f and g. Chebyshev-Grüss inequalities for some kind of operator via discrete oscillations is presented by Gonska, Raça and Rusu [7].

On the other hand, the notion of tensor product of operators is a key concept in functional analysis and its applications particularly in quantum mechanics. The theory of tensor product of operators has been investigate in the literature; see, e.g., [8,9]. In [10,11], the authors extend the notion of tensor product to the Tracy-Singh product for operators on a Hilbert space, and supply algebraic/order/analytic properties of this product.

In this paper, we establish a number of integral inequalities of Chebyshev type for continuous fields of Hermitian operators relating Tracy-singh products and weighted Pythagorean means. The Pythagorean means considered here are three classical means -the geometric mean, the arithmetic mean, and the harmonic mean. The continuous field considered here is parametrized by a locally compact Hausdorff space Ω endowed with a finite Radon measure. In Section 2, we give basic results on Tracy-Singh products for Hilbert space operators and Bochner integrability of continuous field of operators on a locally compact Housdorff space. In Section 3, we provide Chebyshev type inequalities involving Tracy-Singh products of operators under the assumption of synchronous Tracy-Singh property. In Section 4, we establish Chebyshev integral inequalities concerning operator means and Tracy-Singh products under the assumption of synchronous monotone property. Finally, we prove Chebyshev-Grüss inequalities via oscillations for continuous fields of operators in Section 5. In the case that Ω is a finite space with the counting measure, such integral inequalities reduce to discrete inequalities. Our results include Chebyshev-type inequalities concerning tensor product of operators and Tracy-Singh/Kronecker products of matrices.

2. Preliminaries

In this paper, we consider complex Hilbert spaces $\mathbb{H}$ and $\mathbb{K}$. The symbol $\mathbb{B}(\mathbb{X})$ stands to the Banach space of bounded linear operators on a Hilbert space $\mathbb{X}$. The cone of positive operators on $\mathbb{X}$ is denoted by $\mathbb{B}(\mathbb{X})^+$. For Hermitian operators A and B in $\mathbb{B}(\mathbb{X})$, the situation $A \geq B$ means that $A - B \in \mathbb{B}(\mathbb{X})^+$. Denote the set of all positive invertible operators on $\mathbb{X}$ by $\mathbb{B}(\mathbb{X})^{++}$.

We fix the following orthogonal decompositions:

$$\mathbb{H} \;=\; \bigoplus_{i=1}^{m} \mathbb{H}_i, \quad \mathbb{K} \;=\; \bigoplus_{k=1}^{n} \mathbb{K}_k$$

where all $\mathbb{H}_i$ and $\mathbb{K}_j$ are Hilbert spaces. Such decompositions lead to a unique representation for each operator $A \in \mathbb{B}(\mathbb{H})$ and $B \in \mathbb{B}(\mathbb{K})$ as a block-matrix form:

$$A \;=\; \left[A_{ij}\right]_{i,j=1}^{m,m} \quad \text{and} \quad B \;=\; \left[B_{kl}\right]_{k,l=1}^{n,n}$$

where $A_{ij} \in \mathbb{B}(\mathbb{H}_j, \mathbb{H}_i)$ and $B_{kl} \in \mathbb{B}(\mathbb{K}_l, \mathbb{K}_k)$ for each i, j, k, l.

2.1. Tracy-Singh Product for Operators

Let $A \in \mathbb{B}(\mathbb{H})$ and $B \in \mathbb{B}(\mathbb{K})$. Recall that the tensor product of A and B, denoted by $A \otimes B$, is a unique bounded linear operator on the tensor product space $\mathbb{H} \otimes \mathbb{K}$ such that

$$(A \otimes B)(x \otimes y) = Ax \otimes By, \quad \forall x \in \mathbb{H}, \forall y \in \mathbb{K}.$$

When $\mathbb{H} = \mathbb{K} = \mathbb{C}$, the tensor product of operators becomes the Kronecker product of matrices.

Definition 1. *Let $A = [A_{ij}]_{i,j=1}^{m,m} \in \mathbb{B}(\mathbb{H})$ and $B = [B_{kl}]_{k,l=1}^{n,n} \in \mathbb{B}(\mathbb{K})$. The Tracy-Singh product of A and B is defined to be in the form*

$$A \boxtimes B \;=\; \Big[\left[A_{ij} \otimes B_{kl}\right]_{kl}\Big]_{ij}, \tag{5}$$

which is a bounded linear operator from $\bigoplus_{i=1}^{m} \bigoplus_{k=1}^{n} \mathbb{H}_i \otimes \mathbb{K}_k$ into itself.

When $m = n = 1$, the Tracy-Singh product $A \boxtimes B$ is the tensor product $A \otimes B$. If $\mathbb{H}_i = \mathbb{K}_j = \mathbb{C}$ for all i, j, the above definition becomes the usual Tracy-Singh product for complex matrices.

