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On Some Initial and Initial Boundary Value Problems for Linear and Nonlinear Boussinesq Models

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Abstract: The main concern of this paper is to apply the modified double Laplace decomposition method to a singular generalized modified linear Boussinesq equation and to a singular nonlinear Boussinesq equation. An a priori estimate for the solution is also derived. Some examples are given to validate and illustrate the method.

Keywords: double Laplace transform; inverse Laplace transform; generalized modified linear Boussinesq equation; nonlinear Boussinesq equation

MSC: 35D35; 35L20

1. Introduction

One and higher dimensional Boussinesq equations are generally used in coastal and ocean engineering, modelling tidal oscillations and tsunami wave modelling. These equations are classified as hyperbolic equations, like nonlinear shallow water equations, and they were originally derived as a model for water waves. They in fact describe the irrotational motion of an incompressible fluid in the long wave limit and they are described by the Navier-Stokes equations. Boussinesq equations also appear as acoustic, elastic, electromagnetic or gravitational waves. Some developments of Boussinesq equations for one and multi-dimensional spaces can be found, for example, in Wei et al. [1], Madsen and Schaffer [2], Guido Schneider [3], Nwogu [4] and Kirby [5].

During the last three decades, many methods have been developed and used to solve these equations, such as homotopy analysis and homotopy perturbation methods (Francisco and Fernández [6], Gupta and Saha [7] and Dianhen et al. [8]), the analytic method [9], the modified decomposition method (Wazwaz [10], Fang et al. [11] and Basem and Attili [12]) the Laplace Adomian Decomposition Method (Hardik et al. [13], Zhang et al. [14], Liang et al. [15]) the transformed rational function method (Wang [16], Engui [17]) the integral transform method (Charles et al. [18]) the energy integral method (Joseph [19], Mesloub [20]) the inverse scattering method (Peter et al. [21]) and other different numerical methods were used to investigate problems dealing with Boussinesq equations, see for example, Jang [22], Iskandar and Jain [23], Bratsos [24], Dehghan and Salehi [25], Boussinesq [26], and Onorato et al. [27]. For the bifurcation of solutions and possible applications of Boussinesq equations, we may refer to References [28,29]. The purpose of the main result of this work is to use the modified double Laplace decomposition method for solving a singular generalized modified linear Boussinesq equation and a singular nonlinear Boussinesq equation. We also obtain an a priori estimate for the solution and we provide some examples to validate and illustrate the modified double Laplace decomposition method.

This paper is organized as follows—in Section 2, we introduce some tools to be used in the subsequent sections. In Section 3, we set and pose the first problem dealing with an initial boundary value problem for a singular modified linear Boussinesq equation with Bessel operator. Section 4 is devoted to establishing an a priori bound for the solution of problem (14)–(16) from which we deduce the uniqueness of its solutions in a weighted Sobolev space. In Section 5, we discuss the use of the modified double Laplace decomposition method for solving the posed problem (14)–(16) and an example is considered to illustrate the method. In Section 6, we consider an initial value problem for the one dimensional singular nonlinear Boussinesq equation. We have again used the modified double Laplace decomposition method to obtain the solution of this nonlinear problem and an example is given to confirm the validity of the method in the last section.

2. Preliminaries

(1) Function spaces: Let $L^2_\rho(Q)$ be the weighted $L^2(Q)$ Hilbert space of square integrable functions on $Q = (0,1) \times (0,T)$, $T < \infty$, with scalar product

$$(\varphi, \psi)_{L^2_\rho(Q)} = \int_Q x\varphi\psi dxdt, \quad \rho = x, \tag{1}$$

and with the associated finite norm

$$\|\varphi\|^2_{L^2_\rho(Q)} = \int_Q x\varphi^2 dxdt, \tag{2}$$

and let $W^{1,1}_{2,\rho}$ [30] be the weighted Hilbert space consisting of the elements φ of $L^2_\rho(Q)$ having first order generalized derivatives square summable on Q. The space $W^{1,1}_{2,\rho}(Q)$ is equipped with the scalar product

$$(\varphi, \psi)_{W^{1,1}_{2,\rho}(Q)} = (\varphi, \psi)_{L^2_\rho(Q)} + (\varphi_x, \psi_x)_{L^2_\rho(Q)} + (\varphi_t, \psi_t)_{L^2_\rho(Q)}, \tag{3}$$

and the associated norm is

$$\|\varphi\|^2_{W^{1,1}_{2,\rho}(Q)} = \|\varphi\|^2_{L^2_\rho(Q)} + \|\varphi_x\|^2_{L^2_\rho(Q)} + \|\varphi_t\|^2_{L^2_\rho(Q)}. \tag{4}$$

We also use the weighted spaces on $(0,1)$, such as $L^2_\rho((0,1))$ and $W^1_{2,\rho}((0,1))$, whose definitions are analogous to the spaces on Q.

