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Solving Triangular Intuitionistic Fuzzy Matrix Game by Applying the Accuracy Function Method

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Received: 3 September 2019; Accepted: 1 October 2019; Published: 9 October 2019

Abstract: In this paper, the matrix game based on triangular intuitionistic fuzzy payoff is put forward. Then, we get a conclusion that the equilibrium solution of this game model is equivalent to the solution of a pair of the primal–dual single objective intuitionistic fuzzy linear optimization problems $(IFLOP1)$ and $(IFLOD1)$. Furthermore, by applying the accuracy function, which is linear, we transform the primal–dual single objective intuitionistic fuzzy linear optimization problems $(IFLOP1)$ and $(IFLOD1)$ into the primal–dual discrete linear optimization problems $(GLOP1)$ and $(GLOD1)$. The above primal–dual pair $(GLOP1)$–$(GLOD1)$ is symmetric in the sense the dual of $(GLOD1)$ is $(GLOP1)$. Thus the primal–dual discrete linear optimization problems $(GLOP1)$ and $(GLOD1)$ are called the symmetric primal–dual discrete linear optimization problems. Finally, the technique is illustrated by an example.

Keywords: intuitionistic fuzzy matrix game; equilibrium solution; intuitionistic fuzzy linear optimization problem; accuracy function

1. Introduction

Since the 1940s [1,2], game theory has been extensively studied in a systematic and axiomatic way. This is an important milestone in the development of game theory. Generally, games can be divided into two categories: Zero sum game and non-zero sum game. In this paper, we study the zero sum game model. The zero sum game is also known as the matrix game. The data of the classical matrix game model is discrete and precise. Indeed, in practice, players cannot accurately evaluate game data and accurately express their preferences and choices. Therefore, fuzzy set is introduced in [3], and it provides a very powerful tool to deal with the subjective judgment of players and the inaccuracy of game data. Many researchers discuss the problem of the fuzzy matrix game from different viewpoints. In [4], fuzzy sets are first used in cooperative game theory. Subsequently, However, Butnariu et al. [5] introduced fuzzy set in game theory and proposed fuzzy game. The maximum–minimum equilibrium problem of the fuzzy matrix game is discussed in [6]. Bector et al. [7] discussed the fuzzy matrix game in detail. Dubois [8] studied matrix games on account of fuzzy payoffs. Fuzzy linear optimization problems with fuzzy parameters are studied in [9]. Moreover, they use this optimization problem to study matrix games based on fuzzy goals and fuzzy payoffs. Afterwards, by fuzzy relation, they also propose a generalized matrix game model on account of fuzzy payoffs and fuzzy goals in [10].

However, in some cases, players could only know the payoff approximately based on some imprecise degree. However, it is possible players are not sure about it. In other words, there may be some hesitation about the approximate value of the reward. The fuzzy set uses only a membership function to indicate the degree of belonging, whereas the degree of non-belongingness is a compliment to one. In fact, in practice, it is not easy to describe a satisfactory compensation valve of players due to insufficient information available. So, there remains a part of uncertainty of which hesitation survives. Thus fuzzy sets do not synthesize hesitation degree well. Comparatively speaking, Atanassov [11] proposed the definition of intuitionistic fuzzy set. Intuitionistic fuzzy set is a generalization of fuzzy set theory and is now found to be very suitable for dealing with ambiguous problems. The intuitionistic fuzzy set is characterized by two functions representing the degree of membership and the degree of non-membership. Moreover, the sum of the two values is less than or equal to one. The hesitation degree is equal to 1 minus the degree of membership and the degree of non-membership. Hence, the intuitionistic fuzzy set may express information more abundant and flexible than the fuzzy set when uncertain information is involved. The theory of intuitionistic fuzzy sets has been studied in [12,13]. The theory of intuitionistic fuzzy sets has been widely used in decision making, topology and medical science [14–19].