Lemma 1 ([10,11])**.** *Let A, B, C, D be compatible operators. Then*

1. $(\alpha A) \boxtimes B \;=\; A \boxtimes (\alpha B) \;=\; \alpha(A \boxtimes B)$ *for any $\alpha \in \mathbb{C}$.*
2. $(A + B) \boxtimes (C + D) \;=\; A \boxtimes C + A \boxtimes D + B \boxtimes C + B \boxtimes D.$
3. $(A \boxtimes B)(C \boxtimes D) \;=\; (AC) \boxtimes (BD).$
4. *If A and B are Hermitian, then so is $A \boxtimes B$.*
5. *If A and B are positive and invertible, then $(A \boxtimes B)^\alpha \;=\; A^\alpha \boxtimes B^\alpha$ for any $\alpha \in \mathbb{R}$.*
6. *If $A \geqslant C \geqslant 0$ and $B \geqslant D \geqslant 0$, then $A \boxtimes B \;\geqslant\; C \boxtimes D \;\geqslant\; 0.$*

2.2. Bochner Integration

Let Ω be a locally compact Hausdorff (LCH) space equipped with a finite Radon measure μ. A family $\mathcal{A} = (A_t)_{t \in \Omega}$ of operators in $\mathbb{B}(\mathbb{H})$ is said to be bounded if there is a constant $M > 0$ for which $\|A_t\| \leqslant M$ for all $t \in \Omega$. The family $\mathcal{A}$ is said to be a continuous field if parametrization $t \mapsto A_t$ is norm-continuous on Ω. Every continuous field $\mathcal{A} = (A_t)_{t\in\Omega}$ can have the Bochner integral $\int_\Omega A_t d\mu(t)$ if the norm function $t \mapsto \|A_t\|$ possess the Lebesgue integrability. In this case, the resulting integral is a unique element in $\mathbb{B}(\mathbb{H})$ such that

$$\phi\left(\int_\Omega A_t d\mu(t) \right) = \int_\Omega \phi\left(A_t\right) d\mu(t)$$

for every bounded linear functional ϕ on $\mathbb{B}(\mathbb{H})$.

Lemma 2 (e.g., [12])**.** *Let $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ be a Banach space and (Γ, v) a finite measure space. Then a measurable function $f : \Gamma \to \mathbb{X}$ is Bochner integrable if and only if its norm function $\|f\|$ is Lebesgue integrable.*

Lemma 3 (e.g., [12])**.** *Let $f : \Gamma \to \mathbb{X}$ be a Bochner integrable function. If $\varphi : \mathbb{X} \to \mathbb{Y}$ is a bounded linear operator, then the composition $\varphi \circ f$ is Bochner integrable and*

$$\int_\Gamma (\varphi \circ f) dv = \varphi \int_\Gamma f dv.$$

Proposition 1. *Let $(A_t)_{t\in\Omega}$ be a bounded continuous field of operators in $\mathbb{B}(\mathbb{H})$. Then for any $X \in \mathbb{B}(\mathbb{K})$,*

$$\int_\Omega A_t d\mu(t) \boxtimes X = \int_\Omega (A_t \boxtimes X) d\mu(t).$$

Proof. Since the map $t \mapsto A_t$ is continuous and bounded, it is Bochner integrable on Ω. Note that the map $T \mapsto T \boxtimes X$ is linear and bounded by Lemma 1. Now, Lemma 3 implies that the map $t \mapsto A_t \boxtimes X$ is Bochner integrable on Ω and

$$\int_\Omega A_t d\mu(t) \boxtimes X = \int_\Omega (A_t \boxtimes X) d\mu(t).$$

for all $X \in \mathbb{B}(\mathbb{K})$. □

3. Chebyshev Type Inequalities Involving Tracy-Singh Products of Operators

From now on, let Ω be an LCH space equipped with a finite Radon measure μ. Let $\mathcal{A} = (A_t)_{t\in\Omega}$, $\mathcal{B} = (B_t)_{t\in\Omega}$, $\mathcal{C} = (C_t)_{t\in\Omega}$ and $\mathcal{D} = (D_t)_{t\in\Omega}$ be continuous fields of Hilbert space operators.

Definition 2. *The fields $\mathcal{A}$ and $\mathcal{B}$ are said to have the synchronous Tracy-Singh property if, for all $s, t \in \Omega$,*

$$(A_t - A_s) \boxtimes (B_t - B_s) \geqslant 0. \tag{6}$$

They are said to have the opposite-synchronous Tracy-Singh property if the reverse of (6) holds for all $s, t \in \Omega$.