(2) Double Laplace transform [31] The double Laplace transform $F(p,s)$ of a function $f(x,t)$ is defined by

$$L_x L_t [f(x,t)] = F(p,s) = \int_0^\infty e^{-px} \int_0^\infty e^{-st} f(x,t) dt\, dx, \tag{5}$$

where $x, t > 0$ and p, s are complex values, and further double Laplace transform of the first order partial derivatives for a function u is given by

$$L_x L_t \left[\frac{\partial u(x,t)}{\partial x} \right] = pU(p,s) - U(0,s). \tag{6}$$

where $U(p,s)$ is the double Laplace transform of $u(x,t)$. Similarly, the double Laplace transform for second partial derivative with respect to x and t are defined by

$$\begin{aligned} L_xL_t\left[\frac{\partial^2 u(x,t)}{\partial^2 x}\right] &= p^2U(p,s) - pU(0,s) - \frac{\partial U(0,s)}{\partial x}, \\ L_xL_t\left[\frac{\partial^2 u(x,t)}{\partial^2 t}\right] &= s^2U(p,s) - sU(p,0) - \frac{\partial U(p,0)}{\partial t}. \end{aligned} \tag{7}$$

The double Laplace transform of the functions $x\frac{\partial^2\psi}{\partial t^2}$ and $xf(x,t)$ are respectively given by

$$L_xL_t\left(x\frac{\partial^2\psi}{\partial t^2}\right) = -\frac{d}{dp}\left[s^2\Psi(p,s) - s\Psi(p,0) - \Psi_t(p,0)\right], \tag{8}$$

and

$$L_xL_t\left(xf(x,t)\right) = -\frac{dF(p,s)}{dp}, \tag{9}$$

The double Laplace transform of the non-constant coefficient second order partial derivative $x^n\frac{\partial^2\psi}{\partial t^2}$ and the function $x^nf(x,t)$ are given by

$$L_xL_t\left(x^n\frac{\partial^2\psi}{\partial t^2}\right) = (-1)^n\frac{d}{dp}\left[s^2\Psi(p,s) - s\Psi(p,0) - \Psi_t(p,0)\right], \tag{10}$$

$$L_xL_t\left(x^nf(x,t)\right) = (-1)^n\frac{d}{dp}[L_xL_tf(x,t)] = (-1)^n\frac{d^nF(p,s)}{dp^n}, \tag{11}$$

where $n = 1,2,3,\ldots$

The inverse double Laplace transform $L_p^{-1}L_s^{-1}\left[F(p,s)\right] = f(x,t)$ is defined by the complex double integral formula

$$L_p^{-1}L_s^{-1}\left[F(p,s)\right] = f(x,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{px}dp\frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty} e^{st}ds, \tag{12}$$

where $F(p,s)$ must be an analytic function for all p and s in the region defined by the inequalities $Rep \geq c$ and $Res \geq d$, where c and d are real constants to be chosen suitably.

(3) Young's inequality with ε [30]: For any $\varepsilon > 0$, we have the inequality

$$ab \leq \frac{1}{p}\left|\varepsilon a\right|^p + \frac{p-1}{p}\left|\frac{b}{\varepsilon}\right|^{\frac{p}{p-1}}, \quad a,b\in\mathbb{R}, p>1 \tag{13}$$

which is the generalization of *Cauchy inequality with* ε.

(4) Gronwall's Lemma [32]**:** If $f_i(\tau)(i = 1,2,3)$ are nonnegative functions on $(0,T)$, and $f_1(\tau)$, $f_2(\tau)$ are integrable functions , and $f_3(\tau)$ is non-decreasing on $(0,T)$, then if

$$\Im_\tau f_1 + f_2(\tau) \leq f_3(\tau) + c\Im_\tau f_2$$

then

$$\Im_\tau f_1 + f_2(\tau) \leq \exp(c\tau).f_3(\tau)$$

where

$$\Im_\tau g(t) = \int_0^\tau g(t)dt.$$

(5) Poincaré type inequalities [33]

$$\begin{aligned}
(i) \int_0^l \left(\Im_x (\xi u)\right)^2 dx &\leq \frac{l^3}{2} \|u(.,t)\|^2_{L^2_\rho(0,l)}, \\
(ii) \int_0^l \left(\Im_x^2 (\xi u)\right)^2 dx &\leq \frac{l^2}{2} \|\Im_x (\xi u)\|^2_{L^2(0,l)}, \\
(iii) \int_0^l x \left(\Im_x (\xi u)\right)^2 dx &\leq l \|\Im_x (\xi u)\|^2_{L^2(0,l)},
\end{aligned}$$

where

$$\Im_x (\xi u(\xi,t)) = \int_0^x \xi u(\xi,t)\, d\xi,\ \ \Im_x^2 (\xi u(\xi,t)) = \int_0^x \int_0^\xi \eta u(\eta,t)\, d\eta d\xi.$$