In recent years, many researchers have mainly studied intuitionistic fuzzy matrix games, as in [20–22]. A matrix game model of payoffs based on intuitionistic fuzzy sets is proposed in [20]. Li [21] established a theoretical system of decision making and game theory by utilizing intuitionistic fuzzy sets, and this theoretical system provides effective tools for solving complex management problems. Afterwards, Nan [22] came up with a matrix game in which the payoff matrix is a trapezoidal intuitionistic fuzzy number. Nan [23] proposed a bi-matrix game model with intuitionistic fuzzy goals and intuitionistic fuzzy payoffs. An [24] studied intuitionistic fuzzy bi-matrix games by a mean-area ranking based a non-linear programming approach. Nayak et al. [25] studied bi-matrix games with intuitionistic fuzzy goals. Aggarwal et al. [26] researched matrix games based on intuitionistic fuzzy goals and intuitionistic fuzzy payoffs. Moreover, they [27] solved matrix game models with intuitionistic fuzzy goals by the linear optimization method. Seikh et al. [28] presented a model for studying bi-matrix games based on intuitionistic fuzzy goals. A matrix game model with triangular intuitionistic fuzzy numbers was considered in [29]. Moreover, solutions of the game are transformed into crisp linear programming problems by a suitable defuzzification function method. Yager [30] considered the expected value for membership function by finding the expected interval for a triangular fuzzy number. Singh et al. [31] used the same method to give the expected value of the non-membership function. Moreover, the expected values are called score functions. Taking the average (accuracy function) of two expected values being formed of expected interval as a single quantity can get a better approximation for comparison of the process of Yager. Singh et al.'s [31,32] discussion of the definition of triangular intuitionistic fuzzy number is different from other discussion in the literature. Hence, in this paper, we put forward a new model of a matrix game based on triangular intuitionistic fuzzy numbers. We utilize the accuracy function method [31] and single intuitionistic fuzzy optimization method [32] to solve the new model of matrix game based on the triangular intuitionistic fuzzy payoff.

The framework of this paper is as follows: Section 2 briefly depicts the basic knowledge of crisp matrix game models, intuitionistic fuzzy sets and the intuitionistic fuzzy optimization theory. Section 3 puts forward the matrix game based on the triangular intuitionistic fuzzy payoff. Then, we get a conclusion that the equilibrium solution of this game model is equivalent to the solution of a single objective intuitionistic fuzzy linear optimization problem. Furthermore, by applying the accuracy function, which is linear, we transform the single objective intuitionistic fuzzy linear optimization problem into the discrete linear optimization problem. An example is given in Section 4. Section 5 summarizes the full text.

2. Preliminaries

In this part, we first depict a crisp matrix game model in [7,33].

Definition 1 ([33])**.** *The mixed strategies space of player I is denoted by*

$$S^m = \{x = (x_1, x_2, \cdots, x_m)^T \in \mathbb{R}^m | \sum_{i=1}^{m} x_i = 1, x_i \geq 0, i = 1, 2, \cdots, m.\} \tag{1}$$

Similarly, The mixed strategies space of player II is denoted by

$$S^n = \{y = (y_1, y_2, \cdots, y_n)^T \in \mathbb{R}^n | \sum_{j=1}^{n} y_j = 1, y_j \geq 0, j = 1, 2, \cdots, n.\} \tag{2}$$

where x^T *is the transposition of* x, $\mathbb{R}^m$ *and* $\mathbb{R}^n$ *are* m- *and* n-*dimensional Euclidean spaces. The elements* x_i $(i = 1, 2, \cdots, m)$ *of* S^m *are called mixed strategies of player I. Similarly, the elements* y_j $(j = 1, 2, \cdots, n)$ *of* S^n *are called mixed strategies of player II.*

A payoff matrix of crisp a matrix game is taken as follows [7]:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \tag{3}$$

In order not to lose generality, we suppose that players I and II are maximized players. If player I chooses x_i mixed strategy and player II chooses y_j mixed strategy then a_{ij} is the amount paid by player II to player I. $-a_{ij}$ is the amount paid by player I to player II. That is, the gain of one player is the loss of other player.

A single objective matrix game (G) [7] is defined by

$$G = (S^m, S^n, A). \tag{4}$$

Definition 2 ([7])**.** *If* $x \in S^m$ *and* $y \in S^n$, *a vector*

$$E(x, y, A) = x^T A y \tag{5}$$

is called an expected payoff of player I. As the game G is zero-sum, the payoff of player II is $-x^T Ay$.