Theorem 1. *Let $\mathcal{A}$ and $\mathcal{B}$ be bounded continuous fields of Hermitian operators in $\mathbb{B}(\mathbb{H})$ and $\mathbb{B}(\mathbb{K})$, respectively, and let $\alpha : \Omega \to [0, \infty)$ be a bounded measurable function.*

1. *If $\mathcal{A}$ and $\mathcal{B}$ have the synchronous Tracy-Singh property, then*

$$\int_\Omega \alpha(s) d\mu(s) \int_\Omega \alpha(t)(A_t \boxtimes B_t) d\mu(t) \geqslant \int_\Omega \alpha(t) A_t d\mu(t) \boxtimes \int_\Omega \alpha(s) B_s d\mu(s). \tag{7}$$

2. *If $\mathcal{A}$ and $\mathcal{B}$ have the opposite-synchronous Tracy-Singh property, then the reverse of (7) holds.*

Proof. By using Lemma 1, Proposition 1 and Fubini's Theorem [13], we have

$$\begin{aligned}
&\int_\Omega \alpha(s)d\mu(s)\int_\Omega \alpha(t)(A_t \boxtimes B_t)d\mu(t) - \int_\Omega \alpha(t)A_t d\mu(t) \boxtimes \int_\Omega \alpha(s)B_s d\mu(s) \\
&= \iint_{\Omega^2} \alpha(s)\alpha(t)(A_t \boxtimes B_t)d\mu(t)d\mu(s) - \iint_{\Omega^2} \alpha(t)\alpha(s)(A_t \boxtimes B_s)d\mu(t)d\mu(s) \\
&= \frac{1}{2}\iint_{\Omega^2} \left[\alpha(s)\alpha(t)(A_t \boxtimes B_t) - \alpha(t)\alpha(s)(A_t \boxtimes B_s)\right] d\mu(t)d\mu(s) \\
&\quad + \frac{1}{2}\iint_{\Omega^2} \left[\alpha(t)\alpha(s)(A_s \boxtimes B_s) - \alpha(s)\alpha(t)(A_s \boxtimes B_t)\right] d\mu(s)d\mu(t) \\
&= \frac{1}{2}\iint_{\Omega^2} \alpha(s)\alpha(t)\left[(A_t - A_s) \boxtimes (B_t - B_s)\right] d\mu(t)d\mu(s).
\end{aligned}$$

For the case 1, we have

$$\iint_{\Omega^2} \alpha(s)\alpha(t)\left[(A_t - A_s) \boxtimes (B_t - B_s)\right] d\mu(t)d\mu(s) \geqslant 0 \tag{8}$$

and thus (7) holds. For another case, we get the reverse of (8) and, thus, the reverse of (7) holds. □

Remark 1. *In Theorem 1 and other results in this paper, we may assume that Ω is a compact Hausdorff space. In this case, every continuous field on Ω is automatically bounded.*

The next corollary is a discrete version of Theorem 1.

Corollary 1. *Let A_i, B_i be Hermitian operators and let ω_i be nonnegative numbers for each $i = 1, \ldots, n$. Let $\mathcal{A} = (A_1, \ldots, A_n)$ and $\mathcal{B} = (B_1, \ldots, B_n)$.*

1. *If $\mathcal{A}$ and $\mathcal{B}$ have the synchronous Tracy-Singh property, then*

$$\sum_{i=1}^n \omega_i \sum_{i=1}^n \omega_i (A_i \boxtimes B_i) \geqslant \left(\sum_{i=1}^n \omega_i A_i\right) \boxtimes \left(\sum_{i=1}^n \omega_i B_i\right). \tag{9}$$

2. *If $\mathcal{A}$ and $\mathcal{B}$ have the opposite-synchronous Tracy-Singh property, then the reverse of (9) holds.*

Proof. From the previous theorem, set $\Omega = \{1, \ldots, n\}$ equipped with the counting measure and $\alpha(i) = \omega_i$ for all $i = 1, \ldots, n$. □

4. Chebyshev Integral Inequalities Concerning Weighted Pythagorean Means of Operators

Throughout this section, the space Ω is equipped with a total ordering $\preccurlyeq$.

Definition 3. *We say that a field $\mathcal{A}$ is increasing (decreasing, resp.) whenever $s \preccurlyeq t$ implies $A_s \leqslant A_t$ ($A_s \geqslant A_t$, resp.).*

Definition 4. *Two ordered pairs (X_1, X_2) and (Y_1, Y_2) of Hermitian operators are said to have the synchronous property if either*

$$X_i \leqslant Y_i \text{ for } i = 1, 2, \text{ or } X_i \geqslant Y_i \text{ for } i = 1, 2.$$

The pairs (X_1, X_2) and (Y_1, Y_2) are said to have the opposite-synchronous property if either

$$X_1 \leqslant Y_1 \text{ and } X_2 \geqslant Y_2, \text{ or } X_1 \geqslant Y_1 \text{ and } X_2 \leqslant Y_2.$$

Definition 5. *Let $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$ be continuous fields of Hermitian operators. Two ordered pairs $(\mathcal{A},\mathcal{B})$ and $(\mathcal{C},\mathcal{D})$ are said to have the synchronous monotone property if (A_t,B_t) and (C_t,D_t) have the synchronous property for all $t\in\Omega$. They are said to have the opposite-synchronous monotone property if (A_t,B_t) and (C_t,D_t) have the opposite-synchronous property for all $t\in\Omega$.*