3. Problem Setting for a Singular Generalized Improved Modified Linear Boussinesq Equation

In the rectangle $Q = (0,1) \times (0,T)$, $T < \infty$, we consider an initial boundary value problem for the singular generalized improved modified linear Boussinesq equation with damping and with Bessel operator

$$\begin{aligned}
\mathcal{L}\psi &= \frac{\partial^2 \psi}{\partial t^2} - \frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial \psi}{\partial x}\right) - \frac{1}{x}\frac{\partial^2}{\partial x \partial t}\left(x\frac{\partial \psi}{\partial x}\right) - \frac{1}{x}\frac{\partial^3}{\partial x \partial t^2}\left(x\frac{\partial \psi}{\partial x}\right) \\
&= f(x,t), \qquad (14)
\end{aligned}$$

$$\psi(x,0) = f_1(x),\ \ \frac{\partial \psi(x,0)}{\partial t} = f_2(x),\ \ x \in (0,1), \qquad (15)$$

$$\begin{cases} \psi(1,t) = 0, & t \in (0,T), \\ \psi(0,t) = 0, & t \in (0,T), \end{cases} \qquad (16)$$

where $f_1(x)$, $f_2(x)$, and $f(x,t)$ are given functions that satisfy certain conditions which will be specified later on. We obtain an a priori estimate for the solution of problem (14)–(16) and use the modified double Laplace decomposition method for solving it.

4. A Priori Estimate for the Solution of Problem (14)–(16)

In this section, we establish an a priori estimate for the solution of problem (14)–(16) from which we deduce the uniqueness of the solution.

Theorem 1. *The solution ψ of the initial boundary value problem (14)–(16) satisfies the a priori estimate*

$$\begin{aligned}
&\sup_{0\leq\tau\leq T} \|\psi(.,\tau)\|^2_{W^{1,1}_{2,\rho}(0,l)} \\
\leq\ & 2e^{2T}\left(\|f_1\|^2_{W^1_{2,\rho}(0,l)} + \|f_2\|^2_{W^1_{2,\rho}(0,l)} + \|f\|^2_{L^2_\rho(Q)}\right). \qquad (17)
\end{aligned}$$

Proof. We consider the scalar product in $L^2(Q^\tau)$ of the operators $\mathcal{L}\psi$ and $M\psi$,where $M\psi = x\psi_t$,with $Q^\tau = (0,l) \times (0,\tau)$, $0 \le \tau \le T$, $0 < l < \infty$, we obtain

$$\begin{aligned}(\mathcal{L}\psi, M\psi)_{L^2(Q^\tau)} &= (\psi_{tt}, x\psi_t)_{L^2(Q^\tau)} - ((x\psi_x)_x, \psi_t)_{L^2(Q^\tau)} \\ &\quad - ((x\psi_x)_{xt}, \psi_t)_{L^2(Q^\tau)} - ((x\psi_x)_{xtt}, \psi_t)_{L^2(Q^\tau)} .\end{aligned} \tag{18}$$

By using initial and boundary conditions (15) and (16), terms on the right hand side of (18) can be evaluated as follows:

$$(\psi_{tt}, x\psi_t)_{L^2(Q^\tau)} = \frac{1}{2} \|\psi_t(.,\tau)\|^2_{L^2_\rho(0,1)} - \frac{1}{2} \|f_2\|^2_{L^2_\rho(0,1)} , \tag{19}$$

$$-((x\psi_x)_x, \psi_t)_{L^2(Q^\tau)} = \frac{1}{2} \|\psi_x(.,\tau)\|^2_{L^2_\rho(0,1)} - \frac{1}{2} \left\| \frac{\partial f_1}{\partial x} \right\|^2_{L^2_\rho(0,1)} , \tag{20}$$

$$-((x\psi_x)_{xt}, \psi_t)_{L^2(Q^\tau)} = \|\psi_{xt}\|^2_{L^2_\rho(Q^\tau)} , \tag{21}$$

$$-((x\psi_x)_{xtt}, \psi_t)_{L^2(Q^\tau)} = \frac{1}{2} \|\psi_{xt}(.,\tau)\|^2_{L^2_\rho(0,1)} - \frac{1}{2} \left\| \frac{\partial f_2}{\partial x} \right\|^2_{L^2_\rho(0,1)} . \tag{22}$$