Definition 3 ([7])**.** *If* $x \in S^m$ *and* $y \in S^n$ *satisfy the following requirements:*

$$x^* Ay \geq V^*, \ \forall y \in S^n,$$

$$xAy^* \leq W^*, \ \forall x \in S^m.$$

then, x^* *and* y^* *are called the Nash equilibrium solution for players I and II, respectively. Moreover,* V^* *and* W^* *are called the value of* G.

Given a game G model, its solution can be obtained by solving the following discrete single objective linear optimization problems $(GLOP)$ and $(GLOD)$.

$$(GLOP) \qquad \max \quad V$$

$$\begin{aligned} &\text{such that } \sum_{i=1}^{m} a_{ij}x_i \geq V, (j=1,2,\cdots,n), \\ &\forall x \in S^m, \forall y \in S^n, \end{aligned} \tag{6}$$

$$(GLOD) \qquad \min \quad W$$

$$\begin{aligned} &\text{such that } \sum_{j=1}^{n} a_{ij}y_j \leq W, (i=1,2,\cdots,m), \\ &\forall x \in S^m, \forall y \in S^n. \end{aligned} \tag{7}$$

Next, we give the definition of triangular intuitionistic fuzzy number and its related knowledge.

Definition 4 ([11])**.** *Supposed that $\mathbb{R}$ is a finite universe set. $\widetilde{a}^I$ of $\mathbb{R}$ is said to be an intuitionistic fuzzy set if $\widetilde{a}^I$ of $\mathbb{R}$ satisfies the following condition*

$$\widetilde{a}^I = \{\langle x, u_{\widetilde{a}^I}(x), v_{\widetilde{a}^I}(x)\rangle : x \in \mathbb{R}\} \tag{8}$$

where $u_{\widetilde{a}^I}(x), v_{\widetilde{a}^I}(x) : \mathbb{R} \to [0,1]$ are functions such that $0 \leq u_{\widetilde{a}^I}(x) + v_{\widetilde{a}^I}(x) \leq 0, \forall x \in \mathbb{R}$. $u_{\widetilde{a}^I}(x)$ and $v_{\widetilde{a}^I}(x)$ denote degree of membership and degree of non-membership for $x \in \mathbb{R}$ in $\widetilde{a}^I$, respectively.

Definition 5 ([11])**.** *Supposing that $\widetilde{a}^I$ is an intuitionistic fuzzy set, if $u_{\widetilde{a}^I}(x)$ and $v_{\widetilde{a}^I}(x)$ of $\widetilde{a}^I$ satisfy the following conditions*

$$u_{\widetilde{a}^I}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1}, & a_1 < x \leq a_2, \\ \frac{a_3-x}{a_3-a_2}, & a_2 < x < a_3, \\ 0, & otherwise, \end{cases} \tag{9}$$

and

$$v_{\widetilde{a}^I}(x) = \begin{cases} \frac{a_2-x}{a_2-a_1'}, & a_1' < x \leq a_2, \\ \frac{x-a_2}{a_3'-a_2}, & a_2 < x < a_3', \\ 1, & otherwise. \end{cases} \tag{10}$$

where $a_1' \leq a_1 \leq a_2 \leq a_3 \leq a_3'$, then $\widetilde{a}^I$ is said to be a triangular intuitionistic fuzzy number. Moreover, $\widetilde{a}^I$ is denoted by $\widetilde{a}^I = (a_1, a_2, a_3; a_1', a_2, a_3')$. The set of all triangular intuitionistic fuzzy numbers is denoted by $IF(\mathbb{R})$.

Definition 6 ([11])**.** *Assuming that $\widetilde{a}^I = (a_1, a_2, a_3; a_1', a_2, a_3')$ and $\widetilde{b}^I = (b_1, b_2, b_3; b_1', b_2, b_3')$ are triangular intuitionistic fuzzy numbers, the addition of $\widetilde{a}^I$ and $\widetilde{b}^I$ is defined by*

$$\widetilde{a}^I \oplus \widetilde{b}^I = (a_1+b_1, a_2+b_2, a_3+b_3; a_1'+b_1', a_2+b_2, a_3'+b_3').$$