Let us recall the notions of weighted classical Pythagorean means for operators. Indeed, they are generalizations of three famous symmetric operator means as follows. For any $w\in[0,1]$, the w-weighted arithmetic mean of $A,B\in\mathbb{B}(\mathbb{H})$ is defined by

$$A\,\nabla_w\,B \;=\; (1-w)A+wB.$$

The w-weighted geometric mean and w-weighted harmonic mean of $A,B\in\mathbb{B}(\mathbb{H})^{++}$ are defined respectively by

$$\begin{aligned} A\sharp_w B &= A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^w A^{\frac{1}{2}},\\ A\,!_w\,B &= \left[(1-w)A^{-1}+wB^{-1}\right]^{-1}. \end{aligned}$$

For any $A,B\in\mathbb{B}(\mathbb{H})^{+}$, we define the w-weighted geometric mean and w-weighted harmonic mean of A and B to be

$$\begin{aligned} A\sharp_w B &= \lim_{\varepsilon\to 0^+}(A+\varepsilon I)\sharp_w(B+\varepsilon I).\\ A\,!_w\,B &= \lim_{\varepsilon\to 0^+}(A+\varepsilon I)\,!_w\,(B+\varepsilon I), \end{aligned}$$

respectively. Here, the limits are taken in the strong-operator topology.

Lemma 4 (see e.g., [14])**.** *The weighted geometric means, weighted arithmetic means and weighted harmonic means for operators are monotone in the sense that if $A_1\leqslant A_2$ and $B_1\leqslant B_2$, then $A_1\sigma B_1\leqslant A_2\sigma B_2$ where σ is any of $\nabla_w, !_w, \sharp_w$.*

Lemma 5 ([15])**.** *Let $A,B,C,D\in\mathbb{B}(\mathbb{H})^{+}$ and $w\in[0,1]$. Then*

$$(A\boxtimes B)\sharp_w(C\boxtimes D) \;=\; (A\sharp_w C)\boxtimes(B\sharp_w D).$$

Theorem 2. *Let $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$ be bounded continuous fields in $\mathcal{B}(\mathbb{H})^{+}$ and let $\alpha:\Omega\to[0,\infty)$ be a bounded measurable function.*

1. *If $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$ are either all increasing, or all decreasing then*

$$\begin{aligned} &\int_\Omega \alpha(s)d\mu(s)\int_\Omega \alpha(t)[(A_t\boxtimes B_t)\sharp_w(C_t\boxtimes D_t)]d\mu(t)\\ &\qquad\geqslant \int_\Omega \alpha(t)(A_t\sharp_w C_t)d\mu(t)\boxtimes\int_\Omega \alpha(s)(B_s\sharp_w D_s)d\mu(s). \end{aligned} \tag{10}$$

2. *The reverse of (10) holds if either*
   - 2.1 *$\mathcal{A},\mathcal{C}$ are increasing and $\mathcal{B},\mathcal{D}$ are decreasing, or*
   - 2.2 *$\mathcal{A},\mathcal{C}$ are decreasing and $\mathcal{B},\mathcal{D}$ are increasing.*

Proof. Let $s,t \in \Omega$ and assume without loss of generally that $s \preccurlyeq t$. By applying Lemmas 1 and 5, Proposition 1, and Fubini's Theorem [13], we have

$$\begin{aligned}
&\int_\Omega \alpha(s)d\mu(s) \int_\Omega \alpha(t)[(A_t \boxtimes B_t)\sharp_w (C_t \boxtimes D_t)]d\mu(t) - \int_\Omega \alpha(t)(A_t \sharp_w C_t)d\mu(t) \boxtimes \int_\Omega \alpha(s)(B_s \sharp_w D_s)d\mu(s) \\
&\quad = \iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t \boxtimes B_t)\sharp_w (C_t \boxtimes D_t)]d\mu(t)d\mu(s) \\
&\qquad - \iint_{\Omega^2} \alpha(t)\alpha(s)[(A_t \sharp_w C_t) \boxtimes (B_s \sharp_w D_s)]d\mu(t)d\mu(s) \\
&\quad = \iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t \sharp_w C_t) \boxtimes (B_t \sharp_w D_t)]d\mu(t)d\mu(s) \\
&\qquad - \iint_{\Omega^2} \alpha(t)\alpha(s)[(A_t \sharp_w C_t) \boxtimes (B_s \sharp_w D_s)]d\mu(t)d\mu(s) \\
&\quad = \frac{1}{2} \iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t \sharp_w C_t) \boxtimes (B_t \sharp_w D_t) - (A_t \sharp_w C_t) \boxtimes (B_s \sharp_w D_s)]d\mu(t)d\mu(s) \\
&\qquad + \frac{1}{2} \iint_{\Omega^2} \alpha(t)\alpha(s)[(A_s \sharp_w C_s) \boxtimes (B_s \sharp_w D_s) - (A_s \sharp_w C_s) \boxtimes (B_t \sharp_w D_t)]d\mu(s)d\mu(t) \\
&\quad = \frac{1}{2} \iint_{\Omega^2} \alpha(s)\alpha(t)[A_t \sharp_w C_t - A_s \sharp_w C_s] \boxtimes [B_t \sharp_w D_t - B_s \sharp_w D_s]d\mu(t)d\mu(s).
\end{aligned}$$