Combination of (18)–(22), and Cauchy -ε inequality lead to

$$\begin{aligned}&\|\psi_t(.,\tau)\|^2_{L^2_\rho(0,1)} + \|\psi_x(.,\tau)\|^2_{L^2_\rho(0,1)} \\ &+ \|\psi_{xt}(.,\tau)\|^2_{L^2_\rho(0,1)} + 2\|\psi_{xt}\|^2_{L^2_\rho(Q^\tau)} \\ \le\ & \|f_2\|^2_{W^1_{2,\rho}(0,1)} + \left\| \frac{\partial f_1}{\partial x} \right\|^2_{L^2_\rho(0,1)} + \|\psi_t\|^2_{L^2_\rho(Q^\tau)} + \|f\|^2_{L^2_\rho(Q^\tau)} .\end{aligned} \tag{23}$$

We now consider the elementary inequality

$$\|\psi(.,\tau)\|^2_{L^2_\rho(0,1)} \le \|f_1\|^2_{L^2_\rho(0,1)} + \|\psi_t\|^2_{L^2_\rho(Q^\tau)} + \|\psi\|^2_{L^2_\rho(Q^\tau)} . \tag{24}$$

By summing inequalities (23) and (24) side to side, we obtain

$$\begin{aligned}&\|\psi(.,\tau)\|^2_{W^{1,1}_{2,\rho}(0,1)} + \|\psi_{xt}(.,\tau)\|^2_{L^2_\rho(0,1)} + \|\psi_{xt}\|^2_{L^2_\rho(Q^\tau)} \\ \le\ & 2\left(\|f_2\|^2_{W^1_{2,\rho}(0,1)} + \|f_1\|^2_{W^1_{2,\rho}(0,1)} + \|f\|^2_{L^2_\rho(Q^\tau)} + \|\psi_t\|^2_{L^2_\rho(Q^\tau)} + \|\psi\|^2_{L^2_\rho(Q^\tau)} \right) .\end{aligned} \tag{25}$$

Application of Gronwall's lemma [32] to inequality (25) with

$$\begin{cases} \Im_\tau f_1 = \|\psi_{xt}\|^2_{L^2_\rho(Q^\tau)} , \\ f_2(\tau) = \|\psi_t(.,\tau)\|^2_{L^2_\rho(0,1)} + \|\psi(.,\tau)\|^2_{L^2_\rho(0,1)} , \\ f_3(\tau) = \|f_2\|^2_{W^1_{2,\rho}(0,1)} + \|f_1\|^2_{W^1_{2,\rho}(0,1)} + \|f\|^2_{L^2_\rho(Q^\tau)} , \end{cases}$$

gives

$$\begin{aligned} & \|\psi(.,\tau)\|^2_{W^{1,1}_{2,\rho}(0,1)} + \|\psi_{xt}(.,\tau)\|^2_{L^2_\rho(0,1)} + \|\psi_{xt}\|^2_{L^2_\rho(Q^\tau)} \\ \leq \;& 2e^{2\tau}\left(\|f_1\|^2_{W^1_{2,\rho}(0,1)} + \|f_2\|^2_{W^1_{2,\rho}(0,1)} + \|f\|^2_{L^2_\rho(Q^\tau)}\right). \end{aligned} \tag{26}$$

By discarding the last two terms in the left-hand side of (26) and then taking the upper bound for both sides with respect to τ over $[0, T]$ of the obtained inequality, we obtain the following a priori estimate for the solution of the posed problem (14)–(16)

$$\begin{aligned} & \sup_{0\leq\tau\leq T} \|\psi(.,\tau)\|^2_{W^{1,1}_{2,\rho}(0,1)} \\ \leq \;& 2e^{2T}\left(\|f_1\|^2_{W^1_{2,\rho}(0,1)} + \|f_2\|^2_{W^1_{2,\rho}(0,1)} + \|f\|^2_{L^2_\rho(Q)}\right). \end{aligned} \tag{27}$$

□

5. The Modified Double Laplace Decomposition Method

The main aim of this section is to discuss the use of the modified double Laplace decomposition method for solving the linear initial value problem (14) and (15).

By using (6)–(9), we obtain

$$\begin{aligned} \frac{d\Psi}{dp} = \;& \frac{dF_1(p)}{sdp} + \frac{dF_2(p)}{s^2dp} - \frac{1}{s^2}L_xL_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial\psi}{\partial x}\right) + \frac{\partial^2}{\partial x\partial t}\left(x\frac{\partial\psi}{\partial x}\right)\right] \\ & -\frac{1}{s^2}L_xL_t\left[\frac{\partial^3}{\partial x\partial t^2}\left(x\frac{\partial\psi}{\partial x}\right)\right] + \frac{1}{s^2}\frac{dF(p,s)}{dp}. \end{aligned} \tag{28}$$