Let $k > 0$ be real number. Scalar multiplication of $\widetilde{a}^I$ is defined by

$$k\widetilde{a}^I = (ka_1, ka_2, ka_3; ka_1', ka_2, ka_3').$$

Let $k < 0$ be real number. Scalar multiplication of $\widetilde{a}^I$ is defined by

$$k\widetilde{a}^I = (ka_3, ka_2, ka_1; ka_3', ka_2, ka_1').$$

Definition 7 ([31])**.** *Let $\widetilde{a}^I = (a_1, a_2, a_3; a_1', a_2, a_3')$ be a triangular intuitionistic fuzzy number. The score function of $u_{\widetilde{a}^I}(x)$, denoted by $S(u_{\widetilde{a}^I}(x))$, is defined as $S(u_{\widetilde{a}^I}(x)) = \frac{a_1+2a_2+a_3}{4}$. The score function of $v_{\widetilde{a}^I}(x)$,* *denoted by* $S(v_{\widetilde{a}^I}(x))$, *is defined as* $S(v_{\widetilde{a}^I}(x)) = \frac{a_1^{'}+2a_2+a_3^{'}}{4}$. *Then, the accuracy function of* $\widetilde{a}^I$, *denoted by* $f(\widetilde{a}^I)$, *is defined as* $f(\widetilde{a}^I) = \frac{S(u_{\widetilde{a}^I}(x))+S(v_{\widetilde{a}^I}(x))}{2} = \frac{(a_1+2a_2+a_3)+(a_1^{'}+2a_2+a_3^{'})}{8}$.

Theorem 1 ([31]). *The accuracy function* $f : IF(\mathbb{R}) \to \mathbb{R}$ *is a linear function.*

Definition 8 ([32]). *Assuming that* $\widetilde{a}^I = (a_1, a_2, a_3; a_1^{'}, a_2, a_3^{'})$ *and* $\widetilde{b}^I = (b_1, b_2, b_3; b_1^{'}, b_2, b_3^{'})$ *are triangular intuitionistic fuzzy numbers, if* $f(\widetilde{a}^I) \leq f(\widetilde{b}^I)$, *then we say* $\widetilde{a}^I \preceq \widetilde{b}^I$; *if* $f(\widetilde{a}^I) \geq f(\widetilde{b}^I)$, *then we say* $\widetilde{a}^I \succeq \widetilde{b}^I$.

Finally, we give a single objective intuitionistic fuzzy linear optimization problem model and its relevant knowledge.

A single objective intuitionistic fuzzy linear optimization problem $(IFLOP)$ model is given by [32]

$$(IFLOP) \qquad \min \ \{\sum_{j=1}^{n} (\widetilde{c}_j)^I x_j\}$$

$$\begin{array}{ll} subject\ to & \sum_{j=1}^{n} (\widetilde{a}_{ij})^I x_j \succeq (\widetilde{b}_i)^I \quad (i = 1, 2, \cdots, m_1), \\ & \sum_{j=1}^{n} (\widetilde{a}_{ij})^I x_j \preceq (\widetilde{b}_i)^I \quad (i = m_1+1, m_1+2, \cdots, m_2), \\ & \sum_{j=1}^{n} (\widetilde{a}_{ij})^I x_j = (\widetilde{b}_i)^I \quad (i = m_2+1, m_2+2, \cdots, m), \\ & x_j \geq 0, \quad (j = 1, 2, \cdots, n) \end{array} \tag{11}$$

where $(\widetilde{a}_{ij})^I$, $(\widetilde{b}_i)^I$ and $(\widetilde{c}_j)^I$ are triangular intuitionistic fuzzy numbers.

According to Theorem 1, a single objective intuitionistic fuzzy linear optimization problem $(IFLOP)$ model is converted to a discrete linear optimization problem (LOP) model [32]

$$(LOP) \qquad \min \ \{\sum_{j=1}^{n} (\widetilde{c}_j)^{'} x_j\}$$

$$\begin{array}{ll} subject\ to & \sum_{j=1}^{n} (\widetilde{a}_{ij})^{'} x_j \geq (\widetilde{b}_i)^{'} \quad (i = 1, 2, \cdots, m_1), \\ & \sum_{j=1}^{n} (\widetilde{a}_{ij})^{'} x_j \leq (\widetilde{b}_i)^{'} \quad (i = m_1+1, m_1+2, \cdots, m_2), \\ & \sum_{j=1}^{n} (\widetilde{a}_{ij})^{'} x_j = (\widetilde{b}_i)^{'} \quad (i = m_2+1, m_2+2, \cdots, m), \\ & x_j \geq 0, \quad (j = 1, 2, \cdots, n) \end{array} \tag{12}$$

where $(\widetilde{a}_{ij})^{'} = f((\widetilde{a}_{ij})^I)$, $(\widetilde{b}_i)^{'} = f((\widetilde{b}_i)^I)$ and $(\widetilde{c}_j)^{'} = f((\widetilde{c}_j)^I)$ $(i = 1, 2, \cdots, m; j = 1, 2, \cdots, n)$.