If $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are all increasing, we have by Lemma 4 that $A_t \sharp_w C_t \geqslant A_s \sharp_w C_s$ and $B_t \sharp_w D_t \geqslant B_s \sharp_w D_s$. If $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are all decreasing, we have $A_t \sharp_w C_t \leqslant A_s \sharp_w C_s$ and $B_t \sharp_w D_t \leqslant B_s \sharp_w D_s$. Both cases lead to the same conclusion that

$$(A_t \sharp_w C_t - A_s \sharp_w C_s) \boxtimes (B_t \sharp_w D_t - B_s \sharp_w D_s) \geqslant 0,$$

and hence (10) holds. The cases 2.1 and 2.2 yield the same conclusion that

$$(A_t \sharp_w C_t - A_s \sharp_w C_s) \boxtimes (B_t \sharp_w D_t - B_s \sharp_w D_s) \leqslant 0.$$

and hence the reverse of (10) holds. □

Lemma 6. *Let A, B, C, D be Hermitian operators in $\mathbb{B}(\mathbb{H})$ and $w \in [0,1]$.*

1. *If (A, B) and (C, D) have the synchronous property, then*

$$(A \boxtimes B)\nabla_w (C \boxtimes D) \geqslant (A \nabla_w C) \boxtimes (B \nabla_w D). \tag{11}$$

2. *If (A, B) and (C, D) have the opposite-synchronous property, then the reverse of (11) holds.*

Proof. For the synchronous case, we have by using positivity of the Tracy-Singh product (Lemma 1) that $(A - C) \boxtimes (B - D) \geqslant 0$. Applying Lemma 1, we obtain

$$\begin{aligned}
0 &\leqslant w(1-w)\,[(A_1 - B_1) \boxtimes (A_2 - B_2)] \\
&= w(1-w)\,[A_1 \boxtimes A_2 - A_1 \boxtimes B_2 - B_1 \boxtimes A_2 + B_1 \boxtimes B_2] \\
&= [(1-w)(A_1 \boxtimes A_2) + w(B_1 \boxtimes B_2)] - [(1-w)A_1 + wB_1] \boxtimes [(1-w)A_2 + wB_2] \\
&= [(A_1 \boxtimes A_2)\,\nabla_w\,(B_1 \boxtimes B_2)] - [(A_1 \,\nabla_w\, B_1) \boxtimes (A_2 \,\nabla_w\, B_2)].
\end{aligned}$$

Thus $(A_1 \,\nabla_w\, B_1) \boxtimes (A_2 \,\nabla_w\, B_2) \leqslant (A_1 \boxtimes A_2)\,\nabla_w\,(B_1 \boxtimes B_2)$.

For the opposite-synchronous case, we have $(A_1 - B_1) \boxtimes (A_2 - B_2) \leqslant 0$ and hence the reverse of inequality (11) holds. □

Theorem 3. *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ be bounded continuous fields of operators in $\mathcal{B}(\mathbb{H})^+$, let $\alpha : \Omega \to [0, \infty)$ be a bounded measurable function.*

1. *If $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{C}, \mathcal{D})$ have the synchronous monotone property and all of $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are either increasing or decreasing, then*

$$\int_\Omega \alpha(s) d\mu(s) \int_\Omega \alpha(t)[(A_t \boxtimes B_t) \nabla_w (C_t \boxtimes D_t)] d\mu(t) \geqslant \int_\Omega \alpha(t)(A_t \nabla_w C_t) d\mu(t) \boxtimes \int_\Omega \alpha(s)(B_s \nabla_w D_s) d\mu(s). \tag{12}$$

2. *If $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{C}, \mathcal{D})$ have the opposite-synchronous monotone property and if either*

   2.1 *$\mathcal{A}, \mathcal{C}$ are increasing and $\mathcal{B}, \mathcal{D}$ are decreasing, or*
   2.2 *$\mathcal{A}, \mathcal{C}$ are decreasing and $\mathcal{B}, \mathcal{D}$ are increasing,*

   then the reverse of (12) holds.

Proof. Let $s, t \in \Omega$ and assume without loss of generally that $s \preccurlyeq t$. First, we consider the case 1. We have by using Lemmas 1 and 6, proposition 1, and Fubini's Theorem [13] that