Integration of both sides of Equation (28) from 0 to p with respect to p, yields

$$\begin{aligned} \Psi(p,s) = \;& \frac{F_1(p)}{s} + \frac{F_2(p)}{s^2} - \frac{1}{s^2}\int_0^p L_xL_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial\psi}{\partial x}\right) + \frac{\partial^2}{\partial x\partial t}\left(x\frac{\partial\psi}{\partial x}\right)\right]dp \\ & -\frac{1}{s^2}\int_0^p L_xL_t\left[\frac{\partial^3}{\partial x\partial t^2}\left(x\frac{\partial\psi}{\partial x}\right)\right]dp + \frac{F(p,s)}{s^2}, \end{aligned} \tag{29}$$

where $F(p,s)$, $F_1(p)$ and $F_2(p)$ are Laplace transform of the functions $f(x,t)$, $f_1(x)$ and $f_2(x)$ respectively and the double Laplace transform with respect to x, t is defined by L_xL_t. Operating with the double Laplace inverse on both sides of Equation (29), we obtain

$$\begin{aligned} \psi(x,t) = \;& f_1(x) + tf_2(x) - L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_xL_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial\psi}{\partial x}\right) + \frac{\partial^2}{\partial x\partial t}\left(x\frac{\partial\psi}{\partial x}\right)\right]dp\right] \\ & -L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_xL_t\left[\frac{\partial^3}{\partial x\partial t^2}\left(x\frac{\partial\psi}{\partial x}\right)\right]dp - \frac{F(p,s)}{s^2}\right]. \end{aligned} \tag{30}$$

The modified double Laplace decomposition method (MDLDM) defines the solutions $\psi(x,t)$ by the infinite series

$$\psi(x,t) = \sum_{n=0}^{\infty}\psi_n(x,t). \tag{31}$$

Upon substitution of Equation (31) into (30), we get

$$
\begin{aligned}
\sum_{n=0}^{\infty}\psi_n(x,t) &= f_1(x)+tf_2(x)-L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_xL_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\left(\sum_{n=0}^{\infty}\psi_n(x,t)\right)\right)\right]dp\right] \\
&\quad -L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_xL_t\left[\frac{\partial^2}{\partial x\partial t}\left(x\frac{\partial}{\partial x}\left(\sum_{n=0}^{\infty}\psi_n(x,t)\right)\right)\right]dp\right] \\
&\quad -L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}L_xL_t\left[\frac{\partial^3}{\partial x\partial t^2}\left(x\frac{\partial}{\partial x}\left(\sum_{n=0}^{\infty}\psi_n(x,t)\right)\right)\right]\right] \\
&\quad +L_p^{-1}L_s^{-1}\left[\frac{F(p,s)}{s^2}\right]. \qquad (32)
\end{aligned}
$$

On comparing both sides of (32), we get

$$
\psi_0(x,t) = f_1(x)+tf_2(x)+L_p^{-1}L_s^{-1}\left[\frac{F(p,s)}{s^2}\right]. \qquad (33)
$$

In general, the recursive relation is given by

$$
\begin{aligned}
\psi_{n+1}(x,t) &= -L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_xL_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}(\psi_n(x,t))\right)\right]dp\right] \\
&\quad -L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_xL_t\left[\frac{\partial^2}{\partial x\partial t}\left(x\frac{\partial}{\partial x}(\psi_n(x,t))\right)\right]dp\right] \\
&\quad -L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}L_xL_t\left[\frac{\partial^3}{\partial x\partial t^2}\left(x\frac{\partial}{\partial x}(\psi_n(x,t))\right)\right]\right], \qquad (34)
\end{aligned}
$$

where $L_p^{-1}L_s^{-1}$ is the double inverse Laplace transform with respect to p, s. Here we assume that the double inverse Laplace transform with respect to p and s exists for each term in the right hand side of Equations (33) and (34). To illustrate this method, we consider the following example.

Example 1. *Consider the following singular generalized modified linear Boussinesq equation with Bessel operator:*

$$
\begin{aligned}
&\frac{\partial^2\psi}{\partial t^2}-\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial\psi}{\partial x}\right)-\frac{1}{x}\frac{\partial^2}{\partial x\partial t}\left(x\frac{\partial\psi}{\partial x}\right)-\frac{1}{x}\frac{\partial^3}{\partial x\partial t^2}\left(x\frac{\partial\psi}{\partial x}\right), \\
=\;& -x^2\sin t-4\cos t, \qquad (35)
\end{aligned}
$$

subject to the initial conditions

$$
\psi(x,0) = 0,\ \frac{\partial\psi(x,0)}{\partial t}=x^2. \qquad (36)
$$

By multiplying Equation (35) by x and using the definition of partial derivatives of the double Laplace transform and single Laplace transform for Equations (35) and (36), we obtain

$$
\begin{aligned}
\frac{d\Psi}{dp} &= -\frac{1}{s^2}L_xL_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial\psi}{\partial x}\right)+\frac{\partial^2}{\partial x\partial t}\left(x\frac{\partial\psi}{\partial x}\right)+\frac{\partial^3}{\partial x\partial t^2}\left(x\frac{\partial\psi}{\partial x}\right)\right] \\
&\quad +\frac{6}{p^4s^2(s^2+1)}+\frac{4}{p^2s(s^2+1)}-\frac{6}{p^4s^2}. \qquad (37)
\end{aligned}
$$