Theorem 2 ([32]). *An optimal solution of a discrete linear optimization problem* (LOP) *model is an optimal solution of a single objective intuitionistic fuzzy linear optimization problem* $(IFLOP)$ *model.*

Note that solving a single objective intuitionistic fuzzy linear optimization problem $(IFLOP)$ model is equal to solving a discrete linear optimization problem (LOP) model.

3. A Matrix Game Based on Triangular Intuitionistic Fuzzy Payoff

In this section, we will put forward a matrix game model based on triangular intuitionistic fuzzy payoff. Assuming that S^m and S^n are introduced as in Section 2, because practical problems in real life are complex and tedious, players cannot accurately evaluate game model data and express their preferences. In the following, let us assume that the elements of the payoff matrix $\widetilde{A}^I$ are triangular intuitionistic fuzzy numbers. Supposing that aspiration levels $\widetilde{V}^I$ and $\widetilde{W}^I$ of players I and II are triangular intuitionistic fuzzy numbers; thus, a matrix game based on triangular intuitionistic fuzzy payoff, denoted by $TIFG$, is described as

$$TIFG = (S^m, S^n, \widetilde{A}^I, \widetilde{V}^I, \widetilde{W}^I) \tag{13}$$

where

$$\widetilde{A}^I = \begin{pmatrix} (\widetilde{a}_{11})^I & \cdots & (\widetilde{a}_{1n})^I \\ \vdots & \ddots & \vdots \\ (\widetilde{a}_{m1})^I & \cdots & (\widetilde{a}_{mn})^I \end{pmatrix} \tag{14}$$

Definition 9. *If $x \in S^m$ and $y \in S^n$, a vector*

$$E\left(x, y, \widetilde{A}^I\right) = x^T \widetilde{A}^I y \tag{15}$$

is called an intuitionistic fuzzy expected payoff of player I. As the game $TIFG$ is zero-sum, the intuitionistic fuzzy expected payoff of player II is $-x^T \widetilde{A}^I y$.

Definition 10. *If $x \in S^m$ and $y \in S^n$ satisfy the following requirements:*

$$x^* \widetilde{A}^I y \succeq (\widetilde{V}^I)^*, \ \forall y \in S^n,$$

$$x \widetilde{A}^I y^* \preceq (\widetilde{W}^I)^*, \ \forall x \in S^m.$$

then, $(x, y) \in S^m \times S^n$ is called the equilibrium solution of $TIFG$, where x^* and y^* are called the equilibrium solution of players I and II, respectively.

To solve the game $TIFG$ model, we give the following theorem.

Theorem 3. *Let $(x, y) \in S^m \times S^n$. If (x, y) is the equilibrium solution of $TIFG$ if and only if $(x, \widetilde{V}')$ and $(y, \widetilde{W}')$ are the optimal solution of discrete linear optimization problems $(GLOP1)$ and $(GLOD1)$*

$$(GLOP1) \qquad \max \ \widetilde{V}'$$

$$such \ \ that \ \sum_{i=1}^{m} (\widetilde{a}_{ij})' x_i \geq \widetilde{V}', (j = 1, 2, \cdots, n),$$
$$\forall x \in S^m, \forall y \in S^n,$$

$$(GLOD1) \qquad \min \ \widetilde{W}'$$

$$such \ \ that \ \sum_{j=1}^{n} (\widetilde{a}_{ij})' y_j \leq \widetilde{W}', (i = 1, 2, \cdots, m),$$
$$\forall x \in S^m, \forall y \in S^n.$$

where $(\widetilde{a}_{ij})' = f((\widetilde{a}_{ij})^I)$, $\widetilde{V}' = f(\widetilde{V}^I)$ and $\widetilde{W}' = f(\widetilde{W}^I)$ $(i = 1, 2, \cdots, m; j = 1, 2, \cdots, n)$.