$$\begin{aligned}
&\int_\Omega \alpha(s) d\mu(s) \int_\Omega \alpha(t)[(A_t \boxtimes B_t) \nabla_w (C_t \boxtimes D_t)] d\mu(t) - \int_\Omega \alpha(t)(A_t \nabla_w C_t) d\mu(t) \boxtimes \int_\Omega \alpha(s)(B_s \nabla_w D_s) d\mu(s) \\
&= \iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t \boxtimes B_t) \nabla_w (C_t \boxtimes D_t)] d\mu(t) d\mu(s) \\
&\quad - \iint_{\Omega^2} \alpha(t)\alpha(s)[(A_t \nabla_w C_t) \boxtimes (B_s \nabla_w D_s)] d\mu(t) d\mu(s) \\
&\geqslant \iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t \nabla_w C_t) \boxtimes (B_t \nabla_w D_t)] d\mu(t) d\mu(s) \\
&\quad - \iint_{\Omega^2} \alpha(t)\alpha(s)[(A_t \nabla_w C_t) \boxtimes (B_s \nabla_w D_s)] d\mu(t) d\mu(s) \\
&= \iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t \nabla_w C_t) \boxtimes (B_t \nabla_w D_t) - (A_t \nabla_w C_t) \boxtimes (B_s \nabla_w D_s)] d\mu(t) d\mu(s) \\
&= \frac{1}{2} \iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t \nabla_w C_t) - (A_s \nabla_w C_s)] \boxtimes [(B_t \nabla_w D_t) - (B_s \nabla_w D_s)] d\mu(t) d\mu(s).
\end{aligned}$$

Now, by Lemmas 1 and 4, we have

$$(A_t \nabla_w C_t - A_s \nabla_w C_s) \boxtimes (B_t \nabla_w D_t - B_s \nabla_w D_s) \geqslant 0$$

and hence (12) holds. The case 2 can be similarly proven. □

Lemma 7. *Let A, B, C, D be positive operators in $\mathbb{B}(\mathbb{H})$ and $w \in [0, 1]$.*

1. *If (A, B) and (C, D) are synchronous, then*

$$(A \boxtimes B) \,!_w\, (C \boxtimes D) \leqslant (A \,!_w\, C) \boxtimes (B \,!_w\, D). \tag{13}$$

2. *If (A, B) and (C, D) are opposite-synchronous, then the reverse of (13) holds.*

Proof. Assume that (A, B) and (C, D) are synchronous. By continuity, we may assume that $A, B, C, D > 0$. We have

$$(A^{-1} - C^{-1}) \boxtimes (B^{-1} - D^{-1}) \geqslant 0. \tag{14}$$

Using Lemma 1 and (14), we get

$$\begin{aligned}
0 &\leqslant w(1-w)A^{-1} \boxtimes B^{-1} + w(1-w)C^{-1} \boxtimes D^{-1} - w(1-w)A^{-1} \boxtimes D^{-1} - w(1-w)C^{-1} \boxtimes B^{-1} \\
&= \left[(1-w) - (1-w)^2\right] A^{-1} \boxtimes B^{-1} + (w - w^2)C^{-1} \boxtimes D^{-1} - w(1-w)A^{-1} \boxtimes D^{-1} \\
&\quad - w(1-w)C^{-1} \boxtimes B^{-1} \\
&= \left(A^{-1} \boxtimes B^{-1}\right) \triangledown_w \left(C^{-1} \boxtimes D^{-1}\right) - \left(A^{-1} \triangledown_w C^{-1}\right) \boxtimes \left(B^{-1} \triangledown_w D^{-1}\right).
\end{aligned}$$

This implies that

$$\left(A^{-1} \boxtimes B^{-1}\right) \triangledown_w \left(C^{-1} \boxtimes D^{-1}\right) \geqslant \left(A^{-1} \triangledown_w C^{-1}\right) \boxtimes \left(B^{-1} \triangledown_w D^{-1}\right).$$

Hence,

$$\begin{aligned}
(A \boxtimes B) \,!_w\, (C \boxtimes D) &= \left\{(A \boxtimes B)^{-1} \triangledown_w (C \boxtimes D)^{-1}\right\}^{-1} \\
&= \left\{\left(A^{-1} \boxtimes B^{-1}\right) \triangledown_w \left(C^{-1} \boxtimes D^{-1}\right)\right\}^{-1} \\
&\leqslant \left\{\left(A^{-1} \triangledown_w C^{-1}\right) \boxtimes \left(B^{-1} \triangledown_w D^{-1}\right)\right\}^{-1} \\
&= \left(A^{-1} \triangledown_w C^{-1}\right)^{-1} \boxtimes \left(B^{-1} \triangledown_w D^{-1}\right)^{-1} \\
&= (A \,!_w\, C) \boxtimes (B \,!_w\, D).
\end{aligned}$$

For the opposite-synchronous case, we have

$$(A^{-1} - C^{-1}) \boxtimes (B^{-1} - D^{-1}) \leqslant 0$$

and hence the reverse of (13) holds. □

Theorem 4. *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ be bounded continuous fields of operators in $\mathbb{B}(\mathbb{H})^+$ and $\alpha : \Omega \to [0, \infty)$ be a bounded measurable function.*

1. *If $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{C}, \mathcal{D})$ have the opposite-synchronous monotone property and if all of $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are either increasing or decreasing, then*

$$\begin{aligned}
&\int_\Omega \alpha(s) d\mu(s) \int_\Omega \alpha(t) [(A_t \boxtimes B_t) \,!_w\, (C_t \boxtimes D_t)] d\mu(t) \\
&\qquad \geqslant \int_\Omega \alpha(t) (A_t \,!_w\, C_t) d\mu(t) \boxtimes \int_\Omega \alpha(s) (B_s \,!_w\, D_s) d\mu(s).
\end{aligned} \tag{15}$$

2. *If $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{C}, \mathcal{D})$ have synchronous monotone property and if either*

   2.1 *$\mathcal{A}, \mathcal{C}$ are both increasing, and $\mathcal{B}, \mathcal{D}$ are both decreasing, or*
   2.2 *$\mathcal{A}, \mathcal{C}$ are both decreasing and $\mathcal{B}, \mathcal{D}$ are both increasing,*

   then the reverse of (15) holds.