By integrating both sides of (37) from 0 *to* p *with respect to* p*, we obtain*

$$\begin{aligned} \Psi(p,s) &= -\frac{1}{s^2}\int_0^p L_x L_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial \psi}{\partial x}\right)+\frac{\partial^2}{\partial x\partial t}\left(x\frac{\partial \psi}{\partial x}\right)+\frac{\partial^3}{\partial x\partial t^2}\left(x\frac{\partial \psi}{\partial x}\right)\right]dp \\ &\quad -\frac{2}{p^3 s^2(s^2+1)}-\frac{4}{ps(s^2+1)}+\frac{2}{p^3 s^2}. \end{aligned} \tag{38}$$

Application of the inverse double Laplace transform to (38), yields

$$\begin{aligned} \psi(x,t) &= -L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x L_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial \psi}{\partial x}\right)+\frac{\partial^2}{\partial x\partial t}\left(x\frac{\partial \psi}{\partial x}\right)+\frac{\partial^3}{\partial x\partial t^2}\left(x\frac{\partial \psi}{\partial x}\right)\right]dp\right] \\ &\quad +x^2\sin t+4\cos t-4. \end{aligned} \tag{39}$$

Putting (31) into (39) to have

$$\begin{aligned} \sum_{n=0}^{\infty}\psi_n(x,t) &= -L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x L_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\sum_{n=0}^{\infty}\psi_n(x,t)\right)\right]dp\right] \\ &\quad -L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x L_t\left[\frac{\partial^2}{\partial x\partial t}\left(x\frac{\partial}{\partial x}\sum_{n=0}^{\infty}\psi_n(x,t)\right)\right]dp\right] \\ &\quad -L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x L_t\left[\frac{\partial^3}{\partial x\partial t^2}\left(x\frac{\partial}{\partial x}\sum_{n=0}^{\infty}\psi_n(x,t)\right)\right]dp\right] \\ &\quad +x^2\sin t+4\cos t-4. \end{aligned} \tag{40}$$

By modified Laplace decomposition method, we have

$$\psi_0 = x^2\sin t+4\cos t-4,$$

and

$$\begin{aligned} \psi_{n+1}(x,t) &= -L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x L_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\psi_n(x,t)\right)\right]dp\right] \\ &\quad -L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x L_t\left[\frac{\partial^2}{\partial x\partial t}\left(x\frac{\partial}{\partial x}\psi_n(x,t)\right)\right]dp\right] \\ &\quad -L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x L_t\left[\frac{\partial^3}{\partial x\partial t^2}\left(x\frac{\partial}{\partial x}\psi_n(x,t)\right)\right]dp\right]. \end{aligned}$$

Now the components of the series solution are

$$\begin{aligned} \psi_1 &= -L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x L_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\psi_0(x,t)\right)\right]dp\right] \\ &\quad -L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x L_t\left[\frac{\partial^2}{\partial x\partial t}\left(x\frac{\partial \psi_0}{\partial x}\right)\right]dp\right] \\ &\quad -L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x L_t\left[\frac{\partial^3}{\partial x\partial t^2}\left(x\frac{\partial}{\partial x}\psi_0(x,t)\right)\right]dp\right] \\ \psi_1 &= -L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x L_t\left[4x\cos t\right]dp\right] = -L_p^{-1}L_s^{-1}\left[\frac{-4}{ps(s^2+1)}\right] \\ &= 4-4\cos t. \end{aligned}$$

and,

$$\psi_2 = -L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x L_t\left[0\right]dp\right] = 0.$$

Eventually, the approximate solution of the unknown functions is given by

$$\begin{aligned} \sum_{n=0}^{\infty} \psi_n(x,t) &= \psi_0+\psi_1+\psi_2+\ldots. \\ &= x^2\sin t+4\cos t-4+4-4\cos t+0. \end{aligned}$$

Hence, the exact solution is given by

$$\psi(x,t)=x^2\sin t.$$

6. A Nonlinear Singular Boussinesq Equation with Bessel Operator

In this section, we consider the following nonlinear singular one dimensional **Boussinesq** equation [34]

$$\begin{aligned} &\psi_{tt}-\frac{1}{x}\frac{\partial}{\partial x}(x\psi_x)+a(x)\psi_{xxxx}-b(x)\psi_{xxtt}+c(x)\psi_t\psi_{xx}+d(x)\psi_x\psi_{xt} \\ =\ & f(x,t), \end{aligned} \tag{41}$$

subject to the initial conditions

$$\psi(x,0) = g_1(x), \ \frac{\partial\psi(x,0)}{\partial t}=g_2(x), \tag{42}$$

where $a(x)$, $b(x)$, $c(x)$ and $d(x)$ are given functions.