Proof. By using Definition 10, we get that (x, y) is the equilibrium solution of $TIFG$ if and only if (x, y) is an optimal solution of intuitionistic fuzzy optimization problems $(IFOP)$ and $(IFOD)$.

$$(IFOP) \qquad \text{Find} \ \ x \in S^m \ \ \text{such that}$$

$$x \widetilde{A}^I y \succeq \widetilde{V}^I, \quad \forall y \in S^n,$$

$(IFOD)$ Find $y \in S^n$ such that

$$x\widetilde{A}^I y \preceq \widetilde{W}^I, \quad \forall x \in S^m.$$

since S^m and S^n are convex polytopes. Moreover, the problems $IFOP$ and $IFOD$ are crisp intuitionistic fuzzy linear optimization problems; it is sufficient to consider only the extreme points of S^m and S^n. Thus the problems $IFOP$ and $IFOD$ can be rewritten as

$(IFLOP1)$ $\max \ \widetilde{V}^I$

$$\text{such that } \sum_{i=1}^{m} (\widetilde{a}_{ij})^I x_i \succeq \widetilde{V}^I, (j = 1, 2, \cdots, n),$$
$$\forall x \in S^m, \forall y \in S^n,$$

$(IFLOD1)$ $\min \ \widetilde{W}^I$

$$\text{such that } \sum_{j=1}^{n} (\widetilde{a}_{ij})^I y_j \preceq \widetilde{W}^I, (i = 1, 2, \cdots, m),$$
$$\forall x \in S^m, \forall y \in S^n.$$

By applying the accuracy function, which is linear, and utilizing problems $IFLOP$, LOP and Theorem 2, we find that the problems $(IFLOP1)$ and $(IFLOD1)$ are equivalent to problems $GLOP1$ and $GLOD1$, respectively.

$(GLOP1)$ $\max \ \widetilde{V}'$

$$\text{such that } \sum_{i=1}^{m} (\widetilde{a}_{ij})' x_i \geq \widetilde{V}', (j = 1, 2, \cdots, n),$$
$$\forall x \in S^m, \forall y \in S^n,$$

$(GLOD1)$ $\min \ \widetilde{W}'$

$$\text{such that } \sum_{j=1}^{n} (\widetilde{a}_{ij})' y_j \leq \widetilde{W}', (i = 1, 2, \cdots, m),$$
$$\forall x \in S^m, \forall y \in S^n.$$

where $(\widetilde{a}_{ij})' = f((\widetilde{a}_{ij})^I)$, $\widetilde{V}' = f(\widetilde{V}^I)$ and $\widetilde{W}' = f(\widetilde{W}^I)$ $(i = 1, 2, \cdots, m; j = 1, 2, \cdots, n)$. □

Theorem 4. *Let $(x, y) \in S^m \times S^n$. If (x, y) is an optimal solution of discrete linear optimization problems $(GLOP1)$ and $(GLOD1)$, then $\widetilde{V}' \leq \widetilde{W}'$.*

Proof. Since (x, y) is an optimal solution of discrete linear optimization problems $(GLOP1)$ and $(GLOD1)$, by Theorem 3, we have

$$\sum_{i=1}^{m} (\widetilde{a}_{ij})' x_i \geq \widetilde{V}', (j = 1, 2, \cdots, n)$$

and

$$\sum_{j=1}^{n} (\widetilde{a}_{ij})' y_j \leq \widetilde{W}', (i = 1, 2, \cdots, m)$$

Since $\forall x \in S^m, \forall y \in S^n$, we get

$$\sum_{j=1}^{n}\sum_{i=1}^{m}(\widetilde{a}_{ij})' x_i y_j \geq \sum_{j=1}^{n} \widetilde{V}' y_j = \widetilde{V}'$$

and

$$\sum_{i=1}^{m}\sum_{j=1}^{n}(\widetilde{a}_{ij})' y_j x_i \leq \sum_{i=1}^{m} \widetilde{W}' x_i = \widetilde{W}'$$