Proof. Let $s, t \in \Omega$ with $s \preccurlyeq t$. If the pairs $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{C}, \mathcal{D})$ are opposite-synchronous, then we have by applying Lemmas 1 and 7, Proposition 1, and Fubini's Theorem [13] that

$$\begin{aligned}
&\int_\Omega \alpha(s)d\mu(s) \int_\Omega \alpha(t)[(A_t \boxtimes B_t) \,!_w\, (C_t \boxtimes D_t)]d\mu(t) - \int_\Omega \alpha(t)(A_t \,!_w\, C_t)d\mu(t) \boxtimes \int_\Omega \alpha(s)(B_s \,!_w\, D_s)d\mu(s) \\
&\quad = \iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t \boxtimes B_t) \,!_w\, (C_t \boxtimes D_t)]d\mu(t)d\mu(s) \\
&\qquad - \iint_{\Omega^2} \alpha(t)\alpha(s)[(A_t \,!_w\, C_t) \boxtimes (B_s \,!_w\, D_s)]d\mu(t)d\mu(s) \\
&\quad \geqslant \iint_{\Omega^2} \alpha(s)\alpha(t)[(A_t \,!_w\, C_t) \boxtimes (B_t \,!_w\, D_t)]d\mu(t)d\mu(s) \\
&\qquad - \iint_{\Omega^2} \alpha(t)\alpha(s)[(A_t \,!_w\, C_t) \boxtimes (B_s \,!_w\, D_s)]d\mu(t)d\mu(s) \\
&\quad = \frac{1}{2} \iint_{\Omega^2} \alpha(s)\alpha(t)[A_t \,!_w\, C_t - A_s \,!_w\, C_s] \boxtimes [B_t \,!_w\, D_t - B_s \,!_w\, D_s]d\mu(t)d\mu(s).
\end{aligned}$$

For the case 1, we have, by Lemmas 1 and 4,

$$(A_t \,!_w\, C_t - A_s \,!_w\, C_s) \boxtimes (B_t \,!_w\, D_t - B_s \,!_w\, D_s) \;\geqslant\; 0$$

and hence (15) holds. Another assertion can be proved in a similar manner to that of the second assertion in Theorem 3. □

5. Chebyshev-Grüss Inequaities via Oscillations

Throughout this section, let Ω be an LCH space equipped with a probability Radon measure μ. For any continuous field $\mathcal{A} = (A_t)_{t\in\Omega}$ in $\mathbb{B}(\mathbb{H})$ and $\mathcal{B} = (B_t)_{t\in\Omega}$ in $\mathbb{B}(\mathbb{K})$, we define

$$\begin{aligned}
\mathcal{A} \boxtimes \mathcal{B} &= (A_t \boxtimes B_t)_{t\in\Omega}, \quad \mathcal{I}(\mathcal{A}) = \int_\Omega A_t d\mu(t), \\
\operatorname{osc}(\mathcal{A}) &= \max\{\|A_t - A_s\| : (t,s) \in \operatorname{supp}(\mu \times \mu)\}.
\end{aligned}$$

Here, we recall that the support of the product measure $\mu \times \mu$ is defined by

$$\operatorname{supp}(\mu \times \mu) \;=\; \{(t,s) \in \Omega^2 \,:\, (\mu \times \mu)(G) > 0 \text{ for all open sets } G \subseteq \Omega^2 \text{ containing } (t,s)\}.$$

We call $\operatorname{osc}(\mathcal{A})$ the oscillation of the field $\mathcal{A}$.

Theorem 5. *Let $\mathcal{A} = (A_t)_{t\in\Omega}$ and $\mathcal{B} = (B_t)_{t\in\Omega}$ be continuous fields of Hermitian operators in $\mathbb{B}(\mathbb{H})$ and $\mathbb{B}(\mathbb{K})$, respectively. Then*

$$\mathcal{I}(\mathcal{A} \boxtimes \mathcal{B}) - \mathcal{I}(\mathcal{A}) \boxtimes \mathcal{I}(\mathcal{B}) \;\leqslant\; \frac{1}{2} \operatorname{osc}(\mathcal{A}) \cdot \operatorname{osc}(\mathcal{B})(\mu \times \mu)(\Omega^2 \backslash \Delta)(I_{\mathbb{H}} \boxtimes I_{\mathbb{K}}), \tag{16}$$

where $\Delta = \{(t,t) : t \in \Omega\}$.