Multiplication of Equation (41) by x and application of double Laplace transform, give

$$\begin{aligned} &L_xL_t\left[x\psi_{tt}-\frac{\partial}{\partial x}(x\psi_x)+xa(x)\psi_{xxxx}-xb(x)\psi_{xxtt}+xc(x)\psi_t\psi_{xx}+xd(x)\psi_x\psi_{xt}\right] \\ =\ & L_xL_t[xf(x,t)]. \end{aligned} \tag{43}$$

On using the differentiation property of double Laplace transform and initial conditions (42), we get

$$\begin{aligned} \frac{d\Psi}{dp} &= \frac{dG_1(p)}{sdp}+\frac{dG_2(p)}{s^2dp}-\frac{1}{s^2}L_xL_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial\psi}{\partial x}\right)-xa(x)\psi_{xxxx}+xb(x)\psi_{xxtt}\right] \\ &+\frac{1}{s^2}L_xL_t[c(x)\psi_t\psi_{xx}+d(x)\psi_x\psi_{xt}]+\frac{1}{s^2}\frac{dF(p,s)}{dp}. \end{aligned} \tag{44}$$

By integrating both sides of (44) from 0 to p with respect to p, we have

$$\begin{aligned} \Psi(p,s) &= \frac{G_1(p)}{s}+\frac{G_2(p)}{s^2}+\frac{1}{s^2}\int_0^p\frac{dF(p,s)}{dp} \\ &-\frac{1}{s^2}\int_0^p L_xL_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial\psi}{\partial x}\right)\right]dp \\ &+\frac{1}{s^2}\int_0^p L_xL_t(xa(x)\psi_{xxxx}-xb(x)\psi_{xxtt})\,dp \\ &-\frac{1}{s^2}\int_0^p L_xL_t(xc(x)\psi_t\psi_{xx}+xd(x)\psi_x\psi_{xt})\,dp. \end{aligned} \tag{45}$$

Using double inverse Laplace transform, it follows from (45) that

$$\begin{aligned}\psi(x,t) &= f_1(x) + t f_2(x) + L_p^{-1} L_s^{-1}\left[\frac{1}{s^2}\int_0^p \frac{dF(p,s)}{dp}\right] \\ &-L_p^{-1} L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x L_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial\psi}{\partial x}\right)\right]dp\right] \\ &+L_p^{-1} L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x L_t\left[xa(x)\psi_{xxxx} - xb(x)\psi_{xxtt}\right]dp\right] \\ &-L_p^{-1} L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x L_t\left[xc(x)\psi_t\psi_{xx} + xd(x)\psi_x\psi_{xt}\right]dp\right]. \end{aligned} \tag{46}$$

Moreover, the nonlinear terms $N_1 = \psi_t\psi_{xx}$ and $N_2 = \psi_x\psi_{xt}$ are defined by

$$N_1 = \psi_t\psi_{xx} = \sum_{n=0}^{\infty} A_n, \quad N_1 = \psi_x\psi_{xt} = \sum_{n=0}^{\infty} B_n, \tag{47}$$

where the Adomian polynomials for A_n and B_n are defined by

$$A_n = \frac{1}{n!}\left(\frac{d^n}{d\lambda^n}\left[N_1\sum_{j=0}^{n}\left(\lambda^j\psi_j\right)\right]\right)_{\lambda=0}, n = 0, 1, 2, \ldots. \tag{48}$$

and

$$B_n = \frac{1}{n!}\left(\frac{d^n}{d\lambda^n}\left[N_2\sum_{j=0}^{n}\left(\lambda^j\psi_j\right)\right]\right)_{\lambda=0}, \; n = 0, 1, 2, \ldots. \tag{49}$$

By substitution of Equations (47)–(49) into (46), we obtain:

$$\begin{aligned}\sum_{n=0}^{\infty}\psi_n(x,t) &= f_1(x) + t f_2(x) + L_p^{-1} L_s^{-1}\left[\frac{1}{s^2}\int_0^p \frac{dF(p,s)}{dp}\right] \\ &-L_p^{-1} L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x L_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\left(\sum_{n=0}^{\infty}\psi_n\right)\right)\right]dp\right] \\ &+L_p^{-1} L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x L_t\left[xa(x)\left(\sum_{n=0}^{\infty}\psi_n\right)_{xxxx} - xb(x)\left(\sum_{n=0}^{\infty}\psi_n\right)_{xxtt}\right]dp\right] \\ &-L_p^{-1} L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x L_t\left[xc(x)\sum_{n=0}^{\infty}A_n + xd(x)\sum_{n=0}^{\infty}B_n\right]dp\right], \end{aligned} \tag{50}$$