Thus, we have $\widetilde{V}' \leq \widetilde{W}'$. □

4. Example

Assuming that the demand for wine in the market is essentially the same, wine company I and wine company II aim to obtain their market share. (Where wine companies I and II represent players I and II in the above game $TIFG$, respectively.) The two wine companies come up with two alternative marketing strategies: Advertising investment (α_1) and reducing-price sale (α_2). Due to the uncertainty and imprecision of information, the management of wine companies I and II cannot use any single value to represent the payoff for any one of the marketing strategies. To deal with this uncertainty, a matrix game based on triangular intuitionistic fuzzy payoff of wine company I is considered as follows

$$\widetilde{A}^I = \begin{pmatrix} (7,8,8;6,8,9), (5.5,6,6.5;5,6,7) \\ (5,5,6;5,5,7), (9,9,10;8.5,9,10) \end{pmatrix}$$

where $(7,8,8;6,8,9)$ denotes the profit of wine company I if wine companies I and II choose advertising investment (α_1). Similarly, other elements in $\widetilde{A}^I$ can be given.

In order to solve the above practical problems, by Definition 7, we have

$$f((7,8,8;6,8,9)) = \frac{(7+2\times 8+8)+(6+2\times 8+9)}{8} = 7.75$$

$$f((5.5,6,6.5;5,6,7)) = \frac{(5.5+2\times 6+6.5)+(5+2\times 6+7)}{8} = 6$$

$$f((5,5,6;5,5,7)) = \frac{(5+2\times 5+6)+(5+2\times 5+7)}{8} = 5.375$$

$$f((9,9,10;8.5,9,10)) = \frac{(9+2\times 9+10)+(8.5+2\times 9+10)}{8} = 9.1875$$

By using Theorem 3, we need to solve discrete linear optimization problems ($GLOP2$) and ($GLOD2$).

$$(GLOP2) \qquad \max \ \widetilde{V}'$$

$$\text{such that } \begin{aligned} &7.75x_1 + 5.375x_2 \geq \widetilde{V}', \\ &6x_1 + 9.1875x_2 \geq \widetilde{V}', \\ &x_1 + x_2 = 1, x_1 \geq 0, x_2 \geq 0, \end{aligned}$$

$$(GLOD2) \qquad \min \ \widetilde{W}'$$

$$\text{such that } \begin{aligned} &7.75y_1 + 6y_2 \leq \widetilde{W}', \\ &5.375y_1 + 9.1875y_2 \leq \widetilde{W}', \\ &y_1 + y_2 = 1, y_1 \geq 0, y_2 \geq 0. \end{aligned}$$

Solving the above problems ($GLOP2$) and ($GLOD2$) by utilizing LINGO, we can obtain that $((x_1^* = 0.32, x_2^* = 0.68), (\widetilde{V}')^* = 6.135)$ is an optimal solution of problems ($GLOP2$) and $((y_1^* =$

$0.68, y_2^* = 0.32), (\widetilde{W}')^* = 7.19)$ is an optimal solution of problems $(GLOD2)$. By Theorem 3, $((x_1^* = 0.32, x_2^* = 0.68), (y_1^* = 0.68, y_2^* = 0.32))$ is the equilibrium solution of $TIFG$. Namely, $(x_1^* = 0.32, x_2^* = 0.68)$ and $(y_1^* = 0.68, y_2^* = 0.32)$ are market shares held by wine companies I and II, respectively. $(\widetilde{V}')^* = 6.135$ is the expected profit of wine company I. $(\widetilde{W}')^* = 7.19$ is the expected profit of wine company II.

5. Conclusions

This paper has presented a matrix game based on triangular intuitionistic fuzzy payoff. Moreover, we obtain the conclusion that the equilibrium solution of this game model is equivalent to the solution of a single objective intuitionistic fuzzy linear optimization problem. Furthermore, by applying the accuracy function, which is linear, we transform the single objective intuitionistic fuzzy linear optimization problem into the discrete linear optimization problem. The matrix game theory based on triangular intuitionistic fuzzy payoff in this paper will be applied in other fields such as decision making and management.

Author Contributions: Formal analysis, Y.X.; Methodology, D.Q.

Funding: This work was supported by The National Natural Science Foundations of China (Grant No. 11671001).

Conflicts of Interest: The authors declare that they have no competing interests.

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