Proof. We have by using Lemma 1, Proposition 1 and Fubini's Theorem [13] that

$$\begin{aligned}
\mathcal{I}(\mathcal{A}\boxtimes\mathcal{B})-\mathcal{I}(\mathcal{A})\boxtimes\mathcal{I}(\mathcal{B}) &= \int_\Omega d\mu(s)\int_\Omega A_t\boxtimes B_t d\mu(t)-\int_\Omega A_t d\mu(t)\boxtimes\int_\Omega B_s d\mu(s)\\
&= \iint_{\Omega^2} A_t\boxtimes B_t d\mu(t)d\mu(s)-\iint_{\Omega^2} A_t\boxtimes B_s d\mu(t)d\mu(s)\\
&= \frac{1}{2}\iint_{\Omega^2}(A_t\boxtimes B_t-A_t\boxtimes B_s+A_s\boxtimes B_s-A_s\boxtimes B_t)d\mu(t)d\mu(s)\\
&= \frac{1}{2}\iint\limits_{\Omega^2\setminus\Delta}(A_t-A_s)\boxtimes(B_t-B_s)d\mu(t)d\mu(s)\\
&\leqslant \frac{1}{2}\operatorname{osc}(\mathcal{A})\cdot\operatorname{osc}(\mathcal{B})(\mu\times\mu)(\Omega^2\setminus\Delta)(I_{\mathbb{H}}\boxtimes I_{\mathbb{K}}). \quad \square
\end{aligned}$$

Corollary 2. *Let $A_i\in\mathbb{B}(\mathbb{H})$ and $B_i\in\mathbb{B}(\mathbb{K})$ be Hermitian operators for all $i=1,\dots,n$. Then*

$$\sum_{i=1}^n(A_i\boxtimes B_i)-\left(\sum_{i=1}^n A_i\right)\boxtimes\left(\sum_{i=1}^n B_i\right) \leqslant \frac{n(n-1)}{2}\max_{1\leqslant i,j\leqslant n}\|A_i-A_j\|\cdot\max_{1\leqslant i,j\leqslant n}\|B_i-B_j\|(I_{\mathbb{H}}\boxtimes I_{\mathbb{K}}).$$

Proof. Set $\Omega=\{1,\dots,n\}$ equipped with the counting measure. We have

$$(\mu\times\mu)(\Omega^2\setminus\Delta)=\frac{n(n-1)}{2},\quad \operatorname{supp}(\mu\times\mu)=\Omega\times\Omega$$

and thus

$$\operatorname{osc}(A_1,\dots,A_n) = \max_{1\leqslant i,j\leqslant n}\|A_i-A_j\|,\quad \operatorname{osc}(B_1,\dots,B_n) = \max_{1\leqslant i,j\leqslant n}\|B_i-B_j\|. \quad \square$$

Example 1. *Let $\Omega=[0,1]$, $w\in\Omega$ and $0<\alpha\leqslant 1$. Consider the probability Radon measure $\mu=\alpha\lambda+(1-\alpha)\delta_w$, where λ is Lebesgue measure on Ω and δ_w is the Dirac measure at w. Set*

$$\mathcal{I}(A) := \int_0^1 A_t d\mu(t) = \alpha\int_0^1 A_t d\lambda(t)+(1-\alpha)A_w.$$

We have

$$\mu\times\mu = \alpha^2(\lambda\times\lambda)+\alpha(1-\alpha)(\lambda\times\delta_w)+(1-\alpha)\alpha(\delta_w\times\lambda)+(1-\alpha)^2(\delta_w\times\delta_w).$$

Then $\operatorname{supp}(\mu\times\mu)=[0,1]\times[0,1]$ *and* $(\mu\times\mu)\big([0,1]^2\setminus\Delta\big)=\alpha(2-\alpha)$. *For any continuous fields $\mathcal{A}=(A_t)_{t\in\Omega}$ and $\mathcal{B}=(B_t)_{t\in\Omega}$ of Hermitian operators, the inequality (16) becomes*

$$\mathcal{I}(\mathcal{A}\boxtimes\mathcal{B})-\mathcal{I}(\mathcal{A})\boxtimes\mathcal{I}(\mathcal{B}) \leqslant \frac{1}{2}\alpha(2-\alpha)\max_{0\leqslant s,t\leqslant 1}\|A_t-A_s\|\cdot\max_{0\leqslant t,s\leqslant 1}\|B_t-B_s\|(I_{\mathbb{H}}\boxtimes I_{\mathbb{K}}).$$

6. Conclusions

We establish several integral inequalities of Chebyshev type for continuous fields of Hermitian operators which are parametrized by an LCH space equipped with a finite Radon measure. We also obtain the Chebyshev-Grüss integral inequality via oscillations with respect to a probability Radon measure. These inequalities involve Tracy-Singh products and weighted versions of famous symmetric means. For a particular case that the LCH space is a finite space equipped with the counting measure, such integral inequalities reduce to discrete inequalities. Our results include Chebyshev-type inequalities for tensor product of operators and Tracy-Singh/Kronecker products of matrices.

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