where some few terms of A_n and B_n for $n = 0, 1, 2, 3$ are given by

$$\begin{aligned} A_0 &= \psi_{0t}\psi_{0xx} \\ A_1 &= \psi_{0t}\psi_{1xx} + \psi_{1t}\psi_{0xx} \\ A_2 &= \psi_{0t}\psi_{2xx} + \psi_{1t}\psi_{1xx} + \psi_{2t}\psi_{0xx} \\ A_3 &= \psi_{0t}\psi_{3xx} + \psi_{1t}\psi_{2xx} + \psi_{2t}\psi_{1xx} + \psi_{3t}\psi_{0xx}, \end{aligned} \tag{51}$$

and

$$\begin{aligned} B_0 &= \psi_{0x}\psi_{0xt} \\ B_1 &= \psi_{0x}\psi_{1xt} + \psi_{1x}\psi_{0xt} \\ B_2 &= \psi_{0x}\psi_{2xt} + \psi_{1x}\psi_{1xt} + \psi_{2x}\psi_{0xt}. \\ B_3 &= \psi_{0x}\psi_{3xt} + \psi_{1x}\psi_{2xt} + \psi_{2x}\psi_{1xt} + \psi_{3x}\psi_{0xt}. \end{aligned} \tag{52}$$

Therefore, from (50) above, it follows that

$$\psi_0(x,t) = f_1(x) + tf_2(x) + L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p \frac{dF(p,s)}{dp}\right],$$

and

$$\begin{aligned} \psi_{n+1}(x,t) &= -L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x L_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\psi_n\right)\right]dp\right] \\ &\quad + L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x L_t\left[xa(x)(\psi_n)_{xxxx} - xb(x)(\psi_n)_{xxtt}\right]dp\right] \\ &\quad - L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x L_t\left[xc(x)A_n + xd(x)B_n\right]dp\right]. \end{aligned} \tag{53}$$

To illustrate the used method, we consider the following example, where we let that $a(x) = b(x) = 1$, $c(x) = -4$, $d(x) = 2$ and $f(x,t) = -4t$ in Equation (41).

Example 2. *We consider the nonlinear Boussinesq equation with Bessel operator*

$$\psi_{tt} - \frac{1}{x}\frac{\partial}{\partial x}(x\psi_x) + \psi_{xxxx} - \psi_{xxtt} - 4\psi_t\psi_{xx} + 2\psi_x\psi_{xt} = -4t, \tag{54}$$

subject to the initial conditions

$$\psi(x,0) = 0, \quad \psi_t(x,0) = x^2. \tag{55}$$

The double Laplace decomposition method leads to the following:

$$\psi_0(x,t) = x^2 t - \frac{2}{3}t^3,$$

and

$$\begin{aligned} \psi_{n+1}(x,t) &= -L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x L_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\psi_n\right)\right]dp\right] \\ &\quad - L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x L_t\left[x(\psi_n)_{xxxx} - x(\psi_n)_{xxtt}\right]dp\right] \\ &\quad + L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x L_t\left[4xA_n - 2xB_n\right]dp\right]. \end{aligned} \tag{56}$$

The first iteration is given by

$$\begin{aligned}\psi_1(x,t) &= -L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_xL_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial\psi_0}{\partial x}\right)\right]dp\right] \\ &\quad -L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_xL_t\left[x(\psi_0)_{xxxx}-x\psi_{0xxtt}\right]dp\right] \\ &\quad +L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_xL_t\left[4xA_0-2xB_0\right]dp\right],\end{aligned} \tag{57}$$

$$\psi_1(x,t) = \frac{2}{3}t^3-\frac{4}{5}t^5. \tag{58}$$

The subsequent terms are given by

$$\begin{aligned}\psi_2(x,t) &= -L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_xL_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial\psi_1}{\partial x}\right)\right]dp\right] \\ &\quad -L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_xL_t\left[x(\psi_1)_{xxxx}-x\psi_{1xxtt}\right]dp\right] \\ &\quad +L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_xL_t\left[4xA_1-2xB_1\right]dp\right] \\ &= 0.\end{aligned} \tag{59}$$

and the rest terms are all zeros. Hence

$$\psi(x,t) = x^2t. \tag{60}$$

7. Conclusions

A modified double Laplace decomposition method is presented to study a singular generalized modified linear Boussinesq equation and a singular nonlinear Boussinesq equation. Some examples are given to confirm the validity, efficiency and accuracy of the method. It is found that this method is efficient and easier to apply to the studied linear and nonlinear Boussinesq models.

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